The Efficiency of Fair Division with Connected Pieces

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Abstract

We consider the issue of fair division of goods, using the cake cutting abstraction, and aim to bound the possible degradation in social welfare due to the fairness requirements. Previous work has considered this problem for the setting where the division may allocate each player any number of unconnected pieces. Here, we consider the setting where each player must receive a single connected piece. For this setting, we provide tight bounds on the maximum possible degradation to both utilitarian and egalitarian welfare due to three fairness criteria – proportionality, envy-freeness and equitability.

1 Introduction

Cake Cutting. The problem of fair division of goods is the subject of extensive literature in the social sciences, law, economics, game theory and more. The famous "cake cutting" problem abstracts the fair division problem in the following way. There are n players wishing to divide between themselves a single "cake". The different players may value differently the various sections of the cake, e.g. one player may prefer the marzipan, another the cherries, and a third player may be indifferent between the two. The goal is to obtain a "fair" division of the cake amongst the players. There are several possible definitions to what constitutes a "fair" division, with *proportionality*, *envy-freeness* and *equitability* being the major fairness criteria considered (these notions will be defined in detail later). Many previous works considered the problem of obtaining a fair devision under these (and other) criteria.

Social Welfare. While fairness is clearly a major consideration in the division of goods, another important consideration is the social welfare resulting from the division. Clearly, a division may be envy-free but very inefficient, e.g. in the total welfare it provides to the players. Accordingly, the question arises what, if any, is the tradeoff between these two desiderata? How much social welfare does one have to sacrifice in order to achieve fairness? The answer to this question may, of course, depend on the exact definition of fairness, on the one hand, and the social welfare of interest, on the other.

The first analysis of such questions was provided in [CKKK09], where Caragiannis et al. consider the three leading fairness criteria – proportionality, envy-freeness and equitability – and quantify the possible loss in utilitarian social welfare due to such fairness requirements. Here we continue this line of research, extending the results in two ways. Firstly, the [CKKK09] analysis allows dividing the cake into any number of pieces, possibly even infinite. Thus, each player may get a collection of pieces, rather than a single one. While this may be acceptable in some cases, it may not be so in others, or at least highly undesirable , e.g. in the division of real estate, where players naturally prefer getting a connected plot. Similarly, in the cake scenario itself, allowing unconnected pieces may lead to a situation where, in Stromquist's words [Str80], "a player who hopes only for a modest interval of the cake may be presented instead with a countable union of crumbs". Accordingly, in this work, we focus on divisions in which each player gets a single connected piece of the cake. In addition, we consider both the utilitarian and the egalitarian social welfare functions, whereas Caragiannis et al. considered only utilitarian welfare. For each of these welfare functions, we give tight bounds on the possible loss in welfare due to the three fairness criteria.

1.1 Definitions and Notations

We consider a rectangular cake that can be divided by making parallel cuts. The cake can thus be represented by the interval $[0, 1]$, where each cut is some point $p \in [0, 1]$. The cake needs to be divided to n players (we use the notation $[n]$ for the set $\{1, \ldots, n\}$), each of which has a valuation function $v_i(\cdot)$ assigning a non-negative value to every possible interval of the cake. As customary, we require that for all i, $v_i(\cdot)$ is a nonatomic measure on $[0, 1]$ having $v_i(0, 1) = 1$. Every set of valuation functions $\{v_i(\cdot)\}_{i=1}^n$ defines an *instance* of the cake cutting problem.

Since we consider only divisions in which every player gets a single connected interval, a division of the cake to n players can be represented by a vector

$$x = (x_1, \ldots, x_{n-1}, \pi) \in [0, 1]^{n-1} \times S_n$$

with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq 1$. Here, x_i determines the position of the i-th cut, and π is a permutation that determines which piece is given to which player. For convenience, we denote $x_0 = 0$ and $x_n = 1$, so we can write that player $i \in [n]$ receives the interval $(x_{\pi(i)-1}, x_{\pi(i)})$. We use the notation $u_i(x)$ for the utility that player i gets in the division x, i.e. $u_i(x) = v_i(x_{\pi(i)-1}, x_{\pi(i)})$. We denote by X the set of all possible division vectors, and note that X is a compact set.

Fairness Criteria. We say that a division $x \in X$ is:

- **Proportional** if every player gets at least $\frac{1}{n}$ of the cake (by her own valuation). Formally, x is a proportional division if for all $i \in [n]$, $u_i(x) \geq \frac{1}{n}$.
- **Envy-Free** if no player prefers getting the piece alloted to any of the other players. Formally, x is an envy-free division if for all $i \neq j \in [n]$, $u_i(x) = v_i(x_{\pi(i)-1}, x_{\pi(i)}) \geq v_i(x_{\pi(j)-1}, x_{\pi(j)})$.
- **Equitable** if all the players get the exact same utility in x (by their own valuations). Formally, x is an equitable division if for all $i, j \in [n]$, $u_i(x) = u_j(x)$.

