# A Computational Analysis of the Tournament Equilibrium Set*

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**Abstract**

A recurring theme in the mathematical social sciences is how to select the "most desirable" elements given a binary dominance relation on a set of alternatives. Schwartz's *tournament equilibrium set* (TEQ) ranks among the most intriguing, but also among the most enigmatic, tournament solutions proposed so far in this context. Due to its unwieldy recursive definition, little is known about TEQ. In particular, its monotonicity remains an open problem to date. Yet, if TEQ were to satisfy monotonicity, it would be a very attractive solution concept refining both the Banks set and Dutta's minimal covering set. We show that the problem of deciding whether a given alternative is contained in TEQ is NP-hard. Furthermore, we propose a heuristic that significantly outperforms the naive algorithm for computing TEQ. Early experimental results support the conjecture that TEQ is indeed monotonic.

## 1 Introduction

A recurring theme in the mathematical social sciences is how to select the "most desirable" elements given a binary dominance relation on a set of alternatives. Examples are diverse and include selecting socially preferred candidates in social choice settings (*e.g.*, Fishburn, 1977; Laslier, 1997), finding valid arguments in argumentation theory (*e.g.*, Dung, 1995; Dunne, 2007), determining the winners of a sports tournament (*e.g.*, Dutta and Laslier, 1999), making decisions based on multiple criteria (*e.g.*, Bouyssou et al., 2006), choosing the optimal strategy in a symmetric two-player zero-sum game (*e.g.*, Duggan and Le Breton, 1996), and singling out acceptable payoff profiles in cooperative game theory (Gillies, 1959; Brandt and Harrenstein, 2008). In social choice theory, where dominance-based solutions are most prevalent, the dominance relation can simply be defined as the pairwise majority relation, *i.e.*, an alternative $a$ is said to dominate another alternative $b$ if the number of individuals preferring $a$ to $b$ exceeds the number of individuals preferring $b$ to $a$. As is well known from Condorcet's paradox (de Condorcet, 1785), the dominance relation may contain cycles and thus need not have a maximum, even if each of the underlying individual preferences does. As a consequence, the concept of maximality is rendered untenable in most cases, and a variety of so-called *solution concepts* that take over the role of maximality in non-transitive relations have been suggested (see, *e.g.*, Laslier, 1997).

The *tournament equilibrium set* (TEQ) introduced by Schwartz (1990) ranks among the most intriguing, but also among the most enigmatic, solution concepts that has been proposed for tournaments, *i.e.*, asymmetric and complete dominance relations. Due to its unwieldy recursive definition, however, preciously little is known about TEQ (Dutta, 1990; Laffond et al., 1993). In particular, whether TEQ satisfies the important property of monotonicity remains an open question to date. If it does, TEQ constitutes a most attractive tournament solution, refining both the minimal covering set and the Banks set (Laslier, 1997; Laffond et al., 1993).

Recent work in computer science has addressed the computational complexity of almost

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all common solution concepts (see, *e.g.*, Woeginger, 2003; Alon, 2006; Conitzer, 2006; Brandt et al., 2007). The minimal covering set and the tournament equilibrium set, however, have remained notable exceptions. Laslier writes that "Unfortunately, no algorithm has yet been published for finding the minimal covering set or the tournament equilibrium set of large tournaments. For tournaments of order 10 or more, it is almost impossible to find (in the general case) these sets at hand" (Laslier, 1997, p.8). The minimal covering set has recently been shown to be computable in polynomial time (Brandt and Fischer, 2008). In this paper we prove that the same is not true for TEQ, unless P equals NP. We first give an arguably simpler alternative to Woeginger's (2003) NP-hardness proof for membership in the Banks set. Then the construction used in that proof is modified so as to obtain the analogous result for TEQ. In contrast to the Banks set, there is no obvious reason to suppose that the TEQ membership problem is in NP; it may very well be even harder. In the second part of the paper, we propose and evaluate a heuristic for computing TEQ that performs reasonably well on tournaments with up to 150 alternatives. Experiments further support the conjecture that TEQ is indeed monotonic.

