

Computing the Degree of Manipulability in the Case of Multiple Choice

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Abstract

The problem of the manipulability of known social choice rules in the case of multiple choice is considered. Several concepts of expanded preferences (preferences over the sets of alternatives) are elaborated. As a result of this analysis ordinal and nonordinal methods of preferences expanding are defined. The notions of the degree of manipulability are extended to the case under study. Using the results of theoretical investigation, 22 known social choice rules are studied via computational experiments to reveal their degree of manipulability.

1 Introduction

The problem of manipulation in voting is that the voter can achieve the best social decision for herself by purposely changing her sincere preferences. Theoretical investigations of the manipulation problem were first made in [5] [11]. There, it was shown that if some rather weak conditions hold, any nondictatorial choice rule is manipulable. To which extent social choice rules are manipulable was studied in [1] [7]. However, estimating the degree of manipulability is a very difficult computational problem — to resolve it simplifying assumptions are made. The main and the strongest assumption used is a tie-breaking rule, which allow to consider manipulation problem in the framework of single-valued choice. According to such rule from the set of winning alternatives only one winner is chosen. For example, in [1] [3] in the case of multiple choice the outcome has been chosen with respect to the alphabetical order. This is the most common type of tie-breaking rule because it is simple to implement. But this method also breaks the symmetry between candidates, that can distort the results of computation. The weaker tie-breaking rule was introduced in [9]. According to this rule, a winner is chosen at random in the event of a tie.

Manipulation problem in the case of multiple choice has not been elaborated in detail not only by its computational difficulty, but also because of absence of the common framework allowing to construct preferences over sets of alternatives.

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2 The framework

We use notations from [1]. There is a finite set A consisting of m alternatives ($m > 2$). Let $\mathcal{A} = 2^A \setminus \{\emptyset\}$ denote a set of all not-empty subsets of the set A . Each agent from a finite set $N = \{1, \dots, n\}$, $n > 1$, has preference P_i over alternatives from the set A and expanded preference EP_i over the set \mathcal{A} .

Preferences P_i are assumed to be linear orders, i.e., P_i satisfies the following conditions:

- irreflexivity ($\forall x \in A \ x \bar{P}x$),

- transitivity ($\forall x, y, z \in A \ xPy \text{ and } yPz \Rightarrow xPz$),
- connectedness ($\forall x, y \in A \ x \neq y \text{ either } xPy \text{ or } yPx$).

An ordered n -tuple of preferences P_i is called a profile, \vec{P} . A group decision is made by a social choice rule using \vec{P} and is considered to be an element of the set \mathcal{A} . Let \mathcal{L} denote the set of all linear orders on A . Then the social choice rule can be defined as $F : \mathcal{L}^n \rightarrow \mathcal{A}$. Manipulation in the case of multiple choice can be described as follows. Let

$$\vec{P} = \{P_1, \dots, P_i, \dots, P_n\}$$

be the profile of agents' sincere preferences, while

$$\vec{P}_{-i} = \{P_1, \dots, P_{i-1}, P'_i, P_{i+1}, \dots, P_n\}$$

is a profile in which all agents but i -th declare their sincere preferences, and P'_i is agent i 's deviation from her sincere preference P_i . Let $C(\vec{P}), C(\vec{P}_{-i})$ denote social choice (a subset of the set \mathcal{A}) with respect to profile \vec{P} and profile \vec{P}_{-i} , correspondingly. Then we say that manipulation takes place if for i -th agent $C(\vec{P}_{-i})EP_iC(\vec{P})$, where EP_i is the expanded preference of i -th agent. In other words, we suppose that outcome when the i -th agent deviates from her true preference is more preferable according to her expanded preference (i.e., according to her preferences over sets) than in the case when she reveals her sincere preference.

3 Basic assumptions for preferences expansion

Let us give some basic conditions of the relationship between preferences over alternatives and expanded preferences over outcomes (sets of alternatives).

First condition was introduced in [6] and is also known as Kelly's Dominance axiom. Here we will use the stronger version of Kelly's axiom introduced in [8]

Kelly's Dominance axiom (strong). $\forall i \in N$ and $\forall \vec{P}, \vec{P}' \in \mathcal{L}^n$, if

$$\left(\begin{array}{l} (\forall x \in C(\vec{P}) \text{ and } \forall y \in C(\vec{P}') \Rightarrow xP_i y \text{ or } x = y) \text{ and} \\ (\exists z \in C(\vec{P}) \text{ and } \exists w \in C(\vec{P}') \Rightarrow zP_i w) \end{array} \right),$$

then $C(\vec{P})(EP_i)C(\vec{P}')$.