Stromquist [Str80], showed that for every instance of the cake cutting problem there exists an envy-free division with connected pieces. Since one can easily observe that every envy-free division is in particular proportional, this implies that such proportional divisions also always exist. In this paper we show (Theorem 6) that equitable divisions also always exist for connected pieces (for the case where players need not get a single interval, this is well known).

Social Welfare Functions. For a division $x \in X$, we denote by $u(x)$ the utilitarian social welfare of x, i.e.

$$u(x) = \sum_{i \in [n]} u_i(x) \ .$$

Likewise, we denote by $eg(x)$ the egalitarian social welfare of x, which is

$$eg(x) = \min_{i \in [n]} u_i(x) \ .$$

Note that both these social welfare functions are continuous and thus have maxima in X.

The Price of Fairness. As described above, we aim to quantify the degradation in social welfare due to the different fairness requirements. This is captured by the notion of *Price of Fairness*, in its three forms – *Price of Proportionality*, *Price of Envy-freeness* and *Price of Equitability*, defined as follows. The *Price of Proportionality* (resp. Envy-Freeness, Equitability) of a cake-cutting instance I, with respect to some predefined social welfare function, is defined as the ratio between the maximum possible social welfare for the instance, taken over all possible divisions, and the maximum social welfare attainable when divisions must be proportional (resp. envy-free, resp. equitable). When considering divisions with connected pieces, this restriction is applied to both maximizations. For example, if $X_{EF} \subseteq X$ is the set of all (connected) envy-free divisions of an instance, the egalitarian Price of Envy-Freeness for this instance is

$$\frac{\max_{x \in X} eg(x)}{\max_{y \in X_{EF}} eg(y)} .$$

In this work we show bounds on the maximum utilitarian and egalitarian Price of Proportionality, Envy-Freeness and Equitability of *any* instance.

1.2 Results

We analyze the utilitarian and egalitarian Price of Proportionality, Envy-Freeness and Equitability for divisions with connected pieces. We provide tight bounds (in some cases, up to an additive constant factor) for all six resulting cases. The results are summarized in Table 1; the last row presents the relevant previous results by Caragainnis et al. in [CKKK09], for comparison. The meaning of the upper bounds is that the respective price of fairness of *any* possible instance is never greater than the bound. The meaning of the lower bound is that there *exists* an instance that exhibits at least this price of fairness (for the respective class).

Price of:	Proportionality	Envy-Freeness	Equitability	
Utilitarian	UB: $\frac{\sqrt{n}}{2} + 1 - o(1)$ LB: $\frac{\sqrt{n}}{2}$		UB: n LB: $n - 1 + \frac{1}{n}$	connected pieces (this work)
Egalitarian (tight)	1	$\frac{n}{2}$	1	
Utilitarian	UB: $2\sqrt{n} - 1$ LB: $\frac{\sqrt{n}}{2}$	UB: $n - \frac{1}{2}$ LB: $\frac{\sqrt{n}}{2}$	UB: n LB: $\frac{(n+1)^2}{4n}$	non-connected pieces [CKKK09]

Table 1: All results

Utilitarian Welfare. For the utilitarian social welfare, we show an upper bound of $\frac{\sqrt{n}}{2} + 1 - o(1)$ on the price of envy-freeness, for *any* possible instance. This, we believe, is the first non-trivial upper bound on the Price of Envy-Freeness. It seems that such bounds are hard to obtain since on the one hand we need to consider the "best" possible envy-free division, while on the other hand no efficient method for explicitly constructing any envy-free divisions is known. We show that the same upper bound also applies to the Price of Proportionality.

For the Price of Equitability, we show that it is always bounded by n (though simple, this does require a proof since an equitable division need not even give each player $1/n$). We also provide an almost matching lower bound, showing that for any n there exists an instance with utilitarian Price of Equitability arbitrarily close to $n - 1 + \frac{1}{n}$.

Egalitarian Welfare. When considering the egalitarian social welfare, we show that there is no price for either proportionality or equitability. That is, for any instance there exist both proportional and equitable divisions for which the minimum amount any player gets is no less than if there were no fairness requirements. While perhaps not surprising, the proof for the Price of Equitability is somewhat involved, especially since we require that the divisions be with connected pieces. We note that we are not aware of any previous proof that altogether establishes the existence of an equitable division with connected pieces.