# 2 Preliminaries

A *tournament* $T$ is a pair $(A, \succ)$, where $A$ is a finite set of *alternatives* and $\succ$ an irreflexive, anti-symmetric, and complete binary relation on $A$, also referred to as the *dominance relation*. Intuitively, $a \succ b$ signifies that alternative $a$ beats $b$ in a pairwise comparison. We write $\mathcal{T}$ for the class of all tournaments and have $\mathcal{T}(A)$ denote the set of all tournaments on a fixed set $A$ of alternatives. If $T$ is a tournament on $A$, then every subset $X$ of $A$ induces a tournament $T|_X = (X, \succ|_X)$, where $\succ|_X = \{(x, y) \in X \times X : x \succ y\}$.

As the dominance relation is not assumed to be transitive in general, there need not be a so-called *Condorcet winner*, *i.e.*, an alternative that dominates all other alternatives. A *tournament solution* $S$ is defined as a function that associates with each tournament $T$ on $A$ a subset $S(T)$ of $A$. The definition of a tournament solution commonly includes the requirement that $S(T)$ be non-empty if $T$ is defined on a non-empty set of alternatives and that it select the Condorcet winner if there is one (Laslier, 1997, p.37). For $X$ a subset of $A$, we also write $S(X)$ for the more cumbersome $S(T|_X)$, provided that the tournament $T$ is known from the context. A tournament solution $S$ is said to be *monotonic* if for any two tournaments $T, T' \in \mathcal{T}(A)$ which only differ in that $b \succ a$ in $T$ and $a \succ b$ in $T'$, $a \in S(T)$ implies that also $a \in S(T')$, *i.e.*, reinforcing an alternative cannot cause it to be excluded from the solution set. Monotonicity is a vital property that all reasonable tournament solutions satisfy. In this paper, we will be concerned with two particular tournament solutions, the Banks set and Schwartz's tournament equilibrium set (TEQ). For a proper formal definition, however, we need some auxiliary notions and notations.

Let $R$ be a binary relation on a set $A$. We write $R^*$ for the transitive reflexive closure of $R$. By the *top cycle* $TC_A(R)$ we understand the maximal elements of the asymmetric part of $R^*$. A subset $X$ of $A$ is said to be *transitive* if $R$ is transitive on $X$. For $X \subseteq Y \subseteq A$, $X$ is called *maximal transitive in* $Y$ if $X$ is transitive and no proper superset of $X$ in $Y$ is. Clearly, since $A$ is finite, every transitive set is contained in a maximal transitive set. Given a set $Z = \{Z_i\}_{i \in I}$ of pairwise disjoint subsets of $A$, a subset $X$ of $A$ will be called a *choice set for* $Z$ if it contains precisely one element from each subset $Z_i \in Z$.

In tournaments, maximal transitive sets are also referred to as Banks trajectories. The *Banks set* $BA(T)$ of a tournament $T$ then collects the maximal elements of the Banks trajectories.

**Definition 1 (Banks set)** *Let $T$ be a tournament on $A$. An alternative $a \in A$ is in the* Banks set $BA(T)$ *of $T$ if $a$ is a maximal element of some maximal transitive set in $T$.*

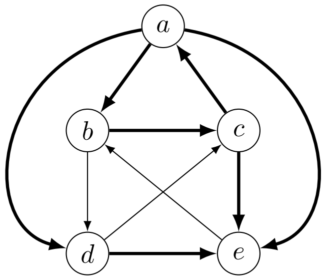


Figure 1: Example due to Schwartz, 1990, where $BA(T) = \{a, b, c, d\}$ and $TEQ(T) = \{a, b, c\}$. The TEQ relation $\longrightarrow$ is indicated by thick edges.

The *tournament equilibrium set* $TEQ(T)$ of a tournament $T$ on $A$ is defined as the top cycle of a particular subrelation of the dominance relation, referred to as the TEQ relation in the following. The underlying idea is that an alternative is only "properly" dominated, *i.e.*, dominated according to the subrelation, if it is dominated by an element that is selected by some tournament solution concept $S$. To make this idea precise, for $X \subseteq A$, we write $\overline{D}_X(a) = \{\, b \in X : b \succ a \,\}$ for the *dominators* of $a$ in $X$, omitting the subscript when $X = A$. Thus, for each alternative $a$ one examines the set $\overline{D}(a)$ of its dominators, and solves the subtournament $T|_{\overline{D}(a)}$ by means of the solution $S$. In the subrelation $a$ is then only dominated by the alternatives in $S(\overline{D}(a))$. This of course, still leaves open the question as to the choice of the solution concept $S$. Now, in the case of $TEQ$, $S$ is taken to be $TEQ$ itself! This recursion is well-defined because for any $X \subseteq A$ and $a \in X$, the set $\overline{D}_X(a)$ is a *proper* subset of $X$. Thus, in order to determine the TEQ relation in a subtournament $T$, one has to calculate the TEQ of smaller and smaller subtournaments of $T$.