We should notice, that this assumption allows us to compare social choices which have at most one alternative in the intersection.

Example. Let $x_1P_ix_6P_ix_7P_ix_9$. Using this condition, we can say that $\{x_1, x_6\}EP_i\{x_6, x_9\}$, but we cannot compare sets $\{x_1, x_6, x_7\}$ and $\{x_6, x_7, x_9\}$.

In other words, if two outcomes are different only by one alternative, the set which has more preferable alternative must be more preferable than another set.

Gärdenfors principle. $\forall i \in N, \forall \vec{P} \in \mathcal{L}^n$ and $\forall y \in A/C(\vec{P})$

- 1) $(C(\vec{P}))EP_i(C(\vec{P}) \cup \{y\})$ whenever $\forall x \in C(\vec{P}) : xP_i y$
- 2) $(C(\vec{P}) \cup \{y\})EP_i(C(\vec{P}))$ whenever $\forall x \in C(\vec{P}) : yP_i x$

This condition is also known as Gärdenfors principle defined in [4]. It can be explained in the following way. If we add to some set an alternative which is more (respectively,

less) preferable than every alternative in the chosen set, new outcome should be more (respectively, less) preferable than the old one.

In the literature, for example in [2], another conditions can be found. But almost all of them do not allow us to compare every possible sets of alternatives. For example, for lexicographic preferences $(x_1 P_i x_2 P_i \dots P_i x_{m-1} P_i x_m)$ we can not compare sets $\{x_1, x_6, x_7\}$ and $\{x_2, x_4\}$ or $\{x_1, x_{100}\}$ and $\{x_{99}, x_{101}\}$. Thus, we should define algorithms of preferences expanding which satisfy conditions mentioned above and allow us to compare all the sets of alternatives.

4 Preference expanding methods

4.1 Lexicographic methods

4.1.1 Leximin

This algorithm of preferences expansion is introduced in [8] and is based on the well-known maximin behaviour approach. Here we will use it in the form given in [10]. This method is based on comparison of the worst alternatives of any two sets. If the worst alternatives are the same, then we should compare second-worst alternatives and so on. If this is impossible, that is, when one social choice is a subset of another social choice, then the greater set is preferred to the lesser one.

Let us describe leximin method of preferences expanding. From preferences $P_i \in \mathcal{L}$ we can receive leximin expanded preferences EP_i by the following algorithm.

Two social choices $X, Y \in \mathcal{A}$ are compared:

1. If $|X| = |Y| = k$, where $k \in \{1, \dots, m-1\}$, then sort alternatives from each social choice from the most preferred to the least one, that is: $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$, where $x_j P_i x_{j+1}$ and $y_j P_i y_{j+1} \forall j \in \{1, \dots, k-1\}$. Then $X EP_i Y$ if and only if $x_h P_i y_h$ for the greatest $h \in \{1, \dots, k\}$ for which $x_h \neq y_h$.
2. If $|X| \neq |Y|$, then sort alternatives from each social choice from the least preferred to the most one, that is: $X = \{x_1, \dots, x_{|X|}\}$ and $Y = \{y_1, \dots, y_{|Y|}\}$, where $x_{j+1} P_i x_j \forall j \in \{1, \dots, |X|-1\}$ and $y_{j+1} P_i y_j \forall j \in \{1, \dots, |Y|-1\}$. There can be two cases:
 - (a) $x_h = y_h \forall h \in \{1, \dots, \min\{|X|, |Y|\}\}$. That is, one social choice is a subset of another social choice. Then, it was already mentioned above, the greater set is preferred to the lesser one, that is, $X EP_i Y$ if and only if $|X| > |Y|$.
 - (b) $\exists h \in \{1, \dots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$. Then $X EP_i Y$ if and only if $x_h P_i y_h$ for the least $h \in \{1, \dots, \min\{|X|, |Y|\}\}$, for which $x_h \neq y_h$.

For example, for three alternatives and preferences $a P_i b P_i c$ over them, leximin expanded preferences EP_i will be

$$\{a\} EP_i \{a, b\} EP_i \{b\} EP_i \{a, c\} EP_i \{a, b, c\} EP_i \{b, c\} EP_i \{c\}$$

4.1.2 Leximax

This preferences expanding method is similar to the leximin one, but in this case the best of any two social choices are compared. If the best alternatives are the same, then we should compare second-best alternatives and so on. If this is impossible, that is, when one social choice is a subset of another social choice, then the lesser set is preferred to the greater one.