For the Price of Envy-Freeness, we show that it is bounded by $n/2$, and provide a matching family of instances that exhibits this price, for any n.

Paper Organization. In Section 2, we present bounds on the Price of Proportionality and the Price of Envy-Freeness. We begin in 2.1 by presenting the upper bound on the utilitarian Price of Envy-Freeness, and complement it by an example already given in Caragiannis et al. [CKKK09], which is tight up to a small additive factor. Both these upper and lower bounds apply also to the utilitarian Price of Proportionality. In 2.2 we show a simple upper bound of $\frac{n}{2}$ for the egalitarian Price of Envy-Freeness, together with a matching (tight) lower bound. We also show that the egalitarian Price of Proportionality is trivially 1. In Section 3 we present bounds on the Price of Equitability. In addition to the (mentioned above) proof that the egalitarian price is 1, we provide a simple upper bound of n on the utilitarian Price of Equitability, together with a lower bound of $n-1+\frac{1}{n}$. In Section 4 we consider the reverse question to that of the Price of Fairness – namely, how much fairness may one have to give up to achieve social optimality. Finally, we conclude this work and present some open questions in Section 5.

1.3 Related Work

The problem of fair division dates back to the ancient times, and takes many forms. The piece of property to be divided may be divisible or indivisible: Divisible goods can be "cut" into pieces of any size without destroying their value (like a cake, a piece of land, or an investment account), while indivisible goods must be given in whole to one person (e.g. a car, a house, or an antique vase). Since such items cannot be divided, the problem is usually to divide a *set* of such goods between a number of players. Fair division may also relate to the allocation of chores (of which every party likes to get as little as possible); this problem is of a somewhat different flavor from goods allocation, and also has the divisible and indivisible variants.

Modern mathematical treatment of fair division started at the 1940s [Ste49], and was initially concerned mainly with finding methods for allocation of divisible goods. Different algorithms – both discrete and continuous ("moving knife algorithms") – were presented (e.g. [Str80, EP84] and [BT95], which also surveys older algorithms), as well as non-constructive existence theorems [DS61, Str80]. In the past fifteen years, several books appeared on the subject [BT96, RW98, Mou04]. Following the evaluation and cut queries model suggested by Robertson and Webb [RW98], much attention was given to the question of lower bounds on the number of steps or cuts required for such divisions in this and other models [MIBK03, EP06, SW03, Str08, Pro09]. In particular, Stromquist [Str08] proves that no finite protocol (even unbounded) can be devised for an envy-free division of a cake among three or more people in which each player receives a connected piece. However, we note that this result applies only to the model presented in that work (which resembles the one suggested by Robertson and Webb), and not for cases where, for example, some mediator has full information of the players' valuation functions and proposes a division based on this information.

Unlike most of the work on cake cutting, the different notions of the price of fairness are not concerned with *procedures* for obtaining divisions, but rather with the *existence* of divisions with different properties (relating to social optimality and fairness). These notions, namely the Price of Proportionality, the Price of Envy-Freeness and the Price of Equitability, were first presented in a recent paper by Caragiannis et al. [CKKK09]. This line of work has some resemblance to the line of work on the Price of Stability [ADK+04], which attracted much attention in the past decade. The work in [CKKK09] analyzes the price of fairness (via the above three measures) with the utilitarian welfare function for divisible and indivisible goods and chores, giving tight bounds (up to a constant multiplicative factor) in most cases. However, unlike in this work, no special attention was given to the case of connected pieces in divisible goods. The results of [CKKK09] for divisible goods are summarized in the last row of Table 1.

2 The Price of Envy-Freeness and Proportionality

2.1 Utilitarian Welfare

Theorem 1. *For every cake-cutting instance with n players, the utilitarian Price of Envy-Freeness with connected pieces is bounded from above by $\frac{\sqrt{n}}{2} + 1 - o(1)$.*

In fact, we prove an even stronger claim: The above bound applies not only to the distance of the "best" envy-free division from utilitarian optimality, but also to the distance from (utilitarian) optimality of *any* envy-free division.

