Optimal Partitions in Additively Separable Hedonic Games[1]

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Abstract

We conduct a computational analysis of partitions in additively separable hedonic games that satisfy standard criteria of fairness and optimality. We show that computing a partition with maximum egalitarian or utilitarian social welfare is NP-hard in the strong sense whereas a Pareto optimal partition can be computed in polynomial time when preferences are strict. Perhaps surprisingly, *checking* whether a given partition is Pareto optimal is coNP-complete in the strong sense, even when preferences are symmetric and strict. We also show that checking whether there exists a partition which is both Pareto optimal and envy-free is Σ_2^p-complete. Furthermore, checking whether there exists a partition which is both envy-free and Nash stable is NP-complete when preferences are symmetric.

1 Introduction

Ever since the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* in 1944, coalitions have played a central role within game theory. The crucial questions in coalitional game theory are which coalitions can be expected to form and how the members of coalitions should divide the proceeds of their cooperation. Traditionally the focus has been on the latter issue, which led to the formulation and analysis of concepts such as the core, the Shapley value, or the bargaining set. Which coalitions are likely to form is commonly assumed to be settled exogenously, either by explicitly specifying the coalition structure, a partition of the players in disjoint coalitions, or, implicitly, by assuming that larger coalitions can invariably guarantee better outcomes to its members than smaller ones and that, as a consequence, the grand coalition of all players will eventually form.

The two questions, however, are clearly interdependent: the individual players' payoffs depend on the coalitions that form just as much as the formation of coalitions depends on how the payoffs are distributed.

Coalition formation games, as introduced by Drèze and Greenberg (1980), provide a simple but versatile formal model that allows one to focus on coalition formation as such. In many situations it is natural to assume that a player's appreciation of a coalition structure only depends on the coalition he is a member of and not on how the remaining players are grouped. Initiated by Banerjee et al. (2001) and Bogomolnaia and Jackson (2002), much of the work on coalition formation now concentrates on these so-called *hedonic games*.

The main focus in hedonic games has been on notions of *stability* for coalition structures such as Nash stability, individual stability, contractual individual stability, or core stability and characterizing conditions under which they are guaranteed to be non-empty (see, e.g., Bogomolnaia and Jackson, 2002; Burani and Zwicker, 2003). The most prominent examples of hedonic games are two-sided matching games in which only coalitions of size two are admissible (Roth and Sotomayor, 1990).

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General coalition formation games have also received attention from the artificial intelligence community, where the focus has generally been on computing partitions that give rise to the greatest social welfare (see, e.g., Sandholm et al., 1999). The computational complexity of hedonic games has been investigated with a focus on the complexity of computing stable partitions for different models of hedonic games (Ballester, 2004; Dimitrov et al., 2006; Cechlárová, 2008). We refer to Hajduková (2006) for a critical overview.

Among hedonic games, *additively separable hedonic games (ASHGs)* are a particularly natural and succinct representation in which each player has a value for any other player and the value of a coalition to a particular player is computed by simply adding his values of the players in his coalition.

Additive separability satisfies a number of desirable axiomatic properties (Barberà et al., 2004). ASHGs are the non-transferable utility generalization of *graph games* studied by Deng and Papadimitriou (1994). Sung and Dimitrov (2010) showed that for ASHGs, checking whether a core stable, strict-core stable, Nash stable, or individually stable partition exists is NP-hard. Dimitrov et al. (2006) obtained positive algorithmic results for subclasses of additively separable hedonic games in which each player divides other players into friends and enemies. Branzei and Larson (2009) examined the tradeoff between stability and social welfare in ASHGs.

Contribution In this paper, we analyze concepts from fair division in the context of coalition formation games. We present the first systematic examination of the complexity of computing and verifying optimal partitions of hedonic games, specifically ASHGs. We examine various standard criteria from the social sciences: *Pareto optimality*, *utilitarian social welfare*, *egalitarian social welfare* and *envy-freeness* (see, e.g., Moulin, 1988).

In Section 3, we show that computing a partition with maximum egalitarian social welfare is NP-hard. Similarly, computing a partition with maximum utilitarian social welfare is NP-hard in the strong sense even if preferences are symmetric and strict.

In Section 4, the complexity of Pareto optimality is studied. We prove that checking whether a given partition is Pareto optimal is coNP-complete in the strong sense even for strict and symmetric preferences. By contrast, we present a polynomial-time algorithm for computing a Pareto optimal partition when preferences are strict. Thus, we identify a natural problem in coalitional game theory where verifying a possible solution is presumably harder than actually finding one.[2] Our computational hardness results imply computational hardness of equivalent problems for *hedonic coalition nets* (Elkind and Wooldridge, 2009).

In Section 5, we consider complexity questions regarding envy-free partitions. We show that checking whether there exists a partition which is both Pareto optimal and envy-free is Σ_2^p-complete. We present an example which exemplifies the tradeoff between satisfying stability (such as Nash stability) and envy-freeness and use the example to prove that checking whether there exists a partition which is both envy-free and Nash stable is NP-complete even when preferences are symmetric.

Our computational hardness results imply computational hardness of equivalent problems for *hedonic coalition nets* (Elkind and Wooldridge, 2009).

2 Preliminaries

In this section, we provide the terminology and notation required for our results.

[2]This is also the case for an unrelated problem in social choice theory (Hudry, 2004).

2.1 Hedonic games

A *hedonic coalition formation game* is a pair $(N, \mathcal{P})$ where N is a set of players and $\mathcal{P}$ is a *preference profile* which specifies for each player $i \in N$ the preference relation $\succsim_i$, a reflexive, complete and transitive binary relation on set $\mathcal{N}_i = \{S \subseteq N \mid i \in S\}$.