For example, for three alternatives and preferences $a P_i b P_i c$ over them, leximax expanded preferences EP_i will be

$$\{a\} EP_i \{a, b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \{b\} EP_i \{b, c\} EP_i \{c\}$$

4.2 Probabilistic methods

These methods of preferences expanding in contrast to lexicographic methods suggest that for voter not only the presence of the alternative in a social choice is important, but the probability that this alternative would be the final outcome is important as well. Here two algorithms are considered: an ordering is constructed based on the probability of the best alternative and an ordering is constructed based on the probability of the worst alternative.

4.2.1 Ordering based on the probability of the best alternative

This preference expanding algorithm is based on the element-wise comparison of two social choices. If the best alternatives of two sets are the same, then the set, in which the probability that this alternative would be the final outcome is higher, is more preferable. In fact, it will be the lesser set. If the best alternatives are the same and have equal probability to be the final outcome, then next alternatives are compared in the same way.

Example. In the set $\{a, b, c\}$ probability that alternative a would be the final outcome equals $\frac{1}{3}$ (we assume that each alternative of the winning set has equal probability to be chosen as final outcome). In the set $\{a, c\}$ this probability equals $\frac{1}{2}$. In other words, there will be $\{a, c\} EP_i \{a, b, c\}$ by expanded preferences based on the probability of the best alternative algorithm.

For example, for three alternatives and preferences aP_ibP_ic over them, expanded preferences EP_i based on the probability of the best alternative will be:

$$\{a\} EP_i \{a, b\} EP_i \{a, c\} EP_i \{a, b, c\} EP_i \{b\} EP_i \{b, c\} EP_i \{c\}$$

4.2.2 Ordering based on the probability of the worst alternative

This preferences expanding method is similar to the previous, but in this case the probability of the worst alternative is consider. The set in which this probability is higher is less preferable.

For example, for three alternatives and preferences aP_ibP_ic over them, expanded preferences EP_i based on the probability of the worst alternative will be:

$$\{a\} EP_i \{a, b\} EP_i \{b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \{b, c\} EP_i \{c\}$$

4.3 Ordinal methods

This approach is based on the assumption of expected utility maximization introduced by von Neuman and Morgenstern. Here we will use a particular case of this assumption.

1. First of all, we will assign utility of each alternative for its place in preferences. In fact, we will rank the alternatives - the best one will receive the rank m , the next one the rank $m - 1$, and so on. The worst alternative has the rank of 1.
2. We assume that each alternative has equal probability to be chosen as the final outcome. It means that utility of the set of alternatives is equal to the average utility value of all alternatives within this set.

In fact, even these assumptions do not allow us to compare all social choices when $m > 2$. For example, for three alternatives and preferences aP_ibP_ic over them, there are sets $\{a, b, c\}$, $\{a, c\}$, $\{b\}$, which have equal utility of 2 according to this approach. So, we need to consider additional assumptions.

4.3.1 Lexicographic expansions

These methods suggest the use of lexicographic approach to the sets which are uncomapred by ordinal method itself. Note that new expanded preferences may differ from lexicographic preferences.

4.3.2 Probabilistic expansions

This methods suggest the use of probabilistic approach to the sets which are uncomapred by ordinal method itself. For example, for four alternatives and preferences $aP_i bP_i cP_i d$ over them, expanded preferences EP_i based on ordinal method with the probability of the worst alternative approach are (the groups of the sets for which expansion is used are underlined):

$$\begin{aligned} & \{a\} EP_i \{a, b\} EP_i \{b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \\ & EP_i \{a, b, d\} EP_i \{b, c\} EP_i \{a, b, c, d\} EP_i \{a, d\} EP_i \\ & EP_i \{a, c, d\} EP_i \{c\} \underline{EP_i \{b, c, d\}} \underline{EP_i \{b, d\}} EP_i \{c, d\} EP_i \{d\} \end{aligned}$$

4.3.3 Attitude to risk expansions

These methods are based on attitude to risk approach. In the case when the expected utility of several sets is equal, the risk-averse voter will prefer the set with the lowest variance and risk-lover voter will prefer the set with the highest variance. For up to 6 alternatives expanded preferences based on ordinal method with this expansions coincide with expanded preferences based on ordinal method with probabilistic expansion. If the number of alternatives is greater than 7 the coincidence does not hold.