**Definition 2 (Tournament equilibrium set)** *Let $T \in \mathcal{T}(A)$. For each subset $X \subseteq A$, define the* tournament equilibrium set *$TEQ(X)$ for $X$ as*

$$TEQ(X) = TC_X(\longrightarrow_X),$$

*where $\longrightarrow_X$ is defined as the binary relation on $X$ such that for all $x, y \in X$,*

$$x \longrightarrow_X y \text{ if and only if } x \in TEQ(\overline{D}_X(y)).$$

Recall that in particular, $TEQ(\emptyset) = \emptyset$. The TEQ relation $\longrightarrow_X$ is a subset of the dominance relation $\succ$, and if $\overline{D}_X(x) \neq \emptyset$, then there is some $y \in \overline{D}_X(x)$ with $y \longrightarrow_X x$. Furthermore, Definition 2 directly yields a recursive algorithm to compute TEQ. Some reflection reveals that this *naive algorithm* requires time exponential in $|A|$ in the worst case.

It can easily be established that the Banks set and TEQ both select the Condorcet winner of a tournament if there is one. Moreover, in a cyclic tournament on three alternatives, the Banks set and TEQ both consist of all alternatives. In more complex tournaments, however, the Banks set and TEQ may differ. Consider, for example, the tournament $T$ depicted in Figure 1. We first calculate the TEQ relation $\longrightarrow$. Thus, *e.g.*, for alternative $e$ we find $\overline{D}(e) = \{a, c, d\}$, which constitutes a three-cycle, and so $TEQ(\overline{D}(e)) = \{a, c, d\}$. Accordingly, $a \longrightarrow e$, $c \longrightarrow e$, as well as $d \longrightarrow e$. Doing this for all alternatives, we find $TEQ(T) = \{a, b, c\}$ as the top cycle $TC(\longrightarrow)$ of the relation $\longrightarrow$. By contrast, the Banks set consists of the four elements $a$, $b$, $c$ and $d$. *E.g.*, $d \in BA(T)$, because $\{d, c, e\}$ is a maximal transitive set with maximal element $d$. Nevertheless, TEQ is always included in the Banks set.

**Proposition 1 (Schwartz, 1990)** *Let $T = (A, \succ)$ be a tournament. Then, $TEQ(T) \subseteq BA(T)$.*

*Proof:* We prove by structural induction on $X$ that $TEQ(X) \subseteq BA(X)$ for all subsets $X$ of $A$. The case $X = \emptyset$ is trivial, as then $TEQ(X) = BA(X) = \emptyset$. So, assume that $TEQ(X') \subseteq BA(X')$, for all $X' \subsetneq X$. We prove that $TEQ(X) \subseteq BA(X)$ as well. To this end, consider an arbitrary $a \in TEQ(X)$. Either $\overline{D}_X(a) = \emptyset$ or $\overline{D}_X(a) \neq \emptyset$. In the former case, $a$ is the Condorcet winner in $X$ and therefore $a \in BA(X)$. In the latter case, $x \longrightarrow_X a$ for some $x \in X$. Having assumed that $a \in TEQ(X)$, *i.e.*, $a \in TC(\longrightarrow_X)$, there is also an $x' \in X$ with $a \longrightarrow_X x'$. Accordingly, $a \in TEQ(\overline{D}_X(x'))$. By the induction hypothesis, also $a \in BA(\overline{D}_X(x'))$. Therefore, there is some maximal transitive set $Y$ in $\overline{D}_X(x')$ of which $a$ is the maximal element. Then, $Y \cup \{x'\}$ is a transitive set in $X$. Now let $Y' \subseteq X$ be a maximal transitive set in $X$ containing $Y \cup \{x'\}$ with $a'$ as maximal element. Observe that $a' \in BA(X)$. Then, $a' \succ x'$ and so $a' \in \overline{D}_X(x')$. Now consider $Y' \cap \overline{D}_X(x')$. Clearly, $Y \cap \overline{D}_X(x')$ is a transitive set in $\overline{D}_X(x')$ which contains $a'$ as its maximal element. Moreover, $Y \subseteq Y' \cap \overline{D}_X(x')$. By maximality of $Y$ it then follows that $Y = Y' \cap \overline{D}_X(x')$ and that $a = a'$. We may conclude that $a \in BA(X)$. □