Proof. Let x be an envy-free division of the cake, and $u(x) = \sum_{i\in[n]} u_i(x)$ its utilitarian social welfare. We show that any other division to connected pieces y has $u(y) \leq \left(\frac{\sqrt{n}}{2} + 1 - \frac{n}{4n^2-4n+2\sqrt{n}}\right) \cdot u(x)$. Our proof is based on the following key observation:

> *Assume that for some $i \in [n]$, $u_i(y) \geq \alpha \cdot u_i(x)$. Since i values any other piece in the division x at most as much as her own, it has to be that in y, i gets an interval that intersects pieces that belonged to at least $\lceil\alpha\rceil$ different players (possibly including i herself).*

We will say that in the division y, player i gets the j-th cut of x if in y, i is given a piece starting at a point $p < x_j$ and ending at the point $p' > x_j$. A more formal statement of our observation is therefore that if in y, i gets at most α cuts of x, it holds that $u_i(y) \leq (\alpha + 1) \cdot u_i(x)$. We can thus bound the ratio $\frac{u(y)}{u(x)}$ by the solution to the following optimization problem, which aims to find values $\{u_i(x)\}_{i=1}^n$ and $\{\alpha_i\}_{i=1}^n$ (the number of cuts of x each player gets) that maximize this ratio.

$$\text{maximize} \quad \frac{\sum_{i=1}^n (\alpha_i + 1)u_i(x)}{\sum_{i=1}^n u_i(x)} \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^n \alpha_i = n - 1$$

$$u_i(x) \geq \frac{1}{n} \qquad \forall 1 \leq i \leq n \tag{2}$$

$$(\alpha_i + 1)u_i(x) \leq 1 \qquad \forall 1 \leq i \leq n \tag{3}$$

$$\alpha_i \in \{0, \ldots, n-1\} \qquad \forall 1 \leq i \leq n$$

(2) is a necessary condition for the envy-freeness of x that provides a lower bound for the denominator, and (3) is equivalent to $u_i(y) \leq 1$.

We therefore concentrate on bounding the solution to the above optimization problem. To this end, the following observations are useful:

1. For any choice of values $\{u_i(x)\}_{i=1}^n$, the optimal assignment for the α_i variables is greedy, i.e. giving each player i, in non-increasing order of $u_i(x)$ the maximum possible value for α_i that does not violate any of the constraints. (This holds since otherwise there are players i, j with $u_i(x) > u_j(x)$ and $\alpha_j \geq 1$ such that increasing α_i by one at the expense of α_j is feasible and yields an increase of $u_i(x) - u_j(x) > 0$ in the numerator of (1), without affecting the denominator.) We thus can divide the players into two groups: Those with "high" $u_i(x)$ values, who receive strictly positive α_i values, and those with "low" $u_i(x)$ values, for which $\alpha_i = 0$.
2. Since the players with low $u_i(x)$ values add the same amount to both the numerator and the denominator in the objective function, maximum is obtained when these values are minimized; i.e. in the optimal solution $u_i(x) = \frac{1}{n}$ for all these players.
3. The solution to the problem above is clearly bounded from above by the solution to the same problem where the α_i variables need not have integral values. Clearly, in the optimal solution to such a problem, all the players with $\alpha_i > 0$ have $(\alpha_i + 1)u_i(x) = 1$.

We can thus bound the solution to our optimization problem by the solution to the following problem. Let K be a variable that denotes the number of players that will have $\alpha_i > 0$; by observation (3) above, for every such player, $(\alpha_i + 1)u_i(x) = 1$, and thus their total contribution to the numerator is K. We therefore seek a solution for:

$$\text{maximize} \qquad \frac{K + (n-K)\cdot\frac{1}{n}}{\sum_{i=1}^{K} u_i(x) + (n-K)\cdot\frac{1}{n}} \tag{4}$$

$$\text{subject to} \qquad \sum_{i=1}^{K}\left(\frac{1}{u_i(x)} - 1\right) = n - 1 \tag{5}$$

$$K \leq n$$

It can be verified (e.g. using Lagrange multipliers) that for any value of $K \leq n$ this is maximized when $u_i(x) = u_j(x)$ for all $i, j \in [K]$, i.e. when $u_i(x) = \frac{K}{n-K+1}$ for all $i \in [K]$. We thus conclude that the maximum solution to the above problem maximizes the ratio

$$\frac{K + (n-K)\cdot\frac{1}{n}}{K\cdot\frac{K}{n+K-1} + (n-K)\cdot\frac{1}{n}}\ ;$$

by elementary calculus this is maximized at $K = \sqrt{n}$, where the value is

$$\frac{(n\sqrt{n} + n - \sqrt{n})(n + \sqrt{n} - 1)}{n^2 + (n - \sqrt{n})(n + \sqrt{n} - 1)} = \frac{(n^2\sqrt{n} - n\sqrt{n} + \frac{1}{2}n) + (2n^2 - 2n + \sqrt{n}) - \frac{1}{2}n}{2n^2 - 2n + \sqrt{n}}$$
$$= \frac{\sqrt{n}}{2} + 1 - \frac{n}{4n^2 - 4n + 2\sqrt{n}} = \frac{\sqrt{n}}{2} + 1 - o(1)\ ,$$

as stated. □

Since every envy-free division is in particular proportional, we immediately get that the bound on the utilitarian Price of Envy-Freeness also applies to the Price of Proportionality:

Corollary 2. *For every cake-cutting instance with n players, the utilitarian Price of Proportionality in connected pieces is bounded from above by $\frac{\sqrt{n}}{2} + 1 - o(1)$.*

We conclude by showing that these bounds are essentially tight (up to a small additive factor). The construction we show is identical to the one in [CKKK09], and we provide it here again for completeness.

