Fractional Solutions for NTU-Games

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Abstract

In this paper we survey some applications of Scarf's Lemma. First, we introduce the notion of fractional core for NTU-games, which is always nonempty by the Lemma. Stable allocation is a general solution concept for games where both the players and their possible cooperations can have capacities. We show that the problem of finding a stable allocation, given a finitely generated NTU-game with capacities, is always solvable by a variant of Scarf's Lemma. Finally, we describe the interpretation of these results for matching games.

1 Introduction

Complex social and economic situations can be described as games where the players may cooperate with each other. Most studies in cooperative game theory focus on the issue of how the participants form disjoint coalitions, and sometimes also on the way the members of coalitions share the utilities of their cooperations among themselves (in case of games with transferable utility). However, in reality, an agent in the market (or any individual in some social situation) may be involved in more than one cooperation at a time, moreover, a cooperation may be performed with different intensities. For instance, an employer can have several employees and their working hours can be different (but within some reasonable limits).

Scarf [20] proved that every balanced NTU-game (i.e, cooperative game with non-transferable utilities) has a nonempty core. His theorem was based on a lemma, which became known as Scarf's Lemma, as its importance has been recognised for its own right.

In this paper, we give a new interpretation of the fractional solutions which are obtained by the Scarf algorithm for different settings. First we consider the original setting of the Lemma for finitely generated NTU-games, and we describe the meaning of the output in terms of *fractional core*. We show the correspondence between this notion and the concept of *fractional stable matchings for hypergraphs*. We conclude Section 2 by explaining how the normality of a hypergraph implies the nonemptiness of the core for the corresponding NTU-games. In Section 3, we define the stable allocation problem for hypergraphs, which corresponds to the problem of finding a fractional core for NTU-games where the players can be involved in more than one coalition and the joint activities can be performed at different intensity levels (up to some capacity constrains). We show that a variant of the Scarf Lemma implies the existence of the latter solution as well. In Section 4, we apply these results for *matching games* and we derive some well-known theorems in this context. Finally, we present some important open problems and new research directions.

2 Fractional core - fractional stable matchings

In this section, first we describe Scarf's Lemma and we give a new interpretation of the fractional results obtained by the Lemma.

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2.1 Definitions, preliminaries

We recall the definition of *n-person games with nontransferable utility* (NTU-game for short).

Definition 1. *An* NTU-game *is given by a pair* (N, V)*, where* $N = \{1, 2, \ldots, n\}$ *is the set of players and* V *is a mapping of a set of feasible utility vectors, a subset* $V(S)$ *of* $\mathbb{R}^S$ *to each coalition of players,* $S \subseteq N$*, such that* $V(\emptyset) = \emptyset$*, and for all* $S \subseteq N$*,* $S \neq \emptyset$*:*

a) $V(S)$ *is a closed subset of* $\mathbb{R}^S$

b) $V(S)$ *is comprehensive, i.e. if* $u^S \in V(S)$ *and* $\tilde{u}^S \leq u^S$ *then* $\tilde{u}^S \in V(S)$

c) The set of vectors in $V(S)$ *in which each player in* S *receives no less than the maximum that he can obtain by himself is a nonempty, bounded set.*

One of the most important solution concepts is the core.

Definition 2. *A utility vector* $u^N \in V(N)$ *is in the core of the game, if there exists no coalition* $S \subseteq N$ *with a feasible utility vector* $\tilde{u}^S \in V(S)$ *such that* $u_i^N < \tilde{u}_i^S$ *for every player* $i \in S$*. Such a coalition is called blocking coalition.*

An NTU-game (N, V) is *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for every pair of disjoint coalitions S and T. In what follows, we restrict our attention to superadditive games.

Partitioning games are special superadditive games. Given a set of *basic coalitions* $\mathcal{B} \subseteq 2^N$, that contain all singletons (i.e. every single player has the right not to cooperate with the others), a partitioning game $(N, V, \mathcal{B})$ is defined as follows: if $\Pi_{\mathcal{B}}(S)$ denotes the set of partitions of S into basic coalitions, then $V(S)$ can be generated as:

$$V(S) = \{u^S \in \mathbb{R}^S | \exists \pi = \{B_1, B_2, \ldots, B_k\} \in \Pi_{\mathcal{B}}(S) : u^S \in V(B_1) \times V(B_2) \times \cdots \times V(B_k)\}$$

This means that u^S is a feasible utility vector of S if there exist a partition π of S into basic coalitions such that each utility vector $u^S|_{B_i}$ can be obtained as a feasible utility vector by basic coalition B_i in π.

Given an NTU-game (N, V), let $U(S)$ be the set of *Pareto optimal* utility vectors of the coalition S, i.e. $u^S \in U(S)$ if there exists no $\tilde{u}^S \in V(S)$, where $u^S \neq \tilde{u}^S$ and $u^S \leq \tilde{u}^S$.