Otherwise, little is known and much surmised about the theoretical properties of TEQ. For example, Schwartz (1990) conjectured that the top cycle of the TEQ relation is always weakly connected, a property of TEQ we will refer to as *CTC* for *connected top cycle*. Laffond et al. (1993) showed that TEQ satisfying CTC is equivalent to it having a number of useful properties. In particular, TEQ is monotonic if and only if CTC holds. Moreover, CTC implies the inclusion of TEQ in the minimal covering set (see, *e.g.*, Laslier, 1997), another appealing tournament solution. Thus, if TEQ satisfies CTC it might be considered a very strong solution concept. Otherwise, TEQ lacks the vital property of monotonicity and as such it would be severely flawed as a tournament solution.

# 3 An Alternative NP-Hardness Proof for Membership in the Banks Set

We begin our investigation of the computational complexity of the TEQ membership problem by giving an alternative proof for NP-hardness of the analogous problem for the Banks set. The latter was first demonstrated by Woeginger (2003) using a reduction from graph three-colorability. Our proof works by a reduction from *3SAT*, the NP-complete satisfiability problem for Boolean formulas in conjunctive normal form with exactly three literals per clause (see, *e.g.*, Papadimitriou, 1994). It is arguably simpler than Woeginger's, and a much similar construction will be used in the next section to prove NP-hardness of membership in TEQ. The tournaments used in both reductions will be taken from a special class $\mathcal{T}^*$.

**Definition 3 (The class $\mathcal{T}^*$)** *A tournament $(A, \succ)$ is in the class $\mathcal{T}^*$ if it satisfies the following properties. There is some odd integer $n \geq 1$, the number of* layers *in the tournament, such that $A = C \cup U_1 \cup \dots \cup U_n$, where $C, U_1, \dots, U_n$ are pairwise disjoint and $C = \{c_0, \dots, c_n\}$. Each $U_i$ is a singleton if $i$ is even, and $U_i = \{u_i^1, u_i^2, u_i^3\}$ if $i$ is odd. The complete and asymmetric dominance relation $\succ$ is such that for all $c_i \in C_i$, $c_j \in C_j$, $u_i \in U_i$, $u_j \in U_j$ $(0 \leq i, j \leq n)$:*

*(i)* $c_i \succ c_j$, *if* $i > j$,

*(ii)* $u_i \succ c_j$, *if* $i = j$,

*(iii)* $c_j \succ u_i$, *if* $i \neq j$,

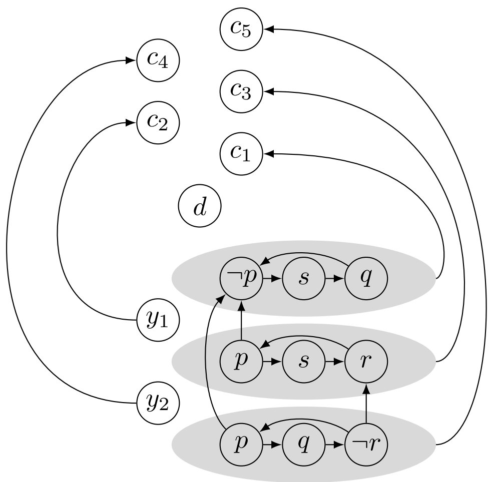


Figure 2: Tournament $T_\varphi^{BA}$ for the *3CNF* formula $\varphi = (\neg p \vee s \vee q) \wedge (p \vee s \vee r) \wedge (p \vee q \vee \neg r)$. Omitted edges are assumed to point downwards.