Proposition 3. *The utilitarian Price of Proportionality (and thus also the utilitarian Price of Envy-Freeness) in connected pieces is larger than* $\frac{\sqrt{n}}{2}$.

Proof. For some integer m, consider $n = m^2$ players with the following valuation functions. For $i = 1, \ldots, \sqrt{n}$, player i assigns a value of 1 to the piece $(\frac{i-1}{\sqrt{n}}, \frac{i}{\sqrt{n}})$ and 0 to the rest of the cake (we call these players the "focused players"). All other players (players $i = (\sqrt{n}+1), \ldots, n$, the "indifferent players") assign a uniform value to the entire cake. In any proportional division, the indifferent players must get a total of at least $\frac{n-\sqrt{n}}{n}$ of the physical cake, and their total utility is less than 1. This leaves the focused players with at most $\frac{1}{\sqrt{n}}$ of the physical cake, and so they obtain (together) a total utility of at most 1; the utilitarian value of a proportional division is therefore less than 2. On the other hand, the division giving each of the focused players the entire interval they desire (and leaving nothing to the indifferent players) has a utilitarian social welfare of $\sqrt{n}$. The Price of Proportionality for this case is therefore larger than $\frac{\sqrt{n}}{2}$, as stated. □

2.2 Egalitarian Welfare

Proposition 4. *For every cake-cutting instance, the egalitarian Price of Proportionality is 1.*

Proof. Let x be a proportional division, and y the egalitarian optimal division. By proportionality, every player i has $u_i(x) \geq \frac{1}{n}$, and thus $eg(x) \geq \frac{1}{n}$. Since y is the egalitarian optimal division, we have that for every $i \in [n]$, $u_i(y) \geq eg(y) \geq eg(x) \geq \frac{1}{n}$; this implies that y is proportional as well. □

Theorem 5. *The egalitarian Price of Envy-Freeness for cake-cutting instances with n players and connected pieces is $\frac{n}{2}$. In particular, this is also an upper bound on the egalitarian Price of Envy-Freeness for n players and non-connected pieces.*

Proof. First, note that if the egalitarian optimal division is itself envy-free, the Price of Envy-Freeness is 1, and that every division with egalitarian welfare of $\frac{1}{2}$ is envy-free. We therefore assume that this is not the case, and that in the egalitarian optimal y division some player i has $u_i(y) < \frac{1}{2}$. Let x be some envy-free division, then x is in particular proportional and thus has $u_i(x) \geq \frac{1}{n}$; the upper bound follows.

It remains to show a lower bound for the connected case. Let $\epsilon > 0$ be an arbitrarily small constant, and consider n players with the following valuation functions. For $i = 1, \ldots, (n-1)$, player i assigns a value of $\frac{1}{2} + \epsilon$ to the piece $(i - \epsilon, i + \epsilon)$ (her "favorite piece"), a value of $\frac{1}{2} - \epsilon$ to the piece $(1 - \frac{2i+1}{2n} - \epsilon, 1 - \frac{2i+1}{2n} + \epsilon)$ (her "second-favorite piece"), and value of 0 to the rest of the cake. Finally, player n assigns a uniform value to the entire cake.

In order for player n to get utility of α, this player needs to receive an α fraction of the cake (in physical size). However, every connected piece of physical size at least $\frac{1}{n} + 2\epsilon$ necessarily contains some other player's "favorite piece", and it is immediate that if a single player receives the entire favorite piece of another player, there is envy. Thus, in every envy-free division of the cake, player n gets utility of less than $\frac{1}{n} + 2\epsilon$. However, there exists a division in which every player gets utility of at least $\frac{1}{2} - \epsilon$. Such a division is achieved by giving players $i = 1 \ldots \lfloor \frac{n-1}{2} \rfloor$ their favorite pieces, players $i = (\lfloor \frac{n-1}{2} \rfloor + 1) \ldots (n-1)$ their second-favorite pieces, and player n the interval $(\frac{1}{2} + \epsilon, 1)$ (the remaining parts of the cake can be given to any of the players closest to them). The stated bound follows as ϵ approaches zero. □