A utility vector $u^S \in V(S)$ is *separable* if there exist a proper partition π of S into subcoalitions $S_1, S_2, \ldots, S_k$ such that $u^S|_{S_i}$ is in $V(S_i)$ for every $S_i \in \pi$. A utility vector that is non-separable, Pareto-optimal and in which each player receives no less than the maximum that he can obtain by himself is called an *efficient* vector. A coalition S is *essential* if $V(S)$ contains an efficient utility vector. In other words, a coalition S is essential, if its members can obtain an efficient utility vector that is not achievable independently by its subcoalitions. The set of essential coalitions is denoted by $\mathcal{E}(N, V)$.

We say that a coalition S is not *relevant* if for every utility vector $u^S \in V(S)$ there exists a proper subcoalition $T \subset S$ such that $u^S|_T$ is in $V(T)$. The set of relevant coalitions is denoted by $\mathcal{R}(N, V)$. The idea behind this notion is that if a non-relevant coalition S is blocking with a utility vector u^S, then one of its subcoalitions, say T_1, must be also blocking with utility vector $u^{T_1} = u^S|_{T_1}$. Moreover, if T_1 is not relevant or u^{T_1} is separable, then we can find another coalition $T_2 \subset T_1$, such that $u^{T_2} = u^{T_1}|_{T_2} = u^S|_{T_2}$, an so on. Continuing this argument, it is clear that there must be a relevant coalition $T_i \subset S$, that is blocking with a non-separable vector $u^{T_i} = u^S|_{T_i}$. This observation implies the following Proposition:

Proposition 1. *A utility vector* $u^N \in V(N)$ *is in the core if and only if it is not blocked by any relevant coalition with an efficient utility vector.*

Obviously, if a coalition is not essential, then it cannot be relevant either. In a partitioning game, the set of essential coalitions must be a subset of the basic coalitions by definition.

Proposition 2. *For every partitioning game* $(N, V, \mathcal{B})$, $\mathcal{R}(N, V, \mathcal{B}) \subseteq \mathcal{E}(N, V, \mathcal{B}) \subseteq \mathcal{B}$ *holds.*

Scarf [20] observed that the previously introduced notions are purely ordinal in character: they are invariant under a continuous monotonic transformation of the utility function of any individual. Hence, without loss of generality, we may assume that $U^{\{i\}} = \{0\}$ for every singleton, and all the efficient utility vectors are nonnegative. Moreover, the discussion can be carried out on an abstract level with the outcomes for each individual represented by arbitrary ordered sets, as we describe this in detail below.

Suppose that in order to obtain a particular non-separable vector $u^{S,k}$ in $U(S)$, the members of S have to perform a joint activity, say $\mathfrak{a}^{S,k}$. Let $\mathcal{A}^S$ denote the set of activities that yield efficient utility vectors in $U(S)$. The preference of a player over the possible activities in which he can be involved is determined by the utilities that he obtains in these activities. Formally, we suppose that $\mathfrak{a}^{S,k} \leq_i \mathfrak{a}^{T,l} \iff u_i^{S,k} \leq u_i^{T,l}$ for any pair of activities $\mathfrak{a}^{S,k}$ and $\mathfrak{a}^{T,l}$, where $i \in S$ and $i \in T$.

Considering an efficient utility vector $u^{N,l}$ of the grandcoalition N, the non-separability implies that $u^{N,l}$ corresponds to a joint activity $\mathfrak{a}^{N,l}$ of the entire set of players. Otherwise, if $u^{N,l}$ is separable, then $u^{N,l}$ can be obtained as a direct sum of independent efficient utility vectors of essential subcoalitions that form a partition of the grandcoalition. This can be considered as a set of independent activities of the subcoalitions. An *outcome of the game*, denoted by X then can be regarded as a partition π of the players and a set of activities $\mathcal{A}^\pi$ performed independently by the coalitions in π, so let $X = (\pi, \mathcal{A}^\pi)$. An outcome X is judged by a player i according to the activity he is involved in, denoted by $\mathfrak{a}_i(X)$. An outcome is in the core of the game, or in other words, it is *stable* if there exist no blocking coalition S and an activity $\mathfrak{a}^{S,k}$ that is strictly preferred by all of its members, i.e., $\mathfrak{a}^{S,k} >_i \mathfrak{a}_i(X)$ for every $i \in S$. (This is equivalent to the blocking condition $u_i^{N,l} < \tilde{u}_i^S$, if the outcome X corresponds to the utility vector $u^{N,l}$.)