Example. Let us consider lexicographic preferences $x_1 P_i x_2 P_i \dots P_i x_n$, where $n \geq 7$. There are sets $\{x_1, x_5, x_6\}$ and $\{x_2, x_3, x_7\}$ which have the equal rank and the equal variance. So, these sets are uncomapred by ordinal method with attitude to risk expansions.

For three alternatives these methods yield the same results as probabilistic methods, but for four alternatives this fact does not hold. For example, for four alternatives and the preferences $aP_i bP_i cP_i d$ over them, expanded preferences EP_i based on ordinal method with risk-lover expansion are (the groups of the sets for which expansion is used are underlined):

$$\begin{aligned} & \{a\} EP_i \{a, b\} EP_i \{a, c\} EP_i \{a, b, c\} EP_i \{b\} EP_i \\ & EP_i \{a, b, d\} EP_i \{a, d\} EP_i \{a, b, c, d\} EP_i \{b, c\} EP_i \\ & EP_i \{a, c, d\} EP_i \{b, d\} \underline{EP_i \{b, c, d\}} \underline{EP_i \{c\}} EP_i \{c, d\} EP_i \{d\} \end{aligned}$$

4.3.4 Cardinality expansions

This approach is based on comparison of the cardinality of sets which are uncomapred by ordinal method itself. We assume, that when expected utility of several sets is equal, then for voter a cardinality is important. There are two methods: one assumes that the greater set is preferred to smaller one in case of the same rank, and the other assume that the smaller set is preferred to the greater one. Note that these assumptions are rather non-binding. It allows us to compare all sets only when there are three alternatives. However, even in this case this method do not give different results. For example, for three alternatives and preferences $aP_i bP_i c$ over them, expanded preferences EP_i based on ordinal method with greater set approach yield the same result as leximax method and ordinal method with leximax expansion:

$$\{a\} EP_i \{a, b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \{b\} EP_i \{b, c\} EP_i \{c\}$$

For more than four alternatives cardinality expansion itself don't allows to compare all sets of alternatives. So, additional expansions mentioned above should be added in this case.

5 Indices of manipulability

Number of alternatives being m , the total number of possible linear orders is obviously equal to $m!$, and total number of profiles with n agents is equal to $(m!)^n$. In [7] to measure a degree of manipulability of social choice rules the following index was introduced (we call it Kelly's index and denote as K) :

$$K = \frac{d_0}{(m!)^n},$$

where d_0 is the number of profiles in which manipulation takes place¹.

In [1] index of freedom of manipulation is introduced. We also introduce two similar indices: the degree of nonsensitivity to preference change and probability of getting worse. Let us note that for an agent there are $(m! - 1)$ linear orders to use instead of her sincere preference. Denote as κ_i^+ ($i = 1, \dots, n$; $0 \leq \kappa_i^+ \leq m! - 1$) the number of orderings in which voter is better off, κ_i^0 - the number of orderings when the result of voting remain the same and κ_i^- - the number of orderings in which voter is worse off. It is obvious that $\kappa_i^+ + \kappa_i^0 + \kappa_i^- = (m! - 1)$. Dividing each κ_i to $(m! - 1)$ one can find the share of each type of orderings for an agent i in this profile. Summing up each share over all agents and dividing it to n one can find the average share in the given profile. Summing the share over all profiles and dividing this sum to $(m!)^n$ we obtain three indices

$$I_1 = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^n \kappa_i}{(m!)^n \cdot n \cdot (m! - 1)}$$

where κ_i is κ_i^+ , κ_i^0 or κ_i^- . It is obvious that $I_1^+ + I_1^0 + I_1^- = 1$.

These indices K and I_1 (as well as index J) measure the degree of manipulability in terms of the share of manipulable profiles or the share of orderings using which an agent can manipulate.

The following two indices show the *efficiency* of manipulation, i.e., to which extent an agent can be better off via manipulating her sincere ordering. Let under a profile \vec{P} social decision be the set $C(\vec{P})$ which stands at k -th place from the top in the expanded preferences of i -th agent. Let after her manipulation the social decision be a set $C(\vec{P}')$ which stands in the expanded preferences of the i -th agent at j -th place from the top, and let $j < k$. Then $\theta = j - k$ shows how is the i -th agent better off. Let us sum up θ for all advantageous orderings κ_i^+ (defined above), and let us divide the obtained value to κ_i^+ . Denote this index through Z_i , which shows an average "benefit" (in terms of places) of manipulation of the agent i gained via manipulation κ_i^+ orderings from $(m! - 1)$. Summing up this index over all agents and over all profiles, we obtain the index under study

$$I_2 = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^n Z_i}{(m!)^n \cdot n}$$