The statement $S \succ_i T$ means that i strictly prefers S over T. Also $S \sim_i T$ means that i is indifferent between coalitions S and T. A *partition* π is a partition of players N into disjoint coalitions. By $\pi(i)$, we denote the coalition in π which includes player i.

A game $(N, \mathcal{P})$ is *separable* if for any player $i \in N$ and any coalition $S \in \mathcal{N}_i$ and for any player j not in S we have the following: $S \cup \{j\} \succ_i S$ if and only if $\{i, j\} \succ_i \{i\}$; $S \cup \{j\} \prec_i S$ if and only if $\{i, j\} \prec_i \{i\}$; and $S \cup \{j\} \sim_i S$ if and only if $\{i, j\} \sim_i \{i\}$.

We consider utility-based models rather than purely ordinal models. In *additively separable preferences*, a player i gets value $v_i(j)$ for player j being in the same coalition as i and if i is in coalition $S \in \mathcal{N}_i$, then i gets utility $\sum_{j \in S \setminus \{i\}} v_i(j)$.

A game $(N, \mathcal{P})$ is *additively separable* if for each player $i \in N$, there is a utility function $v_i : N \to \mathbb{R}$ such that $v_i(i) = 0$ and for coalitions $S, T \in \mathcal{N}_i$, $S \succsim_i T$ if and only if $\sum_{j \in S} v_i(j) \geq \sum_{j \in T} v_i(j)$.

A preference profile is *symmetric* if $v_i(j) = v_j(i)$ for any two players $i, j \in N$ and is *strict* if $v_i(j) \neq 0$ for all $i, j \in N$ such that $i \neq j$. We consider ASHGs (additively separable hedonic games) in this paper. Unless mentioned otherwise, all our results are for ASHGs. For any player i, let $F(i) = \{j \mid v_i(j) > 0\}$ be the set of players which i strictly likes. Similarly, let $E(i) = \{j \mid v_i(j) < 0\}$ be the set of players which i strictly dislikes.

2.2 Fair and optimal partitions

In this section, we formulate concepts from the social sciences especially the economics and the fair division literature for the context of hedonic games. For a utility-based hedonic game $(N, \mathcal{P})$ and partition π, we will denote the utility of player $i \in N$ by $u_\pi(i)$. The different notions of fair or optimal partitions are defined as follows.[3]

1. The *utilitarian social welfare* of a partition is defined as the sum of individual utilities of the players: $u_{ut}(\pi) = \sum_{i \in N} u_\pi(i)$. A *maximum utilitarian partition* maximizes the utilitarian social welfare.

2. The *elitist social welfare* is given by the utility of the player that is best off: $u_{el}(\pi) = \max\{u_\pi(i) \mid i \in N\}$. A *maximum elitist partition* maximizes the utilitarian social welfare.

3. The *egalitarian social welfare* is given by the utility of the agent that is worst off: $u_{eg}(\pi) = \min\{u_\pi(i) \mid i \in N\}$. A *maximum egalitarian partition* maximizes the egalitarian social welfare.

4. An *ordered utility vector* associated with partition π is given by $(u_\pi(p(1)), \ldots, u_\pi(p(n)))$ where p is a permutation of players such that $u_\pi(p(i)) \leq u_\pi(p(j))$ where $p(i) \leq p(j)$. Then a partition π with the *maximum leximin social welfare* is one which has lexicographically the greatest ordered utility vector. We refer to π as a *maximum leximin partition.*

5. A partition π of N is *Pareto optimal* if there exists no partition π' of N which *Pareto dominates* π, that is for all $i \in N$, $\pi'(i) \succsim_i \pi(i)$ and there exists at least one player $j \in N$ such that $j \in N$, $\pi'(j) \succ_j \pi(j)$.

[3]All welfare notions considered in this paper (utilitarian, elitist, egalitarian, and leximin) are based on the interpersonal comparison of utilities. Whether this assumption can reasonably be made is debatable.

6. *Envy-freeness* is a notion of fairness. In an *envy-free* partition, no player has an incentive to replace another player.

For the sake of brevity, we will consider all the notions described above as optimality criteria although envy-freeness is more concerned with fairness. We consider the following computational problems with respect to the optimality criteria defined above.

OPTIMALITY: Given $(N, \mathcal{P})$ and a partition π of N, is π optimal?
EXISTENCE: Does an optimal partition for a given $(N, \mathcal{P})$ exist?
SEARCH: If an optimal partition for a given $(N, \mathcal{P})$ exists, find one.

EXISTENCE is trivially true for all criteria of optimality concepts. By the definitions, it follows that there exist partitions which satisfy maximum utilitarian social welfare, elitist social welfare, egalitarian social welfare and leximin ordering respectively. The partition consisting of the grand coalition and the partition of singletons satisfy envy-freeness. During our computational analysis, we will assume familiarity of the reader with basic computational complexity classes. We recall that a problem is said to be NP-hard in the strong sense if it remains so even when its numerical parameters are bounded by a polynomial in the length of the input.

3 Complexity of maximizing social welfare

In this section, we examine the complexity of maximizing social welfare in ASHGs. Our first result is the following one.