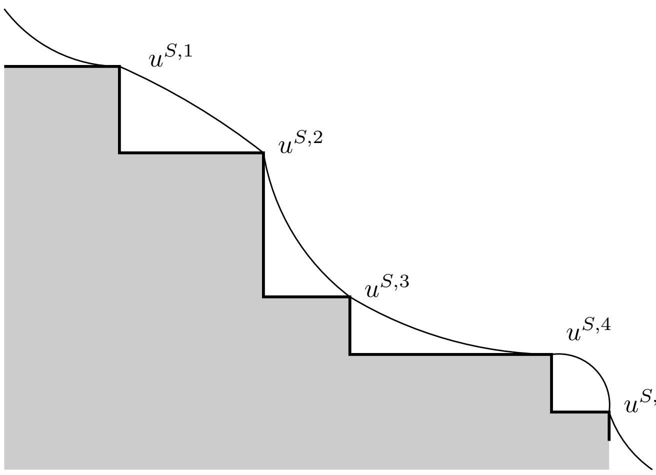
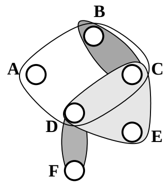


Figure 1: Approximation with finite number of efficient utility vectors.

An NTU-game is *finitely generated* if for every essential coalition S, $U(S)$ contains a finite number of vectors. Here, the preference order of a player over the set of activities, in which he can be involved, can be represented by preference lists. As Scarf observed in [20] and [21], a general NTU-game can be approximated by a finitely generated NTU-game (see an illustration in Figure 1). Here, we will not discuss this question in details.

If for every essential coalition S, in a given NTU-game, $U(S)$ contains only one single vector, u^S then an outcome of the game is simply a partition, since each essential coalition has only one activity to perform. So here, instead of activities, each player has a preference order over the essential coalitions in which he can be a member. These games are called *coalition formation games* (CFG for short), and an outcome that is in the core of the game is called a *core-partition*. The following example illustrates a CFG.

Example 1. *Suppose that we are given 6 players: A, B, C, D, E and F, and 4 possible basic coalitions with corresponding joint activities. The first activity, b (bridge) can be played by A, B, C and D, the second one, p (poker) can be played by C, D and E. Finally, B can play chess with C (denoted by c_1) and D can play chess with F (denoted by c_2). The preferences of the players over the joint activities are as follows.*



Activities	Participants	Players	Preference lists
b :	$\{A, B, C, D\}$	B :	$b\ c_1$
p :	$\{C, D, E\}$	C :	$p\ b\ c_1$
c_1 :	$\{B, C\}$	D :	$b\ p\ c_2$
c_2 :	$\{D, F\}$		

Here, $\{p, \{A\}, \{B\}, \{F\}\}$ is a core-partition, since b is not blocking because C prefers his present coalition p to b, similarly, c_1 is not blocking because C prefers playing poker with D and E to playing chess with B, and c_2 is not blocking because D also prefers playing poker to playing chess with F. One can easily check that $\{b, \{E\}, \{F\}\}$ is also a core-partition, but the partition $\{c_1, c_2, \{A\}, \{E\}\}$ is not in the core, since p and b are both blocking coalitions.

2.2 Fractional core by Scarf's Lemma

First, we present Scarf's Lemma [20] and then we introduce the notion of fractional core. The following description of the Lemma is due to Aharoni and Fleiner [1] (here $[n]$ denotes the set of integers $1, 2, \ldots, n$, and $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise).

Theorem 3 (Scarf, 1967)**.** *Let n, m be positive integers, and b be a vector in $\mathbb{R}^n_+$. Also let $A = (a_{i,j}), C = (c_{i,j})$ be matrices of dimension $n \times (n + m)$, satisfying the following three properties: the first n columns of A form an $n \times n$ identity matrix (i.e. $a_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}^{n+m}_+ : Ax = b\}$ is bounded, and $c_{i,i} < c_{i,k} < c_{i,j}$ for any $i \in [n], i \neq j \in [n]$ and $k \in [n + m] \setminus [n]$.*

Then there is a nonnegative vector x in $\mathbb{R}^{n+m}_+$ such that $Ax = b$ and the columns of C that correspond to supp(x) form a dominating set, that is, for any column $i \in [n + m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \leq c_{k,j}$ for any $j \in$ supp(x).

Let the columns of A and C correspond to the efficient utility vectors (or equivalently to some activities) of the essential coalitions in a finitely generated NTU-game as follows. If the k-th columns of A and C correspond to the utility vector $u^{S,l}$, then let $a_{i,k}$ be 1 if $i \in S$ and 0 otherwise, (so the k-th column of A is the *membership vector* of coalition S). Furthermore, let $c_{i,k} = u^{S,l}_i$ if $i \in S$ and $c_{i,k} = M$ otherwise, where M is a sufficiently large number. We set $c_{i,i} = u^{\{i\}}_i = 0$ and $c_{i,j} = 2M$ if $i \neq j \leq n$. Finally, let $b = 1^N$. By applying Scarf's Lemma for this setting, we obtain a solution x that we call a *fractional core element* of the game. We refer to the set of fractional core elements as the *fractional core* of the game.