The next criterion I_3 is a modification of I_2 . Instead of evaluating the "average" benefit Z_i for i -th agent, we evaluate the value

$$Z_i^{\max} = \max(Z_1, \dots, Z_{\kappa_i}).$$

In other words, the value Z_i^{\max} show the maximal benefit which can be obtained by agent i . Summing up this index over all agents and over all profiles, we obtain our next index under study

¹In [1] an extended version of Kelly's index was introduced. Denote by λ_k the number of profiles in which exactly k voters can manipulate. Construct index $J_k = \frac{\lambda_k}{(m!)^n}$ which shows the share of profiles in which exactly k voters can manipulate. Obviously, $K = J_1 + J_2 + \dots + J_n$. Then one can consider the vectorial index $J = (J_1, J_2, \dots, J_n)$.

$$I_3 = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^n Z_i^{\max}}{(m!)^{n \cdot n}}$$

The indices K, I_1, I_2, I_3, J have been calculated for each of the rules introduced in the next section. The indices I_1, I_2 and I_3 were introduced in [1].

6 Social Choice Rules

The calculation of indices is performed for up to $m = 5$ alternatives for 22 social choice rules. In this work the results only for 5 rules will be given.

1. Plurality Rule

Choose alternatives, that have been admitted to be the best by the maximum number of agents, i.e.

$$a \in C(\vec{P}) \iff [\forall x \in A \quad n^+(a, \vec{P}) \geq n^+(x, \vec{P})],$$

where $n^+(a, \vec{P}) = \text{card}\{i \in N \mid \forall y \in A \quad a P_i y\}$

2. Approval Voting.

Let us define

$$n^+(a, \vec{P}, q) = \text{card}\{i \in N \mid \text{card}\{D_i(a)\} \leq q - 1\},$$

i.e., $n^+(a, \vec{P}, q)$ means the number of agents for which a is placed on q 'th place in their orderings. Thus, if $q = 1$, then a is the first best alternative for i -th voter; if $q = 2$, then a is either first best or second best option, etc. The integer q can be called as degree of procedure.

Now we can define Approval Voting Procedure with degree q

$$a \in C(\vec{P}) \iff [\forall x \in A \quad n^+(a, \vec{P}, q) \geq n^+(x, \vec{P}, q)],$$

i.e., the alternatives are chosen that have been admitted to be between q best by the maximum number of agents.

It can be easily seen that Approval Voting Procedure is a direct generalization of Plurality Rule; for the latter $q = 1$.

3. Borda's Rule.

Put to each $x \in A$ into correspondence a number $r_i(x, \vec{P})$ which is equal to the cardinality of the lower contour set of x in $P_i \in \vec{P}$, i.e. $r_i(x, \vec{P}) = |L_i(x)| = |\{b \in A : x P_i b\}|$. The sum of that numbers over all $i \in N$ is called Borda's count for alternative x ,

$$r(a, \vec{P}) = \sum_{i=1}^n r_i(a, P_i).$$

Alternative with maximum Borda's count is chosen., i.e.

$$a \in C(\vec{P}) \iff [\forall b \in A, \quad r(a, \vec{P}) \geq r(b, \vec{P})].$$

4. Black's Procedure.

If Condorset winner exists, it is to be chosen. Otherwise, Borda's Rule is applied.

5. Threshold rule.

Let $v_1(x)$ be the number of agents for which the alternative x is the worst in their ordering, $v_2(x)$ is the number of agents placing the x second worst, and so on, $v_m(x)$

the number of agents considering the alternative x the best. Then we order the alternatives lexicographically. The alternative x is said to V -dominate the alternative y if: $v_1(x) < v_1(y)$ or, if there exists k not more than m , s.t. $v_i(x) = v_i(y)$, $i = 1, \dots, k - 1$, and $v_k(x) < v_k(y)$. In other words, first, the number of worst places are compared if these numbers are equal then the number of second worst places are compared and so on. The alternatives which are not dominated by other alternatives via V are chosen.

7 Computation scheme

The calculation of indices is performed for up to $m = 5$ alternatives. For small number of voters n all possible profiles are checked for manipulability and respective indices are evaluated. For greater number of voters the statistical scheme is used.

For each profile under consideration in both exhaustive and statistical schemes, all $m!-1$ manipulating orderings for each voter are generated and the respective choice sets of manipulating profiles are compared with the choice of the original profile, using all introduced methods of the preference extensions.