3 The Price of Equitability

In order to talk about the Price of Equitability, we first have to make sure that the concept is well-defined. When non-connected pieces are concerned, it is known that every cake cutting instance has an equitable division [DS61]. However, the proof of Dubins and Spanier allows a "piece" of the cake to be any member of the σ-algebra of subsets, which is quite far from our restricted case of pieces that are all single intervals. Another result by Alon [Alo87] establishes the existence of an equitable division giving every player exactly $\frac{1}{n}$ by each measure; however, such a division may require up to $n^2 - 1$ cuts. The question thus arises whether equitable divisions with connected pieces always exist; to the best of our knowledge, this question has not been addressed before, and we answer it here to the affirmative. Furthermore, we show that such a division requires no sacrifice of egalitarian welfare.

Theorem 6. *For every cake-cutting instance there exists an equitable division of the cake with connected pieces. Furthermore, there always exists such a division in which the egalitarian social welfare is as high as possible in any division with connected pieces.*
This holds even for cake cutting instances that do not have $v_i(0, 1) = 1$ *for all* i *(i.e. even if some players' valuation of the entire cake is not* 1*).*

Proof. Recall that the egalitarian welfare is a continuous function and X is compact, and thus $eg(\cdot)$ has a maximum in X; we denote $OPT = \max_{x \in X} eg(x)$. We also denote by $Y \subset X$ the set of divisions with egalitarian value OPT, i.e.

$$Y = \left\{ y = (y_1, \ldots, y_{n-1}, \pi) \in X \;\middle|\; eg(y) = OPT \right\}.$$

We note that Y is a compact set; this follows from the fact that it is a closed subset of X (which is compact itself). To show that Y is closed, we show that $\overline{Y} = X \setminus Y$ is open. Let $z \in \overline{Y}$ be some division not in Y; then the division z must have egalitarian value smaller than OPT and in particular there must exist a player i and $\epsilon > 0$ such that $u_i(z) \leq OPT - \epsilon$. Since player i's valuation of the cake is a nonatomic measure, there must exist $\delta_L, \delta_R > 0$ such that extending i's piece to the interval $(z_{\pi(i)-1} - \delta_L, z_{\pi(i)} + \delta_R)$ increases i's utility (compared to the original division z) by less than ϵ. Therefore, in the ball of radius $\delta = \min\{\delta_L, \delta_R\}$ around z (e.g. in L_∞), every division still gives i utility smaller than OPT, and thus this ball does not intersect Y. It thus follows that $\overline{Y}$ is an open set, and so Y is closed and compact.

Recall that our aim is to show that Y contains an equitable division; to that end, we define a function $\Delta : Y \to \mathbb{R}$ by setting

$$\Delta(y) = \max_{i,j \in [n]} \left\{ u_i(y) - u_j(y) \right\} = \max_{i \in [n]} \left\{ u_i(y) - OPT \right\}.$$

We complete the proof by showing that for any ϵ, there exists a devision $y^{(\epsilon)} \in Y$, such that $\Delta(y^{(\epsilon)}) \leq \epsilon$. Since Y is a compact set and $\Delta(\cdot)$ is continuous, the image of Y is also compact. We therefore conclude that there must be some $y^* \in Y$ with $\Delta(y^*) = 0$ (since the image of Y is in particular a closed subset of $\mathbb{R}$ containing a point $p < \epsilon$ for every $\epsilon > 0$); such y^* is clearly equitable.

It remains to prove that for any ϵ, $y^{(\epsilon)}$ exists. We prove this by induction on the number of players n. For $n = 1$ there is only one possible division, which obtains exactly OPT for the single player. Assume for $n - 1$, we prove for n. Let y be any division in Y (assuming w.l.o.g. that y uses the identity permutation). We first construct a division y' such that for $i = 1, \ldots, n-1$, $u_i(y') = OPT$, by sequentially moving the border y'_i (between players i and $i+1$) to the left as far as possible while keeping that $u_i(y') \geq OPT$. This is possible since in y, $u_i(y) \geq OPT$ and the borders only need to move to the left. Consider the resulting

y'. If $u_n(y') \leq OPT + \epsilon$ we are finished; otherwise, let y'' be the division obtained from y' by moving the border y''_{n-1} (between players $n-1$ and n) as far right as necessary so that $u_n(y'') = OPT + \epsilon$. Now, omit the rightmost piece (that of player n), and consider the $(n-1)$-player cake cutting problem, on the remaining cake. (Note that the players' valuation of the entire new cake need not be identical to their valuation of the original cake, and that the new cake has a different set Y' of egalitarian-optimal divisions.)