What is the meaning of a fractional core element? Let us suppose first, that a fractional core element x is integer, so $x_i \in \{0,1\}$ for all i. In this case we show that x gives a utility vector u^N that is in the core of the game. Let u^N be the utility vector of N received by summing up those independent essential utility vectors for which $x(u^{S,k}) = 1$, then u^N is obviously in $V(N)$ by superadditivity. To prove that u^N must be in the core of the game, let $u^{S,k}$ be an essential utility vector, with $x(u^{S,k}) = 0$. By the statement of Scarf's Lemma, there must be a player i and an essential utility vector $u^{T,l}$, such that $i \in T$, $x(u^{T,l}) = 1$ and $u_i^{S,k} \leq u_i^{T,l}$, so S cannot be a blocking coalition with the efficient utility vector $u^{S,k}$.

In other words, the $Ax = 1^N$ condition of the solution says that x gives a partition π of N and a set of activities $\mathcal{A}^\pi$ that are performed (we say that $\mathfrak{a}^{S,k}$ is *performed*, i.e. $\mathfrak{a}^{S,k} \in \mathcal{A}^\pi$, if $x(u^{S,k}) = 1$, implying that S is a coalition in partition π). Let $X = (\pi, \mathcal{A}^\pi)$ be the corresponding outcome, and let $\mathfrak{a}^{S,k}$ be an activity not performed, (i.e. $\mathfrak{a}^{S,k} \notin \mathcal{A}^\pi$). Then, by Scarf's Lemma there must be a player i of S for which the performed activity, $\mathfrak{a}_i(X)$ he is involved in is not worse than $\mathfrak{a}^{S,k}$, i.e., $\mathfrak{a}^{S,k} \leq_i \mathfrak{a}_i(X)$, thus S cannot be a blocking coalition with activity $\mathfrak{a}^{S,k}$.

In the non-integer case, we shall regard $x(u^{S,k})$ as the *intensity* at which the activity $\mathfrak{a}^{S,k}$ is performed by coalition S. The $Ax = 1^N$ condition means that each player participates in activities with total intensity 1, including maybe the activity that this player performs alone. The domination condition says that for each activity, which is not performed with intensity 1, there exists a member of the coalition who is not interested in increasing the intensity of this activity, since he is satisfied by some other preferred activities that fill his remaining capacity. Formally, if $x(u^{S,k}) < 1$ then there must be a player i in S such that $\sum_{a^{T,l} \geq_i a^{S,k}} x(u^{T,l}) = 1$.

In Example 1, $x(p) = \frac{1}{3}$, $x(b) = \frac{2}{3}$ is a fractional core element, since for each activity there is at least one player who is not interested in increasing the intensity of that activity. In our corresponding technical report [5] we illustrate with an example that the fractional core of a game may admit a unique fractional core element where the intensities of certain activities can be arbitrary small nonnegative values.

2.3 Fractional stable matching for hypergraphs

For a finitely generated NTU-game, the problem of finding a stable outcome is equivalent to the *stable matching problem* (SM for short) *for a hypergraph*, as defined by Aharoni and Fleiner [1]. Here, the vertices of the hypergraph correspond to the players, the edges correspond to the efficient vectors (or to activities being performed by the players concerned), and the preference of a vertex over the edges it is incident with comes from the preference of the corresponding player over the activities he can be involved in. This is called a *hypergraphic preference system*. A *matching* corresponds to a set of joint activities performed by certain coalitions that form a partition of the grandcoalition together with the singletons (i.e. with the vertices not covered by the matching). A matching M is *stable* if there exist no *blocking edge*, i.e. an edge $e \notin M$ such for that every vertex v covered by e, either v is unmatched in M or strictly prefers e to the edge that covers v in M. The corresponding set of activities gives a stable outcome, since there exist no blocking coalition with an activity that is strictly preferred by all of its members. Note that different activities performed by the same players are represented by *multiple edges* in the corresponding hypergraph. A hypergraph which represents the efficient utility vectors of a CFG is simple (i.e, does not contain multiple edges and loops). [3]

[3] We shall note that Aharoni and Fleiner [1] supposed in their model that the preferences of the players are strict (i.e., no player is indifferent between any pair of activities). In the literature on stable matching, the setting where players may have *ties* in their lists is referred to as *stable matching problem with ties.* In this case, a matching M is (weakly) stable if it does not admit a blocking edge (where the definition of a

The notion of a *fractional stable matching* for an instance of SM for a hypergraph was defined by Aharoni and Fleiner [1] as follows. A function x assigning non-negative weights to edges of the hypergraph is called a *fractional matching* if $\sum_{v\in h} x(h) \leq 1$ for every vertex v. A fractional matching x is called *stable* if every edge e contains a vertex v such that $\sum_{v\in h, e\leq_v h} x(h) = 1$. The existence of a fractional stable matching can be verified by Scarf's Lemma just like the existence of a fractional core element. Actually, these two notions are basically equivalent.