8 Results

In the case of 3 alternatives there are 4 different types of extended preferences. For example, if preferences over are $aP_i bP_i c$ then the extended preferences are as follows:

1. Leximin method, Ordinal method with leximin or greater set extensions.

$$\{a\} EP_i \{a, b\} EP_i \{b\} EP_i \{a, c\} EP_i \{a, b, c\} EP_i \{b, c\} EP_i \{c\}$$

2. Leximax method, Ordinal method with leximax or lesser set extensions.

$$\{a\} EP_i \{a, b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \{b\} EP_i \{b, c\} EP_i \{c\}$$

3. Probabilistic method based on the probability of the worst alternative, Ordinal method with risk-averse extension.

$$\{a\} EP_i \{a, b\} EP_i \{b\} EP_i \{a, b, c\} EP_i \{a, c\} EP_i \{b, c\} EP_i \{c\}$$

4. Probabilistic method based on the probability of the best alternative, Ordinal method with risk-lover extension.

$$\{a\} EP_i \{a, b\} EP_i \{a, c\} EP_i \{a, b, c\} EP_i \{b\} EP_i \{b, c\} EP_i \{c\}$$

The results of Kelly's index calculation for 3 and 4 voters are presented in the tables 1 and 2 correspondingly. In the brackets near the name of the rule the results from [1] are given. One can see that in most cases, especially in the case of 4 voters, the degree of manipulability in the case of single-valued choice is underestimated. We also can state that for almost all rules Method 1 and Method 3 have the same Kelly's index as well as Methods 2 and 4.

In Figures 1 and 2 the results of calculation for the larger number of voters are given. Kelly's index is shown on the Y-axis and the logarithm of the number of voters is shown on the X-axis. The calculation was made for each number of voters from 3 to 25 and then for 29, 30, 39, 40 and so on up to 100. That explains changes at the figures when number of voters is more than 25.

We can make several conclusions from these figures.

	Method 1	Method 2	Method 3	Method 4
Plurality (0,1667)	0,2222	0	0,2222	0
Approval q=2	0,1111	0,6111	0,1111	0,6111
Borda (0,2361)	0,3056	0,4167	0,3056	0,4167
Black (0,1111)	0,0556	0,1667	0,0556	0,1667
Threshold	0,3056	0,4167	0,3056	0,4167

Table 1: The case of 3 alternatives and 3 voters

	Method 1	Method 2	Method 3	Method 4
Plurality (0,1852)	0,3333	0,3333	0,3333	0,3333
Approval q=2	0,2963	0,2963	0,2963	0,2963
Borda (0,3102)	0,3611	0,4028	0,3611	0,4028
Black (0,1435)	0,2361	0,2778	0,2778	0,2361
Threshold	0,4028	0,4028	0,4028	0,4028

Table 2: The case of 3 alternatives and 4 voters

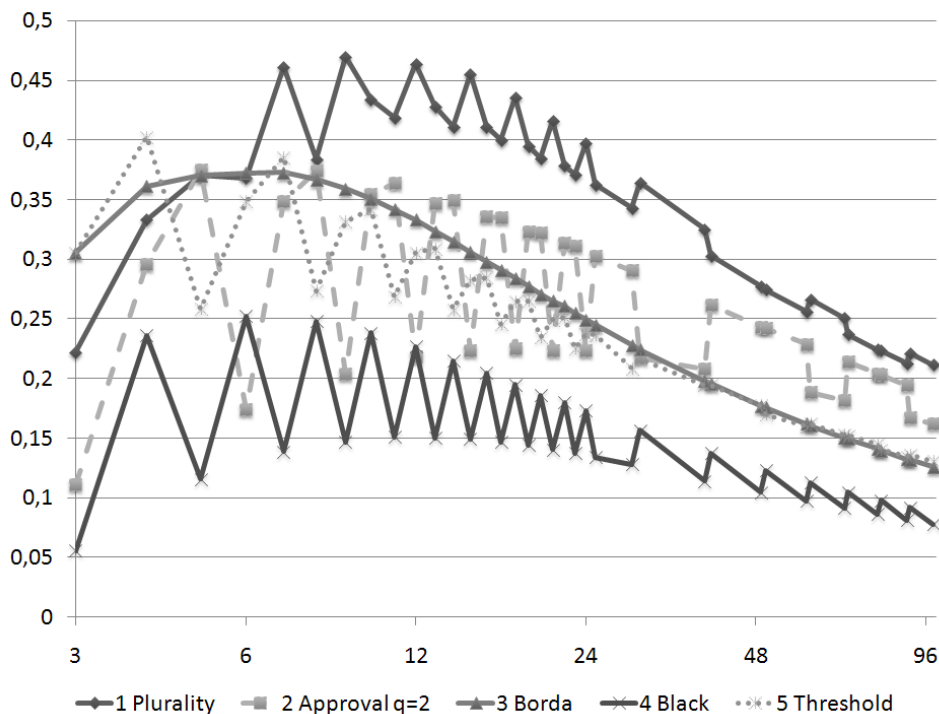


Figure 1: Kelly's index for Leximin extension method.