Now, it cannot be the case that for this new problem the egalitarian maximum is more than OPT, as that would induce an egalitarian maximum greater than OPT for the entire problem. On the other hand, egalitarian value of OPT is clearly attainable, as it is obtained by y'' (reduced to the first $n-1$ players). Hence, OPT is also the egalitarian maximum for the new $(n-1)$-player problem. Thus, by the inductive hypothesis, there exists a division for this problem that obtains egalitarian welfare OPT and such that no player gets more than $OPT+\epsilon$. Combining this solution with the piece $(y''_{n-1}, 1)$ given to player n, we obtain $y^{(\epsilon)} \in Y$, such that no player gets more than $OPT + \epsilon$. □

Theorem 7. *The utilitarian Price of Equitability in connected pieces is upper-bounded by n, and for any n there is an example in which it is arbitrarily close to $n - 1 + \frac{1}{n}$.*

Proof. We begin by showing an upper bound on the utilitarian Price of Equilibility. From Theorem 6 we have that there always exists an equitable egalitarian-optimal division with connected pieces. Since there also always exists a proportional division (whose egalitarian social welfare is at least $\frac{1}{n}$), the egalitarian-optimal division must have an egalitarian social welfare of at least $\frac{1}{n}$ and thus a utilitarian social welfare of at least 1. Clearly, the maximum utilitarian social welfare attainable in any non-equitable division is less than n, and thus the utilitarian Price of Equitability is also less than n.

For the lower bound, fix some small $\epsilon > 0$ and consider n players with the following valuation functions. For $i = 1, \ldots, (n-1)$, player i assigns value of 1 to the interval $(\frac{i}{n} - \epsilon, \frac{i}{n} + \epsilon)$ and 0 to the rest of the cake. Finally, player n assigns uniform value to the entire cake.

Since any connected piece of (physical) size $\frac{1}{n} + 2\epsilon$ necessarily contains the entire desired piece of at least one player $i \in [n-1]$, the utility of player n in any equitable division must be strictly smaller than $\frac{1}{n} + 2\epsilon$; the utilitarian welfare of such a division is therefore smaller than $1 + 2n\epsilon$. Now, consider the following (non-equitable) division: give player 1 the interval $(0, \frac{1}{n} + \epsilon)$, players $i = 2, \ldots, (n-1)$ the interval $(\frac{i-1}{n} + \epsilon, \frac{i}{n} + \epsilon)$, and player n the interval $(\frac{n-1}{n} + \epsilon, 1)$. The utilitarian welfare of this division is $n - 1 + \frac{1}{n} - \epsilon$. By appropriately choosing ϵ, the Price of Equitability can be arbitrarily close to $n - 1 + \frac{1}{n}$. □

4 Trading Fairness for Efficiency

The work on the Price of Fairness is concerned with the trade-off between two goals of cake division: Fairness, and efficiency (in terms of social welfare). However, the results we presented so far, as well as the results in [CKKK09], concentrate on one direction of this trade-off, namely *how much efficiency may have to be sacrificed to achieve fairness.* We now turn to look at the analogue question of *how much fairness may have to be given up to achieve social optimality*; sadly, it seems that at least for the connected-pieces case, the results are somewhat pessimistic, except for equitability and proportionality with the egalitarian welfare.

In order to answer such questions, one first has to quantify unfairness. The following definitions seem natural:

We say that a division x:

- is α-*unproportional* if some player $i \in [n]$ has $u_i(x) \leq \frac{1}{\alpha \cdot n}$.

- *has envy of* α if there exist players $i, j \in [n]$ for which

$$v_i(x_{\pi(j)-1}, x_{\pi(j)}) \geq \alpha \cdot v_i(x_{\pi(i)-1}, x_{\pi(i)}) = \alpha \cdot u_i(x) ,$$

  i.e. if some i feels that $j \neq i$ received a piece worth α-times more than the one she got.

- is α-*inequitable* if there are players $i, j \in [n]$ with $u_i(x) \geq \alpha \cdot u_j(x)$.