*(iv)* $u_i \succ u_j$, *if* $i < j$ *and at least one of* $i$ *and* $j$ *is even,*

*(v)* $u_i^k \succ u_i^l$, *if* $i$ *is odd and* $k \equiv l - 1 \pmod 3$

*We also refer to* $c_0$ *by* $d$*, for "decision node" and to* $\bigcup_{1 \le i \le n} U_n$ *by* $U$*. For* $i = 2k$*, we have as a notational convention* $U_i = Y_k = \{y_k\}$ *and set* $Y = \bigcup_{1 \le 2k \le n} Y_k$*.*

Observe that this definition fixes the dominance relation between any two alternatives except for some pairs of alternatives that are both in $U$.

As a next step in the argument, we associate with each instance of *3SAT* a tournament in the class $\mathcal{T}^*$. An instance of *3SAT* is given by a formula $\varphi$ in 3-*conjunctive normal form (3CNF), i.e.,* $\varphi = (x_1^1 \vee x_1^2 \vee x_1^3) \wedge \cdots \wedge (x_m^1 \vee x_m^2 \vee x_m^3)$, where each $x \in \{x_i^1, x_i^2, x_i^3 \colon 1 \le i \le m\}$ is a literal. For each clause $x_i^1 \vee x_i^2 \vee x_i^3$ we assume $x_i^1$, $x_i^2$ and $x_i^3$ to be distinct literals. We moreover assume the literals to be indexed and by $X_i$ we denote the set $\{x_i^1, x_i^2, x_i^3\}$. For literals $x$ we have $\bar{x} = \neg p$ if $x = p$, and $\bar{x} = p$ if $x = \neg p$, where $p$ is some propositional variable. We may also assume that if $x$ and $y$ are literals in the same clause, then $x \neq \bar{y}$. We say a *3CNF* $\varphi = (x_1^1 \vee x_1^2 \vee x_1^3) \wedge \cdots \wedge (x_m^1 \vee x_m^2 \vee x_m^3)$ is *satisfiable* if there is a choice set $V$ for $\{X_i\}_{1 \le i \le m}$ such that $v' = \bar{v}$ for no $v, v' \in V$. Next we define for each *3SAT* formula $\varphi$ the tournament $T_\varphi^{BA}$.

**Definition 4 (Banks construction)** *Let* $\varphi$ *be a 3CNF* $(x_1^1 \vee x_1^2 \vee x_1^3) \wedge \cdots \wedge (x_m^1 \vee x_m^2 \vee x_m^3)$*. Define* $T_\varphi^{BA} = (C \cup U, \succ)$ *as the tournament in the class* $\mathcal{T}^*$ *with* $2m - 1$ *layers such that for all* $1 \le j < 2m$*,*

$$U_j = \begin{cases} X_i & \text{if } j = 2i - 1, \\ \{y_i\} & \text{if } j = 2i \end{cases}$$

*and such that for all* $x \in X_i$ *and* $x' \in X_j$ $(1 \le i, j \le m)$*,*

$$x \succ x' \quad \text{if both } j < i \text{ and } x' = \bar{x} \text{ or both } i < j \text{ and } x' \neq \bar{x}.$$

*Observe that in conjunction with the other requirements on the dominance relation of a tournament in* $\mathcal{T}^*$*, this completely fixes the dominance relation* $\succ$ *of* $T_\varphi^{BA}$*.*

An example of a tournament $T_\varphi^{BA}$ for a *3CNF* $\varphi$ is shown in Figure 2. We are now in a position to present our alternative proof that the Banks membership problem is NP-complete.

**Theorem 1** *The problem of deciding whether a particular alternative is in the Banks set of a tournament is NP-complete.*

*Proof: Membership* in NP is obvious. For a fixed alternative $d$, we can simply guess a transitive subset of alternatives $V$ with $d$ as maximal element and verify that $V$ is also maximal with respect to set inclusion.

For NP-hardness, we show that $T_\varphi^{BA}$ contains a maximal transitive set with maximal element $d$ if and only if $\varphi$ is satisfiable. First observe that $V$ is a maximal transitive subset with maximal element $d$ in $T_\varphi^{BA}$ only if both

($i$) for all $1 \leq i < 2m$ there is a $u \in U_i$ such that $u \in V$, and

($ii$) there are no $1 \leq i < j < 2m$, $u \in U_i$, $u' \in U_j$ with $u, u' \in V$ such that $u_j \succ u_i$.