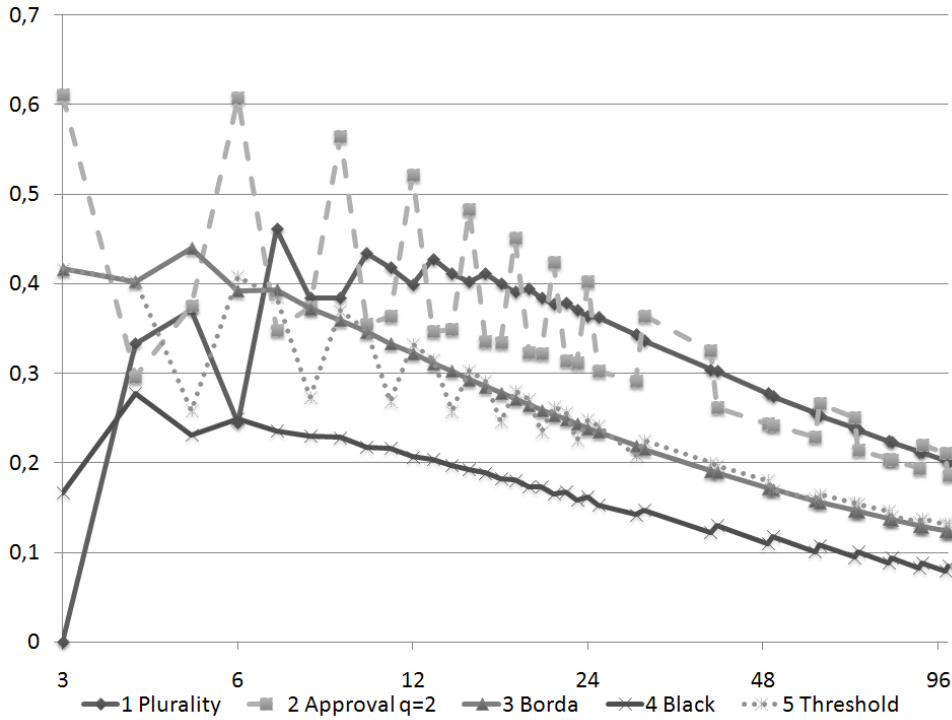


Figure 2: Kelly's index for Leximax extension method.

1. The answer on the question which rule is less manipulable depends on the method of preferences extension. For example, for the number of voters divisible by 3 Approval voting rule is less manipulable than Plurality rule for Method 1 and is more manipulable for Method 2.
2. If the number of voters is small Threshold rule is less manipulable than Borda rule. But when the number of voters is high enough Borda rule is better in Kelly's sense. The exact minimum number of voters needed depends on the method used.
3. Black's Procedure is least manipulable almost for any number of voters and for any method.
4. Kelly's index for Black's Procedure and Method 1 depends on even or odd number of the voters considered. At the same time for rules such as Plurality, Approval voting and Threshold, there is a cycle length of m . In this case there is a cycle length of 3. The dependence from the number of alternatives is explained by differences in number and cardinality of ties produced by rules. For example, the set $\{a, b, c\}$ can appear as the result of plurality voting only in the case when the number of voters is divisible by the number of alternatives.

In Figure 3 the results of calculation of I_1 index for 3 alternatives, 3 voters and Method 1 are given. The left part of each row is the degree of freedom of manipulation. The right part is the probability of getting worse. The middle part is the degree of nonsensitivity to preference change. In Figure 4 the results of calculation of I_1 index for 3 alternatives, 100 voters and Method 1 are given. We can see that the bigger the number of voters is, the less is freedom of manipulation as well as the probability of getting worse.