Using these "unfairness" notions, we can obtain the following simple results:

Proposition 8. *There are cake-cutting instances where an utilitarian-optimal division is necessarily infinitely unfair, by all three measures above.*

Proof. Consider the cake cutting instance from the proof of Proposition 3. In this instance, the unique utilitarian-optimal division gives no cake at all to the "indifferent players"; it follows that this division is infinitely unproportional and inequitable, and has inifinite envy. ☐

We already know (Proposition 4 and Theorem 6) that egalitarian optimality is not in conflict with neither proportionality nor equitability. However, this is not the case for envy:

Proposition 9. *There are cake-cutting instances where an egalitarian-optimal division necessarily has envy arbitrarily close to* $n-1$*, and this is the maximum possible envy for such divisions.*

Proof. Let $\epsilon > 0$ be an arbitrarily small constant, and consider n players with the following valuation functions, which are fairly similar to those in the proof of Theorem 5. For $i = 1, \ldots, (n-1)$, player i assigns a value of $1 - \frac{1}{n} - \epsilon$ to the piece $(i - \frac{\epsilon}{2}, i + \frac{\epsilon}{2})$ (her "favorite piece"), a value of $\frac{1}{n} + \epsilon$ to the piece $(1 - \frac{2i+1}{2n} - \frac{\epsilon}{2}, 1 - \frac{2i+1}{2n} + \frac{\epsilon}{2})$ (her "second-favorite piece"), and value of 0 to the rest of the cake. Finally, player n assigns uniform value to the entire cake.

It is clear that there is no way for the egalitarian value to exceed $\frac{1}{n} + \epsilon$: In order for that to happen, player n must get a connected piece of physical size larger than $\frac{1}{n} + \epsilon$, which must contain the entire favorite piece of some player $i < n$, and so player i can get utility at most $\frac{1}{n} + \epsilon$. However, egalitarian welfare of $\frac{1}{n} + \epsilon$ can be easily achieved, and in such case player n indeed devours the entire favorite piece of some player $i < n$; this player receives a piece worth (in her eyes) only $\frac{1}{n} + \epsilon$ while she values the piece n receives as worth $1 - \frac{1}{n} - \epsilon$. The envy in every egalitarian-optimal division is therefore $\frac{n-1-\epsilon n}{1+\epsilon n}$, which can be arbitrarily close to $n-1$ with an appropriate choice of ϵ.

Since the egalitarian-optimal division is always proportional, every player must get at least $\frac{1}{n}$ of the cake in it; therefore, in this player's view, another player may get at most $\frac{n-1}{n}$. It thus follows that in every such division the maximum possible envy is $n-1$. ☐

5 Conclusions and Open Problems

In this work we analyzed the possible degradation in social welfare due to fairness requirements, when requiring that each player obtain a single connected piece. We obtain that the results vary considerably, depending on the fairness criteria used, and the social welfare function in consideration. The bounds range from provably no degradation for proportionality and equitability under the egalitarian welfare, through an $O(\sqrt{n})$ degradation for envy-freeness and proportionality under the utilitarian welfare, to an $O(n)$ degradation for equitability under the utilitarian welfare and for envy-freeness under the egalitarian welfare. We have also seen that if we seek to trade fairness to achieve social optimality, the "exchange rate" may (at the worst case) be infinite for utilitarian welfare (for all three fairness criteria), or linear for egalitarian welfare and envy-freeness.

Many open questions await further research, including:

- *Small number of connected pieces.* One motivation for considering cake cutting with connected pieces is the desire to avoid situations where a player receives "a pile of crumbs" for his fair share of the cake. On the other hand, requiring that each player receives a single connected interval may be too strict a requirement. A natural middle ground is to require that each player receives only a small number of pieces, e.g. a constant number. The question thus arises to bound the degradation to the social welfare under such requirements. In such an analysis it would be interesting to see how the bounds on degradation behave as a function of the number of permissible pieces.

- *The Egalitarian Price of Fairness with non-connected pieces.* [CKKK09] provide bounds on the Price of Fairness using the utilitarian welfare function, for the setting that non-connected pieces are permissible. Bounding the egalitarian Price of Fairness in this setting remains open. A trivial upper bound on the Price of Envy-freeness is $\frac{n}{2}$, and we have examples of instances where this price is strictly larger than 1, but obtaining tight bounds seems to require additional work and techniques.

- *The egalitarian Price of Proportionality and Price of Equitability for indivisible goods.* [CKKK09] provide analysis for the utilitarian Price of Fairness for such goods. A simple example can be constructed to show a tight bound of $\frac{n}{2}$ for the egalitarian Price of Envy-Freeness for this case. It thus remains open to determine the egalitarian Price of Proportionality and Equitability for such goods.

- *The Price of Fairness for connected chores.* As we already mentioned, fair division of chores has a somewhat different flavor from division of goods, and may require somewhat different techniques. One possible motivation for requiring connected division of chores may be, for example, a case in which a group of gardeners need to maintain a large garden, and so would like to give each of them one (connected) area to be responsible for.

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