To show the equivalence formally, we consider the polytope of intensity vectors $\{x|Ax = 1^N, x \geq 0\}$ on the one hand, where A is the membership-matrix of the efficient utility vectors (or the corresponding activities) of dimension $n \times (n+m)$ as defined by Scarf's Lemma. On the other hand, the fractional matching polytope is $\{x|Bx \leq 1^N, x \geq 0\}$, where B is the vertex-edge incidence matrix of the hypergraph of dimension $n \times m$. Obviously, $A = (I_n|B)$, so the difference is only the $n \times n$ identity matrix, i.e. the membership-matrix of the singletons. So, there is a natural one-to-one correspondence between the elements of the two polytopes: if x^m is a fractional matching of dimension m, then let $\bar{x}^v = 1^N - Ax^m$ be a vector of dimension n, that gives the unfilled intensities of the players (or in other words, the intensities of the single activities). The direct sum of these two independent vectors, x is an intensity vector of dimension $n+m$, and vice versa. The stability condition is equivalent to the domination condition of Scarf's Lemma.

Aharoni and Fleiner [1] showed that a fractional stable matching can be assumed to be an extremal point of the fractional matching polytope. This fact comes from a statement similar to the following Proposition:

Proposition 4. *If x is a fractional core element of a finitely generated NTU-game, and $x = \sum \alpha_i x^i$, where $\alpha_i > 0$ for all i, $\sum \alpha_i = 1$ and x^i satisfies the $Ax^i = 1^N$ and $x^i \geq 0$ conditions, then each x^i must be a fractional core element.*

The proof of this Proposition is obvious, since $supp(x^i) \subseteq supp(x)$, that implies the dominating property of the fractional core element.

Corollary 5. *For any finitely generated NTU-game, there exists a fractional core element that is an extremal point of the polytope $\{x|Ax = 1^N, x \geq 0\}$.*

Corollary 5 implies that if, for a given finitely generated NTU-game, all the extremal points of the above polytope are integers (or, in other words, the polytope has the *integer property*) then the game has a nonempty core.

2.4 Normality implies the nonemptiness of the core

The definition of a normal hypergraph is due to Lovász [19]. If H is a hypergraph and H' is obtained from H by deleting edges, then H' is called a *partial hypergraph* of H. The *chromatic index* $\chi_e(H)$ of a hypergraph H is the least number of colours sufficient to colour the edges of H so that no two edges with the same colour have a vertex in common. Note that the maximum degree, $\Delta(H)$ (that is, the maximum number of edges containing some one vertex) is a lower bound for the chromatic index. A hypergraph H is *normal* if every partial hypergraph H' of H satisfies $\chi_e(H') = \Delta(H')$. Obviously, the normality is preserved by adding or deleting multiple edges or loops. The following theorem of Lovász [19] gives an equivalent description of normal hypergraphs.

blocking edge is the same as described above). However, an instance of SM with ties can be always derived to another instance of SM (with no ties) by simply breaking the ties arbitrary, and a matching that is stable for the derived instance is (weakly) stable for the original one. The same applies for the core and fractional core in the context of NTU-games. In fact, the Scarf algorithm starts with a perturbation of matrix C in the case that any player is indifferent between two activities she may be involved in (i.e., when her utilities in these two activities are the same for her).

Theorem 6 (Lovász). *The fractional matching polytope of a hypergraph H has the integer property if and only if H is normal.*

Suppose that for a finitely generated NTU-game the set of essential coalitions forms a normal hypergraph. The hypergraph of the corresponding SM must be also normal, since it is obtained by adding multiple edges and by removing the loops. By Theorem 6, the fractional matching polytope, $\{x|Bx \leq 1^N, x \geq 0\}$ has the integer property, and so has the polytope of intensity vectors, $\{x|Ax = 1^N, x \geq 0\}$ as it was discussed previously. This argument and Corollary 5 verify the following Lemma 7.

Lemma 7. *If, for a finitely generated NTU-game, the set of essential coalitions, $\mathcal{E}(N, V)$ forms a normal hypergraph, then the core of the game is nonempty.*

By Lemma 7 and Proposition 2 the following holds.

Theorem 8. *If the set of basic coalitions, $\mathcal{B}$ forms a normal hypergraph, then every finitely generated NTU-game $(N, V, \mathcal{B})$ has a nonempty core.*

Let $A^{\mathcal{B}}$ denote the membership-matrix of the set of basic coalitions $\mathcal{B}$. The fact that the integer property of the polytope $\{x|A^{\mathcal{B}}x = 1^N, x \geq 0\}$ implies the nonemptiness of every NTU-game $(N, V, \mathcal{B})$ was proved first by Vasin and Gurvich [23], and independently, by Kaneko and Wooders [14]. Later, Le Breton *et al.* [18], Kuipers [17] and Boros and Gurvich [8] observed independently that the integer property of the polytope $\{x|A^{\mathcal{B}}x = 1^N, x \geq 0\}$ is equivalent to the integer property of the matching polytope $\{x|A^{\mathcal{B}}x \leq 1^N, x \geq 0\}$, and to the normality of the corresponding hypergraph.