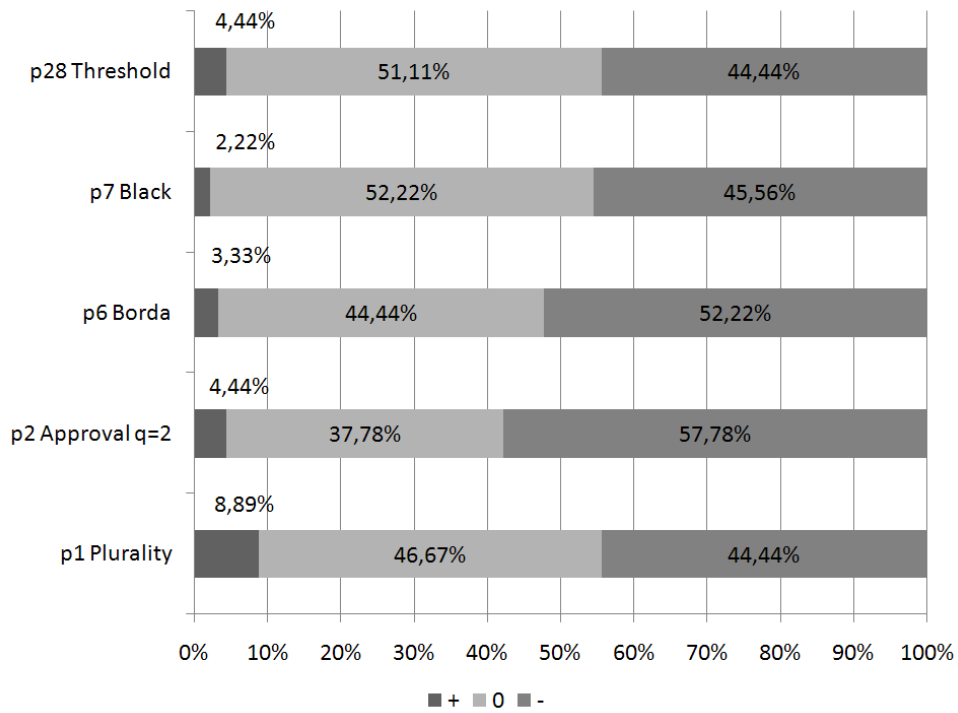


Figure 3: I_1 for Leximin extension method and 3 alternatives.

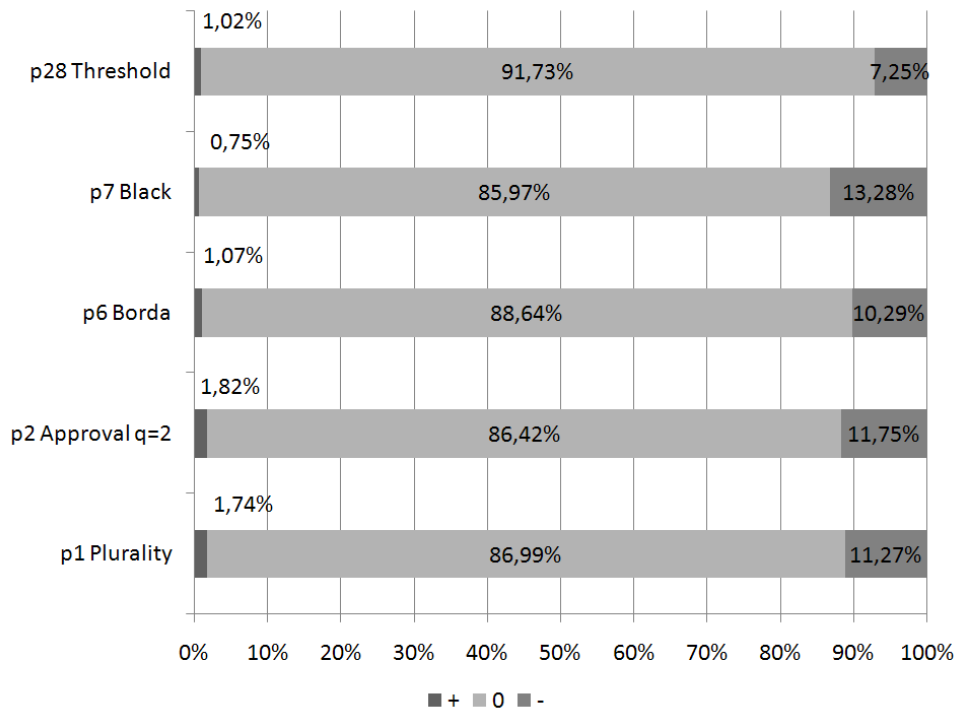


Figure 4: I_1 for Leximin extension method and 100 alternatives.

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