Theorem 1. *Computing a maximum utilitarian partition is NP-hard in the strong sense even with symmetric and strict preferences.*

Proof. We prove Theorem 1 by a reduction from the MAXCUT problem. Before defining the MAXCUT problem, recall that a *cut* is a partition of the vertices of a graph into two disjoint subsets. The *cut-set* of the cut is the set of edges whose end points are in different subsets of the partition. In a weighted graph, the *weight of the cut* is the sum of the weights of the edges in the cut-set. Then, MAXCUT is the following problem:

MAXCUT
INSTANCE: An undirected weighted graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}^+$ and an integer k.
QUESTION: Does there exist a cut of weight at least k in G?

We present a polynomial-time reduction from MAXCUT to UTILSEARCH, the problem of computing a maximum utilitarian partition. Consider an instance I of MAXCUT with a connected undirected graph $G = (V, E)$ and positive weights $w(i, j)$ for each edge (i, j). Let $W = \sum_{(i,j)\in E} w(i, j)$. We show that if there is there a polynomial-time algorithm for computing a maximum utilitarian social welfare partition, then we have a polynomial-time algorithm for MAXCUT.

Consider the following method which in polynomial time reduces I to an instance I' of UTILSEARCH. I' consists of $|V| + 2$ players $N = \{m_1, \ldots, m_{|V|}, s_1, s_2\}$. For any two players m_i and m_j, $v_{m_i}(m_j) = v_{m_j}(m_i) = -w(i, j)$. For any player m_i and player s_j, $v_{m_i}(s_j) = v_{s_j}(m_i) = W$. Also $v_{s_1}(s_2) = v_{s_2}(s_1) = -W(|V| + 1)$.

We first prove that partition π^* with maximum utilitarian social welfare u^* consists of exactly two coalitions with s_1 and s_2 in different coalitions. We do so by proving two claims. The first claim is that every player m_i is either in a coalition with s_1 or s_2. Assume this is not true and there exists a partition π such that $u_{ut}(\pi) = u^*$ and m_i is not in the same coalition with s_1 or s_2. Then, if m_i joins $\pi(s_1)$, $u_{ut}(\pi)$ increases at least by $2W$ and it decreases by at most $2\sum_{j\in N} w(i,j) < 2W$. Therefore, $u_{ut}(\pi)$ increases which is a contradiction. The second claim is that s_1 and s_2 are in different coalitions in π^*. Assume this is not true and there exists a partition π with utilitarian social welfare u^* such that s_1 and s_2 are together in a coalition. Then the welfare of π can be increased by at least $2(|V|+1)(W) - 2|V|W = 2W$ if s_2 breaks up and forms a singleton coalition. This is a contradiction.

We are now ready to present the reduction. Assume there exists a polynomial-time algorithm which computes a feasible maximum utilitarian social welfare partition π. From the two claims above, we can assume that partition π has two coalitions with s_1 and s_2 in different coalitions. Then, $u_{ut}(\pi) = 2(X + \sum_{m_i \notin \pi(m_j)} -v_{m_i}(m_j))$ where $X = -W + (|V|+1)W \geq 2W$ if $|V| \geq 2$. We also know that $\sum_{m_i \notin \pi(m_j)} -v_{m_i}(m_j)) < W$. We can obtain a cut (A,B) from π where $A = \{i \mid m_i \in \pi(s_1)\}$ and $B = \{i \mid m_i \in \pi(s_2)\}$. Let the weight of the cut (A,B) be c. We know that $c \leq c^*$ where c^* is the weight of the maxcut for instance I. It is now shown that (A,B) is a maxcut if and only if $u_{ut}(\pi) = u^*$. Assume $u_{ut}(\pi) = u^*$ but (A,B) is a not a maxcut. In that case there exists a maxcut (C,D) such that $\sum_{i\in C, j\in D} w(i,j) > \sum_{i\in A, j\in B} w(i,j)$. Therefore, there exists a partition $\pi' = \{\{s_1 \cup \{m_i \mid i \in A\}\}, \{s_2 \cup \{m_i \mid i \in B\}\}\}$ where $u_{ut}(\pi') = 2(X + \sum_{m_i \notin \pi'(m_j)} -v_{m_i}(m_j)) > 2(X + \sum_{m_i \notin \pi(m_j)} -v_{m_i}(m_j))$. This is a contradiction as $u_{ut}(\pi) = u^*$.

Now assume that (A,B) is a maxcut but $u_{ut}(\pi) < u^*$. Then there exists another partition π^* such that $u_{ut}(\pi') = 2(X + \sum_{m_i \notin \pi'(m_j)} -v_{m_i}(m_j)) = u^*$. Therefore, the graph cut corresponding to π^* has a bigger maxcut value than (A,B) which is a contradiction. □

Computing a maximum elitist partition is much easier.

Proposition 1. *There exists a polynomial-time algorithm to compute a maximum elitist partition.*

Proof. Recall that for any player i, $F(i) = \{j \mid v_i(j) > 0\}$. Let $f(i) = \sum_{j\in F(i)} v_i(j)$. Both $F(i)$ and $f(i)$ can be computed in linear time. Let $k \in N$ be the player such that $f(k) \geq f(i)$ for all $i \in N$. Then $\pi = \{\{\{k\} \cup F(k)\}, N \setminus \{\{k\} \cup F(k)\}\}$ is a partition which maximizes the elitist social welfare. □

As a corollary, we can verify whether a partition π has maximum elitist social welfare by computing a partition π^* with maximum elitist social welfare and comparing $u_{el}(\pi)$ with $u_{el}(\pi^*)$. Just like maximizing the utilitarian social welfare, maximizing the egalitarian social welfare is hard:

Theorem 2. *Computing a maximum egalitarian partition is NP-hard in the strong sense.*

Proof. We provide a polynomial-time reduction from the following NP-hard problem (Woeginger, 1997):

MaxMinMachineCompletionTime
INSTANCE: A set of m identical machines $M = \{M_1, \ldots, M_m\}$, a set of n independent jobs $J = \{J_1, \ldots, J_n\}$ where job J_i has processing time p_i.
OUTPUT: Allot jobs to the machines such that the minimum processing time (without machine idle times) of all machines is maximized.