3 Fractional b-core with capacities - stable allocations

In what follows, we introduce the notion of fractional b-core element as a solution of Scarf's Lemma with the original settings. Let the same matrices A and C of dimension $n \times (n+m)$ correspond to the set of effective utility vectors (or activities) in a finitely generated NTU-game as it was described in the previous section. The only modification is that now b is an arbitrary vector of $\mathbb{R}^n_+$ (instead of 1^N). Let $x \in \mathbb{R}^{n+m}_+$ be referred to as a *fractional b-core element* if x is a solution of the Scarf Lemma for the above setting.

Here, $b(i)$ is an upper bound for the total intensity at which player i is capable to perform activities, since $\sum_{i\in S} x(u^{S,l}) = b(i)$. The domination condition of the Lemma says that for every activity $\mathfrak{a}^{T,k}$, there exists some player i who is not interested in increasing the intensity of $\mathfrak{a}^{T,k}$, because his remaining intensity is filled with better activities, i.e., if $u^{T,k}$ corresponds to activity $\mathfrak{a}^{T,k}$, then $\sum_{u_i^{S,l} \geq u_i^{T,k}} x(u^{S,l}) = b(i)$.

In fact, to produce a fractional core element (in other words, a fractional 1^N-core element) with the algorithm of Scarf, we perturb not just matrix C (in case of indifferences), but also the vector 1^N, to avoid the degeneracy. The standard nondegeneracy assumption provides that all variables associated with the n columns of a feasible basis for the equations $A\tilde{x} = \tilde{b} = 1^N + \varepsilon^N$ are strictly positive. Thus, the perturbation uniquely determines the steps of Scarf algorithm. By rounding the final fractional $\tilde{b}$-core element $\tilde{x}$, a fractional core element x is found. The following simple Lemma says that the fractional b-core element has the *scaling property.*

Lemma 9. *Given a finitely generated NTU-game, and a positive constant λ. Suppose that $b' = \lambda \cdot b$, then x is a fractional b-core element if and only if $x' = \lambda \cdot x$ is a fractional b'-core element.*

Let us suppose that the intensities of the activities in the finitely generated NTU-game are constrained by *capacities*. Formally, for each joint activity $\mathfrak{a}^{S,l}$ and for the corresponding utility vector $u^{S,l}$, there may exist a nonnegative capacity $c(u^{S,l})$ for which $x(u^{S,l}) \leq c(u^{S,l})$ is required.

The stable allocation problem can be defined for hypergraphs as follows. Suppose that we are given a hypergraph H and for each vertex v a strict preference order over the edges incident with v (again, this corresponds to the preferences of the players over the activities in which they can be involved). Suppose, that we are given nonnegative *bounds* on the vertices $b : V(H) \to \mathbb{R}_+$ and nonnegative *capacities* on the edges $c : E(H) \to \mathbb{R}_+$. A nonnegative function x on the edges, is an *allocation* if $x(e) \leq c(e)$ for every edge e and $\sum_{v\in h} x(h) \leq b(v)$ for every vertex v. An allocation is *stable* if every *unsaturated* edge e (i.e., every edge e with $x(e) < c(e)$) contains a vertex v such that $\sum_{v\in h, e\leq_v h} x(h) = b(v)$. In this case we say that e is *dominated at* v. If every bound and capacity is integral then we refer to this problem as the *integral stable allocation problem.*

Theorem 10. *Every stable allocation problem for hypergraphs is solvable.*

Proof. Suppose that we are given a given a hypergraph H. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices and let $E(H) = \{e_1, e_2, \ldots, e_m\}$ be the set of edges. We define the extended membership-matrix A, and the extended preference-matrix C of size $(n + m) \times (n + 2m)$ as follows.

The left part of A is an identity matrix of size $(n + m) \times (n + m)$, (i.e. $a_{i,j} = \delta_{i,j}$ for $i, j \in [n + m]$). At the bottom of the right side there is another identity matrix of size $m \times m$, so $a_{n+i,n+m+j} = \delta_{i,j}$ for $i, j \in [m]$. Finally, at the top of the right side we have the vertex-edge incidence matrix of H (i.e. $a_{i,n+m+j} = 1$ if $v_i \in e_j$ and 0 otherwise for $i \in [n]$ and $j \in [m]$).