Regarding ($i$), if there is an $1 \leq i < 2m$ such that no element of $U_i$ is contained in $V$, we can always add $c_i$ to $V$ in order to obtain a larger transitive set. If ($ii$) were not to hold, both $i$ and $j$ have to be odd for $u_j$ to dominate $u_i$. However, in light of ($i$), there has to be $k$ with $i < k < j$ and $u'' \in U_k$ such that $u'' \in V$. It follows that $V$ is not transitive because $u$, $u''$, and $u'$ form a cycle. If there is maximal transitive set $V$ with maximal element $d$ complying with both ($i$) and ($ii$), a satisfying assignment of $\varphi$ can be obtained by letting all literals contained in $X \cap V$ be true.

For the opposite direction, assume that $\varphi$ is satisfiable. Then there is a choice set $W$ for $\{X_i\}_{1 \leq i \leq m}$ such that $x' = \bar{x}$ for no $x, x' \in W$. Obviously $V = W \cup \{y_1, \dots, y_{m-1}\} \cup \{d\}$ does not contain any cycles and thus is transitive with maximal element $d$. In order to obtain a larger transitive set with a different maximal element, we need to add $c_i$ for some $1 \leq i \leq m$ to $V$. However, $V \cup \{c_i\}$ always contains a cycle consisting of $c_i$, $d$, and $u$ for some $u \in U_i$, contradicting the transitivity of $V \cup \{c_i\}$. We have thus shown that $d$ is the maximal element of some maximal transitive set in $T_\varphi^{BA}$ containing $V$ as a subset. $\square$

# 4 NP-hardness of Membership in TEQ

In this section we prove that the problem of deciding whether a particular alternative is in the TEQ of a tournament is NP-hard. To this end, we refine the construction that was used in the previous section to prove NP-completeness of membership in the Banks set.

**Definition 5 (TEQ construction)** *Let $\varphi$ be a 3CNF $(x_1^1 \vee x_1^2 \vee x_1^3) \wedge \dots \wedge (x_m^1 \vee x_m^2 \vee x_m^3)$. Further for each $1 \leq i < m$, let there be a set $Z_i = \{z_i^1, z_i^2, z_i^3\}$. Define $T_\varphi^{TEQ}$ as the tournament $(A, \succ)$ in $\mathcal{T}^*$ with $4n-3$ layers such that $A = C \cup U_1 \cup \dots \cup U_{4n-3}$ and for all $1 \leq i \leq m$,*

$$U_j = \begin{cases} X_i & \text{if } j = 4i-3, \\ Z_i & \text{if } j = 4i-1, \\ \{y_i\} & \text{otherwise.} \end{cases}$$

*As in the Banks construction, we let for all $x \in X_i$ and $x' \in X_j$ $(1 \leq i, j \leq m)$*

$$x \succ x' \quad \text{if both } j < i \text{ and } x' = \bar{x} \text{ or both } i < j \text{ and } x' \neq \bar{x}.$$

*Finally, for all $1 \leq i, j \leq m$, $x_i^k \in X_i$ and $z_j^l \in Z_j$,*

$$x_i^k \succ z_j^l \text{ if and only if } i < j \text{ or both } i = j \text{ and } k = l.$$