Let I be an instance of MaxMinMachineCompletionTime and let $P = \sum_{i=1}^{n} p_i$. From I we construct an instance I' of EgalSearch. The ASHG for instance I' consists of $N = \{i \mid M_i \in M\} \cup \{s_i \mid J_i \in J\}$ and the preferences of the players are as follows: for all $i = 1, \dots m$ and all $j = 1, \dots, n$ let $v_i(s_j) = p_j$ and $v_{s_j}(i) = P$. Also, for $1 \leq i, i' \leq m, i \neq i'$ let $v_i(i') = -(P+1)$ and for $1 \leq j, j' \leq n, j \neq j'$ let $v_{s_j}(v_{s_{j'}}) = 0$. Each player i corresponds to machine M_i and each player s_j corresponds to job J_j.

Let π be the partition which maximizes $u_{eg}(\pi)$. We show that players $1, \dots, m$ are in separate coalitions and each player s_j is in $\pi(i)$ for some $1 \leq i \leq m$. We do so by proving two claims. The first claim is that for $i, j \in \{1, \dots m\}$ such that $i \neq j$, we have that $i \notin \pi(j)$. Assume there exist exactly two players i and j for which this is not the case. Then we know that $u_\pi(i) = -(P+1) + \sum_{s_j \in \pi(i)} p_j$. Since $\sum_{s_j \in \pi(i)} p_j \leq P$, we know that $u_\pi(i) = u_\pi(j) < 0$, $u_\pi(a) \geq 0$ for all $a \in N \setminus \{i, j\}$ and thus $u_{eg}(\pi) < 0$. However, if i deviates and forms a a singleton coalition in new partition π', then $u_{\pi'}(i) = 0$ and $u_{\pi'}(j) \geq 0$ and the utility of other players has not decreased. Therefore, $u_{eg}(\pi') \geq 0$ which is a contradiction.

The second claim is that each player s_j is in a coalition with a player i. Assume this was not the case so that there exists at least one such player s_j. Since we already know that all is are in separate coalitions, then $u_\pi(a) > 0$ for all $a \in N \setminus \{s_j\}$ and $u_{eg}(\pi) = u_\pi(s_j) = 0$. Then s_j can deviate and join $\pi(i)$ for any $1 \leq i \leq m$ to form a new partition π'. By that, the utility of no player decreases and $u_{\pi'}(s_j) > 0$. If this is done for all such s_j, we have $u_{eg}(\pi') > 0$ for the new partition π' which is a contradiction.

A job allocation $\mathsf{Alloc}(\pi)$ corresponds to a partition π where s_j is in $\pi(i)$ if job J_j is assigned to M_i for all j and i. Note that the utility $u_\pi(i) = \sum_{s_j \in \pi(i)} v_i(s_j) = \sum_{s_j \in \pi(i)} p_j$ of a player corresponds to the total completion time of all jobs assigned to M_i according to $\mathsf{Alloc}(\pi)$. Let π^* be a maximum egalitarian partition. Assume that there is another partition π' and $\mathsf{Alloc}(\pi')$ induces a strictly greater minimum completion time. We know that $u_{\pi^*}(s_j) = u_{\pi''}(s_j) = P$ for all $1 \leq j \leq n$ and $u_{\pi^*}(i) \leq P$ for all $1 \leq i \leq m$. But then from the assumption we have $u_{eg}(\pi') > u_{eg}(\pi^*)$ which is a contradiction. □

Since a maximum leximin partition is also a maximum egalitarian partition, we have the corollary that computing a partition with maximum leximin social welfare is NP-hard.

4 Complexity of Pareto optimality

We now consider the complexity of computing a Pareto optimal partition. The complexity of Pareto optimality has already been considered in several settings such as house allocation (Abraham et al., 2005). Bouveret and Lang (2008) examined the complexity of Pareto optimal allocations in resource allocation problems. We show that checking whether a partition is Pareto optimal is hard even under severely restricted settings.

Theorem 3. *The problem of checking whether a partition is Pareto optimal is coNP-complete in the strong sense, even if preferences are symmetric and strict.*

Proof. The reduction is from E3C (EXACT-3-COVER) to deciding whether a given partition is Pareto dominated by another partition or not. We recall the E3C problem.

E3C (EXACT-3-COVER):
INSTANCE: A pair (R, S), where $R = \{1, \dots, r\}$ is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$.
QUESTION: Is there a sub-collection $S' \subseteq S$ which is a partition of R?