The top-right part of C correspond to the preference of the vertices (that is the preference of the players over the activities). We require the following two conditions:

- $c_{i,n+m+j} < c_{i,n+m+k}$ whenever $v_i \in e_j \cap e_k$ and $e_j <_{v_i} e_k$;
- $c_{i,n+m+j} < c_{i,n+m+k}$ whenever $v_i \in e_j \setminus e_k$.

Furthermore, suppose that $c_{n+i,n+m+i} < c_{n+i,n+m+j}$ for every $i \neq j \in [m]$ in the bottom-right part of C. Finally, let the left part of C be such that it satisfies the conditions of Scarf's Lemma. The constant vector, $b \in R_+^{n+m}$ is given by the bounds and capacities, so let $b_i = b(v_i)$ for $i \in [n]$ and $b_{n+j} = c(e_j)$ for $j \in [m]$.

We shall prove that the fractional core element x, obtained by Scarf's Lemma, gives a stable allocation, x^e by simply taking the last m coordinates of x. Here, x^e_j is equal to $x^e(e_j)$ that is the weight of the edge e_j (or equivalently, this is the intensity at which the corresponding activity is performed). If $\bar{x}^v$ and $\bar{x}^e$ are the vectors obtained by taking the $[1, \ldots, n]$ and $[n + 1, \ldots, n + m]$ coordinates of x, then these vectors correspond to the remaining weights of the vertices and edges (or the remaining intensities of the players and the activities), respectively.

Obviously, x^e is an allocation by $Ax = b$, since the first n equations preserve the $\sum_{v\in h} x^e(h) \leq b(v)$ condition for every vertex v, and the last m equations preserve $x^e(e) \leq c(e)$ for every edge e.

To prove stability, let us consider an unsaturated edge e_j and let us suppose that the corresponding dominating row by the lemma has index k. First we show, that $i \in [n]$. From $Ax = b$, obviously $\bar{x}^e(e_k) + x^e(e_k) = c(e_k)$ for every edge e_k. Since $x^e(e_j) < c(e_j)$, then $\bar{x}^e(e_j) > 0$, thus the assumptions on C imply that $i \neq n + j$, for other $i \in [n + m] \setminus [n]$ the contradiction is trivial. If $i \in [n]$, then e is dominated at v_i, since $\bar{x}^v(v_i) = 0$ by the assumptions on C, and the $Ax = b$ condition for the i-th row together with the statement of the lemma imply $\sum_{v_i\in h, e_j\leq_{v_i} h} x^e(h) = b(v_i)$. ☐

4 Matching games

Matching games can be defined as partitioning NTU-games, where the cardinality of each basic coalition is at most 2. For simplicity, in this section we suppose that no player is indifferent between two efficient utility vectors, so their preferences over the joint activities are strict. If a matching game is finitely generated, then the problem of finding an outcome that is in the core is equivalent to a SM for a graphic preference system, where the edges of the graph correspond to efficient utility vectors (and to joint activities).

4.1 Stable matching problem

If the graph of a matching game is simple (i.e, if it contains no multiple edges and loops) then the problem of finding a core-partition for the resulting CFG is called *stable roommates problem*. Otherwise, if the graph has multiple edges then we may refer to SM as *stable roommates problem with multiple activities*.

Let us suppose the set of players N can be divided into two parts, say M and W, such that every two-member basic coalition contains one member from each side (so if $\{m, w\} \in \mathcal{B}$ then $m \in M$ and $w \in W$). In this case, we get a *two-sided matching game* (in the general nonbipartite case the matching game is called *one-sided*).

If a two-sided matching game is finitely generated then the corresponding graphic representation of the SM is bipartite. For bipartite graphs, the following Proposition is well-known.

Proposition 11. *Every bipartite graph is normal.*

Proposition 11 and Theorem 8 imply the following result.

Theorem 12. *Every finitely generated two-sided matching game has a nonempty core.*

Theorem 12 was proved for every two-sided matching game, originally called *central assignment game*, by Kaneko [13]. For the corresponding CFG-s, called *stable marriage problems*, this result was proved by Gale and Shapley [11].

A one-sided matching game can have an empty core, even for a CFG, as Gale and Shapley [11] illustrated with an example. However the half-integer property of the fractional matching polytope implies the existence of stable half-solutions. The following statement is due to Balinski [4].

Theorem 13. *The fractional matching polytope for every graph has only half-integer extremal points.*

As Aharoni and Fleiner [1] showed, Theorem 13 and Corollary 5 imply that for every matching game there exists a so-called *half-core element*, that is a fractional core element x with the half-integer property, i.e. $x_i \in \{0, \frac{1}{2}, 1\}$.

Theorem 14. *If a matching game is finitely generated then it always has a half-core element.*

For CFG-s, the fact that for every instance of SM there exists a stable half-matching was proved by Tan [22]. Finally we note that an easy consequence of Theorem 13 and Lemma 9 is that for every finitely generated matching game, there always exists an integer 2^N-core element.