It is known that E3C remains NP-complete even if each $r \in R$ occurs in at most three members of S. Let (R, S) be an instance of E3C where R is a set and S is a collection of

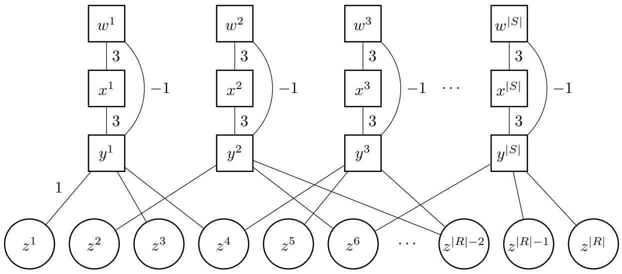

Figure 1: A graph representation of an ASHG derived from an instance of E3C. The (symmetric) utilities are given as edge weights. Some edges and labels are omitted: All edges between any y^s and z^r have weight 1 if $r \in s$. All $z^{r'}, z^{r''}$ with $r' \neq r''$ are connected with weight $\frac{1}{|R|-1}$. All other edges missing in the complete undirected graph have weight -4.

subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$. (R, S) can be reduced to an instance $((N, \mathcal{P}), \pi)$, where $(N, \mathcal{P})$ is an ASHG defined in the following way. Let $N = \{w^s, x^s, y^s \mid s \in S\} \cup \{z^r \mid r \in R\}$. The players preferences are symmetric and strict and are defined as follows:

- $v_{w^s}(x^s) = v_{x^s}(y^s) = 3$ for all $s \in S$
- $v_{y^s}(w^s) = v_{y^s}(w^{s'}) = -1$ for all $s, s' \in S$
- $v_{y^s}(z^r) = 1$ if $r \in s$ and $v_{y^s}(z^r) = -1$ if $r \notin s$ and
- $v_{z^r}(z^{r'}) = 1/(|R| - 1)$ for any $r, r' \in R$
- $v_a(b) = -4$ for any $a, b \in N$ and $a \neq b$ for which $v_a(b)$ is not already defined,

The partition π in the instance $((N, \mathcal{P}), \pi)$ is $\{\{x^s, y^s\}, \{w^s\} \mid s \in S\}\} \cup \{\{z^r \mid r \in R\}\}$. We see that the utilities of the players are as follows: $u_\pi(w^s) = 0$ for all $s \in S$; $u_\pi(x^s) = u_\pi(y^s) = 3$ for all $s \in S$; and $u_\pi(z^r) = 1$ for all $r \in R$.

Assume that there exists $S' \subseteq S$ such that S' is a partition of R. Then we prove that π is not Pareto optimal and there exists another partition π' of N which Pareto dominates π. We form another partition

$$\pi' = \{\{x^s, w^s\} \mid s \in S'\} \cup \{\{y^s, z_i, z_j, z_k\} \mid s \in S' \wedge i, j, k \in s\} \cup \{\{x^s, y^s\}, \{w^s\} \mid s \in (S \setminus S')\}\}.$$

In that case, $u_{\pi'}(w^s) = 3$ for all $s \in S'$; $u_{\pi'}(w^s) = 0$ for all $s \in S \setminus S'$; $u_\pi(x^s) = u_\pi(y^s) = 3$ for all $s \in S$; and $u_\pi(z^r) = 1 + 2/(|R| - 1)$ for all $r \in R$. Whereas the utilities of no player in π' decreases, the utility of some players in π' is more than in π. Since π' Pareto dominates π, π is not Pareto optimal.

We now show that if there exists no $S' \subseteq S$ such that S' is a partition of R, then π is Pareto optimal. We note that -4 is a sufficiently large negative valuation to ensure that if $v_a(b) = v_b(a) = -4$, then $a, b \in N$ cannot be in the same coalition in a Pareto optimal partition. For the sake of contradiction, assume that π is not Pareto optimal and there exists a partition π' which Pareto dominates π. We will see that if there exists a player $i \in N$ such that $u_{\pi'} > u_\pi$, then there exists at least one $j \in N$ such that $u_{\pi'} < u_\pi$. The only players whose utility can increase are $\{x^s \mid s \in S\}$, $\{w^s \mid s \in S\}$ or $\{z^r \mid r \in R\}$. We consider these player classes separately. If the utility of player x^s increases, it can only increase from 3 to 6 so that x^s is in the same coalition as y^s and w^s. However, this means that y^s gets a decreased utility. The utility of y^s can increase or stay the same only if it forms a coalition with some z^rs. However in that case, to satisfy all z^rs, there needs to exist an $S' \subseteq S$ such that S' is a partition of R.

Assume the utility of a player w^s for $s \in S$ increases. This is only possible if w^s is in the same coalition as x^s. Clearly, the coalition formed is $\{w^s, x^s\}$ because coalition $\{w^s, x^s, y^s\}$ brings a utility of 2 to y^s. In that case y^s needs to form a coalition $\{y^s, z_i, z_j, z_k\}$ where $s = \{i, j, k\}$. If y^s forms a coalition $\{y^s, z_i, z_j, z_k\}$, then all players $y^{s'}$ for $s' \in (S \setminus \{s\})$ need to form coalitions of the form $\{y^{s'}, z_{i'}, z_{j'}, z_{k'}\}$ such that $s' = \{i', j', k'\}$. Otherwise, their utility of 3 decreases. This is only possible if there exists a set $S' \subseteq S$ of R such that S' is a partition of R.

Assume that there exists a partition π' that Pareto dominates π and utility of a player $u_{\pi'}(z^r) > u_\pi(z^r)$ for some $r \in R$. This is only possible if each z^r forms the coalition of the form $\{z^r, z^{r'}, z^{r''}, y^s\}$ where $s = \{r, r', r''\}$. This can only happen if there exists a set $S' \subseteq S$ of R such that S' is a partition of R. □

The fact that checking whether a partition is Pareto optimal is coNP-complete has no obvious implications on the complexity of *computing* a Pareto optimal partition. In fact we present a polynomial-time algorithm to compute a partition which is Pareto optimal for strict preferences.

Theorem 4. *For strict preferences, a Pareto optimal partition can be computed in polynomial time.*

Proof. We first describe the algorithm. Set RemainingPlayers to N and set i to 1. Take any player $l_i \in$ RemainingPlayers and form a coalition S_i in which players $j \in$ RemainingPlayers such that $v_{l_i}(j) > 0$ are added. Player l_i will be called the *leader* of coalition S_i. Remove S_i from RemainingPlayers. Increment i by 1 and repeat until RemainingPlayers $= \emptyset$. Return $\{S_1, \ldots, S_m\}$.

We now prove the correctness of the algorithm via induction on the number of coalitions formed. The induction hypothesis is: *Consider the kth first formed coalitions $S_1, \ldots, S_k$. Assume, there exists a partition $\pi' \neq \pi$, such that π' Pareto dominates π. Then $S_1, \ldots, S_k \in \pi'$.* Less formally and in other words, the hypothesis can be stated as follows: *Assume that the first k coalitions $S_1, \ldots, S_k$ have formed. Then neither of the following can happen:*

1. *Some players from $S_1, \ldots, S_k$ move out of their respective coalitions and cause a Pareto improvement.*

2. *Some players from $N \setminus \bigcup_{i \in \{1,\ldots,k\}} S_i$ move to players in coalitions $S_1, \ldots, S_k$ and cause a Pareto improvement.*

Base case: Consider the coalition S_1. Then l_1, the leader of S_1 has no incentive to leave. If he leaves with a subset of players in S_1, he can only become less happy. Other players from S_1 cannot leave S_1 because their leaving makes at least one player less happy. The only possibility left is if S_1 joins $B \subseteq (N \setminus S_1)$ to cause a Pareto improvement. We know that this is not possible as player l_1 would be worse off. Similarly, no player j can move from $N \setminus S_1$ and cause a Pareto improvement because l_1 becomes worse off.
Induction step: Assume that the hypothesis is true. Then we prove that the same holds for the formed coalitions $S = S_1, \ldots, S_k, S_{k+1}$. By the hypothesis, we know that player cannot leave coalitions $S_1, \ldots, S_k$ and cause a Pareto improvement and since preferences are strict, no player can move from $N \setminus \bigcup_{i \in \{1,\ldots,k\}} S_i$ move to coalitions in $S_1, \ldots, S_k$ and cause a Pareto improvement as at least one player in S_{k+1} dislike him.

Now consider S_{k+1}. The leader of S_{k+1} is l_{k+1}. We first show that l_{k+1} cannot cause a Pareto improvement by moving to a coalition outside of S_{k+1}. This is clear because l_{k+1} can only lose utility when he leaves coalition S_{k+1} with a subset of or all of the players.

Similarly, other players in S_{k+1} cannot move out of S_{k+1} without decreasing the payoff of some player in S_{k+1}. Similarly, since the preferences are strict, no player can move from $N \setminus \bigcup_{i\in\{1,\dots,k+1\}} S_i$ and cause a Pareto improvement. □

A standard criticism of Pareto optimality is that it can lead to inherently unfair allocations. To address this criticism, the algorithm can be modified to obtain less lopsided partitions. Whenever an arbitrary player is selected to become the 'leader' among the remaining players, choose a player that does not get extremely high elitist social welfare among the remaining players. Nevertheless, even this modified algorithm may output an partition that fails to be individually rational.[4]

Another natural algorithmic question is to check whether it is possible for all players to attain their maximum possible utility at the same time. We observe that this problem can be solved in polynomial time for any separable game. We will omit the details of the algorithm but the general idea behind the algorithm is to build up coalitions and ensure that a player i and $F(i)$, all the player i likes are in the same coalition. While ensuring this, if there is a player j and a player $j' \in E(j)$ (disliked by j), then return 'no.'

5 Complexity of envy-freeness

Envy-freeness is a well-sought criterion in resource allocation, especially *cake cutting*. Lipton et al. (2004) proposed envy-minimization in different ways and examined the complexity of minimizing envy in resource allocation settings. Bogomolnaia and Jackson (2002) mentioned envy-freeness in hedonic games but focused on stability in hedonic games. We already know that envy-freeness can be easily achieved by the partition of singletons.[5] Therefore, in conjunction with envy-freeness, we seek to satisfy other properties such as stability or Pareto optimality. A partition is *Nash stable* is there is no incentive for a player to be deviate to another (possibly empty) coalition. For symmetric ASHGs, it is known that Nash stable partitions always exists and they correspond to partitions for which the utilitarian social welfare is a local optimum (see, e.g., Bogomolnaia and Jackson, 2002). We now show that for symmetric ASHGs, there may not exist any partition which is both envy-free and Nash stable.

Example 1. Consider an ASHG $(N, \mathcal{P})$ where $N = \{1, 2, 3\}$ and $\mathcal{P}$ is defined as follows: $v_1(2) = v_2(1) = 3$, $v_1(3) = v_3(1) = 3$ and $v_2(3) = v_3(2) = -4$. Then there exists no partition which is both envy-free and Nash stable.

We use the game in Example 1 as a gadget to prove the following.[6]

Theorem 5. *For symmetric preferences, checking whether there exists a partition which is both envy-free and Nash stable is NP-complete in the strong sense.*

Proof. The problem is clearly in NP since envy-freeness and Nash stability can be verified in polynomial time. We reduce the problem from E3C. Let (R, S) be an instance of E3C where R is a set and S is a collection of subsets of R such that $|R| = 3m$ for some positive integer m and $|s| = 3$ for each $s \in S$. (R, S) can be reduced to an instance $(N, \mathcal{P})$ where $(N, \mathcal{P})$ is an ASHG defined in the following way. Let $N = \{y^s \mid s \in S\} \cup \{z_1^r, z_2^r, z_3^r \mid r \in R\}$. We set all preferences as symmetric. The players preferences are as follows:

[4] It can be shown that, for general preferences, computing a partition that is Pareto optimal and individually rational at the same time is weakly NP-hard.