4.2 Stable allocation problem for graphs

The stable allocation problem was introduced by Baïou and Balinski [3] for bipartite graphs. The integer version, where the allocation x is required to be integer on every edge for integer bounds and capacities, was called the *stable schedule problem* by Alkan and Gale [2] (however they considered a more general model, the case of so-called *substitutable* preferences).

Biró and Fleiner [6] generalised the algorithm of Baïou and Balinski [3] for nonbipartite graphs, resulting in a weakly polynomial algorithm that finds a half-integral stable allocation for any given instance of integral stable allocation problem. Dean and Munshi [9] strengthened this result by giving a strongly polynomial algorithm for the same problem. But we shall note that the existence of a stable half-integer allocation is a consequence of Theorem 10.

Theorem 15. *For every integral stable allocation problem in a graph there exists a half-integral stable allocation. If the graph is bipartite, then every integral stable allocation problem is solvable.*

Proof. Suppose that we have a stable allocation x that has some weights that are not half-integers. We create another stable allocation x' with half-integer weights as follows. If $x(e)$ is not integer then let v be the vertex where e is dominated. Since $b(v)$ is integer, there must be another edge, f that is incident with v and has non-integer weight. Moreover, f cannot be dominated at v. By this argument, it can be verified that the edges with non-integer weights form vertex-disjoint cycles, moreover, in each such a cycle the fractional parts of the weights are ε and $1 - \varepsilon$ alternately. If a cycle is odd, then ε must be $\frac{1}{2}$. If a cycle is even, then ε can be modified to be 0 (or 1) in such a way that the obtained allocation x' remains stable and has only half-integer weights.

If the graph is bipartite, thus has no odd cycle, then x' has only integer weights, so x' is an integral stable allocation . □

In [5] we give an integral stable allocation problem for a graph and we illustrate how a half-integer stable allocation can be obtained with the Scarf algorithm.

5 Further directions

Guarantees for solvability. The original goal of Scarf [20] was to give a necessary condition for the nonemptiness of the core for general NTU-games (and this condition was the balancedness of the game). As we described in Section 3, if the coalition structure of an NTU-game can be represented with a normal hypergraph then the core of the game is always nonempty (regardless of the players' preferences). The bipartite graph is an easy example for normal hypergraphs, and so every two-sided matching game has nonempty core. But what other games have this property? Our claim is that certain network games also have a coalition structure where the underlying hypergraph is normal.

Understanding Scarf's results. Scarf proved his Lemma in an algorithmic way. Is there some deeper reasons for the correctness of the Lemma (and a more general interpretation of the algorithm)? What is the relation of this result to other fundamental theorems, such as the Sperner Theorem? There are some recent papers [15, 10] attempting to answer this question, but yet, there still are many open problems regarding this issue.

Also, it would be interesting to know how the Scarf algorithm works for special games. For instance, does the Scarf algorithm run in polynomial time for matching games?

At the beginning of the Scarf algorithm we perturb matrix C and vector b. By doing so, the steps in the algorithm and the final output are fully determined. Can we output every core element of a given NTU-game by using a suitable perturbation? How does the perturbation effect the solution, we obtain by the algorithm? For stable marriage problem we observed that using small epsilons for men and larger epsilons for women we always get the man-optimal stable matching. Can we output each stable matching by a suitable perturbation? Is that true that the smaller epsilon we give to a woman the better partner she is going to get in the resulting stable matching?

Further application of Scarf Lemma. It is possible that the contribution of the participants are not equal in a cooperation. Imagine an internal project of a company where the hours allocated to the employees involved can be different (e.g., a project manager may have less work load than an engineer in terms of working hours). We can facilitate this option easily for any stable matching or stable allocation problem (that we may call *stable allocation problem with contributions*). We only need to use *contribution vectors* rather than membership vectors when defining matrix A in Scarf's Lemma, and the existence of a stable solution is guaranteed. But can we find a stable integral solution in polynomial time for, say, two-sided matching games?

Practical applications. As Gale and Shapley [11] envisaged, stable matching problems turned out to be very useful models for real applications in two-sided markets. Centralised matching schemes have been established worldwide to allocate residents to hospitals, students to schools, and so on. In most cases, a stable solution can be found by the classical Gale-Shapley algorithm. However, there are some special features, such as the presence of *couples* in the residence allocation program, that can make the problem unsolvable (or even if a stable matching exists, the problem of finding one can be NP-hard). Although if the ratio of the couples is relatively small in a large market then a stable matching exists with high probability and sophisticated heuristics may be able to find such solutions (see e.g., [16] and [7]). A new heuristic for this problem could be based on the Scarf algorithm for a stable allocation problem, where a hyperedge would represent an application from a couple to a pair of hospitals. If the solution obtained by the Scarf algorithm is integral then it would correspond to a stable matching. We illustrate this application with an example in [5].

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