[5] The partition of singletons also satisfies individual rationality.

[6] Example 1 and the proof of Theorem 5 also apply to the combination of envy-freeness and individual stability and to that of envy-freeness and contractual individual stability where individual stability and contractual individual stability are variants of Nash stability (Bogomolnaia and Jackson, 2002).

- For all $r \in R$, $v_{z_1^r}(z_2^r) = v_{z_2^r}(z_1^r) = 3$, $v_{z_1^r}(z_3^r) = 3$ and $v_{z_2^r}(z_3^r) = v_{z_3^r}(z_2^r) = -4$.
- For all $s = \{i, j, k\} \in S$, $v_{z_1^i}(z_1^j) = v_{z_1^i}(z_1^k) = v_{z_1^j}(z_1^k) = v_{y^s}(z_1^i) = v_{y^s}(z_1^j) = v_{y^s}(z_1^k) = 1$.
- For all $a, b \in N$ for which valuations have not been defined, $v_a(b) = v_b(a) = -4$

We note that -4 is a sufficiently large negative valuation to ensure that if $v_a(b) = v_b(a) = -4$, then a and b will get negative utility if they are in the same coalition. We show that there exists an envy-free and Nash stable partition for $(N, \mathcal{P})$ if and only if (R, S) is a 'yes' instance of E3C.

Assume that there exists $S' \subseteq S$ such that S' is a partition of R. Then there exists a partition $\pi = \{\{y^s, z_1^i, z_1^j, z_1^k\} \mid s = \{i, j, k\} \in S'\} \cup \{\{z_2^r\}, \{z_3^r\} \mid r \in R\} \cup \{\{s\} \mid s \in S \setminus S'\}$. It is easy to see that partition π is Nash stable and envy-free. Players z_1^r and z_3^r both had an incentive to be with each other when they are singletons. However, each z_1^r now gets utility 3 by being in a coalition with $z_1^{r'}$, $z_1^{r''}$ and y^s where $s = \{r, r', r''\} \in S$. Therefore z_1^r has no incentive to be with z_3^r and z_3^r has no incentive to join $\{z_1^{r'}, z_1^{r'}, z_1^{r''}, y^s\}$ because $v_{z_3^r}(z_1^{r'}) = v_{z_3^r}(z_1^{r''}) = v_{z_3^r}(y^s) = -4$. Similarly, no player is envious of another player.

Assume that there exists no partition $S' \subseteq S$ of R such that S' is a partition of R. Then, there exists at least one $r \in R$ such that z_1^i is not in the coalition of the form $\{z_1^r, z_1^{r'}, z_1^{r''}, y^s\}$ where $s = \{r, r', r''\} \in S$. Then the only individually rational coalitions which z_1^r can form are the following $\{z_1^r\}$, $\{z_1^r, z_3^r\}$, $\{z_1^r, z_2^r\}$ or $\{z_1^r, z_1^{r'}\}$ where $r, r' \in s$ for some $s \in S$. In the first case, z_1^r wants to deviate to $\{z_3^r\}$. In the second case, z_2^r is envious and wants to replace z_3^r. In the third case, z_3^r is envious and wants to replace z_2^r. In the fourth case, z_3^r is envious and wants to replace $z_1^{r'}$. Therefore, there exists no partition which is both Nash stable and envy-free. □

While the existence of a Pareto optimal partition and an envy-free partition is guaranteed, we show that checking whether there exists a partition which is both envy-free and Pareto optimal is hard (Corollary 1). To prove the result, we first define the resource allocation setting. A *resource allocation problem* is a tuple (I, X, w) where I is a set of players (agents), X is the set of indivisible objects and $w : I \times X \to (R)$ is the weight function. A resource allocation $a : I \to 2^X$ is such that for all i and $j \neq i$, $a(i) \cap a(j) = \emptyset$. A resource allocation a dominate a' if and only if 1) for all $a(i) \succsim_i a'(i)$ and 2) there exists i such that $a(i) \succsim_i a'(i)$. A resource allocation is Pareto optimal if it is not dominated by another resource allocation.

Theorem 6. *(Theorem 2, de Keijzer et al. (2009)) The problem ∃-EEF-ADD of checking the existence of an envy-free and Pareto optimal resource allocation is Σ_2^p-complete.*

We can use the result from de Keijzer et al. (2009) to prove the following.

Corollary 1. *Checking whether there exists a partition which is both Pareto optimal and envy-free is Σ_2^p-complete.*

Proof. The problem has a yes instance if there exists an envy-free partition that Pareto dominates every other partition. Therefore the problem is in the complexity class $\mathrm{NP}^{\mathrm{NP}} = \Sigma_2^p$. We now prove that the problem is Σ_2^p-hard. We provide a polynomial-time reduction ∃-EEF-ADD to the problem of checking whether there exists a partition which is both Pareto optimal and envy-free.

Consider an instance (I, X, w) of a resource allocation problem. Let $W = \sum_{i \in I, x_j \in X} |w(i, x_j)|$. The instance (I, X, w) can be reduced to an instance of an ASHG G where $N = I \cup X$ and

- For all $i \in I$, $x_j \in X$, $v_i(x_j) = w(i, x_j)$ and $v_{x_j}(i) = 0$.
- For all x_j, x_k, $v_{x_i}(x_j) = v_{x_j}(x_i) = 0$.
- For all $i, j \in I$, $v_i(j) = v_j(i) = -W|I \cup X|$.

It is clear that for any Pareto optimal partition π, there exist no $i, j \in I \subset N$ such that $i \neq j$ and $j \in \pi(i)$. Assume that this were not the case and there exist $i, j \in I \subset N$ such that $i \neq j$ and $j \in \pi(i)$. Then i and j both get negative value because $\sum_{k\in\pi(i)} v_i(k) = \sum_{k\in(\pi(i)\setminus\{j\})} v_i(k) - W < 0$ and $\sum_{k\in\pi(i)} v_j(k) = \sum_{k\in(\pi(i)\setminus\{i\})} v_j(k) - W < 0$. Then i and j can be separated to form singletons to get another partition π', where the value of every other player $k \in (N \setminus \{i, j\})$ gets the same value while i and j get at least zero value. Therefore there is a one-to-one correspondence between any such partition π and allocation a where $a(i) = \pi(i) \setminus \{i\}$. It now easy to see that π is Pareto optimal and envy-free in G if and only if a is a Pareto optimal and envy-free allocation. □

The results of this section show that, even though envy-freeness can be trivially satisfied on its own, it becomes much more delicate when considered in conjunction with other desirable properties.

6 Conclusions

In this paper, we studied the complexity of partitions in additively separable hedonic games that satisfy standard criteria of fairness and optimality. We showed that computing a partition with maximum egalitarian or utilitarian social welfare is NP-hard in the strong sense whereas a Pareto optimal partition can be computed in polynomial time when preferences are strict. Interestingly, checking whether a given partition is Pareto optimal is coNP-complete even in the restricted setting of strict and symmetric preferences. We also showed that checking the existence of partition which satisfies not only envy-freeness but an additional property like Nash stability or Pareto optimality is computationally hard. The complexity of computing a Pareto optimal partition for ASHGs with general preferences is still open. Since the grand coalition has special significance in coalitional game theory, it would be interesting to study the complexity of checking whether the grand coalition is Pareto optimal. Other directions for future research include approximation algorithms to compute maximum utilitarian or egalitarian social welfare for different representations of hedonic games.

References

D. J. Abraham, K. Cechlárová, D. Manlove, and K. Mehlhorn. Pareto optimality in house allocation problems. In *Proceedings of the 16th International Symposium on Algorithms and Computation*, pages 1163–1175, 2005.

C. Ballester. NP-completeness in hedonic games. *Games and Economic Behavior*, 49(1):1–30, 2004.

S. Banerjee, H. Konishi, and T. Sönmez. Core in a simple coalition formation game. *Social Choice and Welfare*, 18:135–153, 2001.

S. Barberà, W. Bossert, and P. K. Pattanaik. Ranking sets of objects. In S. Barberà, P. J. Hammond, and C. Seidl, editors, *Handbook of Utility Theory*, volume II, chapter 17, pages 893—977. Kluwer Academic Publishers, 2004.

A. Bogomolnaia and M. O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

S. Bouveret and J. Lang. Efficiency and envy-freeness in fair division of indivisible goods: logical representation and complexity. *Journal of Artificial Intelligence Research*, 32(1):525–564, 2008.

S. Branzei and K. Larson. Coalitional affinity games and the stability gap. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI)*, pages 1319–1320, 2009.

N. Burani and W. S. Zwicker. Coalition formation games with separable preferences. *Mathematical Social Sciences*, 45(1):27–52, 2003.

K. Cechlárová. Stable partition problem. In *Encyclopedia of Algorithms*. 2008.

Bart de Keijzer, S. Bouveret, T. Klos, and Y. Zhang. On the complexity of efficiency and envy-freeness in fair division of indivisible goods with additive preferences. In *Proceedings of the First International Conference on Algorithmic Decision Theory*, pages 98–110, 2009.

X. Deng and C. H. Papadimitriou. On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 12(2):257–266, 1994.

D. Dimitrov, P. Borm, R. Hendrickx, and S. C. Sung. Simple priorities and core stability in hedonic games. *Social Choice and Welfare*, 26(2):421–433, 2006.

J. H. Drèze and J. Greenberg. Hedonic coalitions: Optimality and stability. *Econometrica*, 48(4): 987–1003, 1980.

E. Elkind and M. Wooldridge. Hedonic coalition nets. In *Proceedings of the 8th International Joint Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*, pages 417–424, 2009.

J. Hajduková. Coalition formation games: A survey. *International Game Theory Review*, 8(4): 613–641, 2006.

O. Hudry. A note on "Banks winners in tournaments are difficult to recognize" by G. J. Woeginger. *Social Choice and Welfare*, 23:113–114, 2004.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In *Proceedings of the 5th ACM conference on Electronic commerce(ACM-EC)*, pages 125–131, 2004.

H. Moulin. *Axioms of Cooperative Decision Making*. Cambridge University Press, 1988.

A. Roth and M. A. O. Sotomayor. *Two-Sided Matching: A Study in Game Theoretic Modelling and Analysis*. Cambridge University Press, 1990.

T. Sandholm, K. Larson, M. Andersson, O. Shehory, and F. Tohmé. Coalition structure generation with worst case guarantees. *Artificial Intelligence*, 111(1–2):209–238, 1999.

S. C. Sung and D. Dimitrov. Computational complexity in additive hedonic games. *European Journal of Operational Research*, 203(3):635–639, 2010.

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 2nd edition, 1947.

G. J. Woeginger. A polynomial time approximation scheme for maximizing the minimum machine completion time. *Operations Research Letters*, 20(4):149–154, 1997.

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