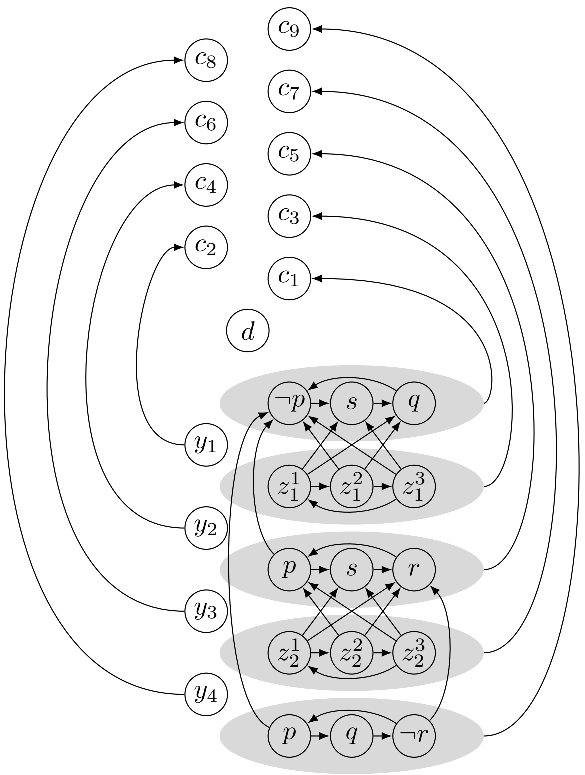


Figure 3: Tournament $T_\varphi^{TEQ}$ for the *3CNF* formula $\varphi = (\neg p \vee s \vee q) \wedge (p \vee s \vee r) \wedge (p \vee q \vee \neg r)$.

An example for such a tournament is shown in Figure 3.

We now proceed to show that a *3SAT* formula $\varphi$ is satisfiable if and only if the decision node $d$ is in the tournament equilibrium set of $T_\varphi^{TEQ}$. We make use of the following lemma.

**Lemma 1** *Let $T = (C \cup U, \succ)$ be a tournament in $\mathcal{T}^*$ and let $B \subseteq C \cup U$ such that $d \in B$. Then, for each $u \in U \cap B$ there exists some $c \in C \cap B$ such that $c \longrightarrow_B^* u$.*

*Proof:* Let $c_i \in C \cap B$ be such that $\overline{D}_B(c_i) \cap C = \emptyset$, *i.e.*, $c_i$ is the alternative in $C$ with the highest index among those included in $B$. Then,

$$c_i \longrightarrow_B c \text{ for all } c \in B \cap C \text{ with } c \neq c_i. \tag{1}$$

For this, merely observe that by construction $c_i$ is the Condorcet winner in $\overline{D}_B(c)$. Hence, $c_i \in \mathit{TEQ}(\overline{D}_B(c))$ and $c_i \longrightarrow_B c$.

The lemma itself then follows from the stronger claim that for each $u \in U \cap B$ there is some $c \in C \cap B$ with both $c \longrightarrow_B^* u$ and $c \in \mathit{TEQ}(B)$. This claim we prove by structural induction on supersets $B$ of $\{d\}$.

If $B = \{d\}$, $U \cap B = \emptyset$ and the claim is satisfied trivially. So let $\{d\}$ be a proper subset of $B$. Again, if $U \cap B = \emptyset$, the claim holds trivially. So we may assume there be some $u \in U \cap B$. Then, $d \in \overline{D}_B(u)$ by construction of $T$. If $\overline{D}_B(u) \cap U = \emptyset$, $\overline{D}_B(u)$ is a non-empty subset of $C \cap B$, and so is $\mathit{TEQ}(\overline{D}_B(u))$. It follows that for some $c \in \mathit{TEQ}(\overline{D}_B(u)) \cap C$ we have $c \longrightarrow_B u$. If, on the other hand, $\overline{D}_B(u) \cap U \neq \emptyset$, the induction hypothesis is applicable and we have $c \in \mathit{TEQ}(\overline{D}_B(u))$ for some $c \in C \cap B$. Hence, $c \longrightarrow_B u$. With $u$ having been chosen arbitrarily, we actually have that for all $u \in U \cap B$, there is some $c \in C \cap B$ with $c \rightarrow_B u$. It remains to be shown that there is some $c \in C \cap \mathit{TEQ}(B)$ with $c \rightarrow_B^* u$.

To this end, again consider $c_i \in C \cap B$ such that $\overline{D}_B(c_i) \cap C = \emptyset$. It suffices to show that $c_i \rightarrow_B^* b$ for all $b \in B$, as then both $c_i \in \mathit{TEQ}(B) \cap C$ and $c_i \rightarrow_B^* u$. So, consider an arbitrary $b \in B$. If $b = c_i$, the case is trivial. If $b \in C \cap B$ but $b \neq c_i$, we are done by (1). If instead $b \in U \cap B$, then $c \rightarrow_B^* b$ for some $c \in C \cap B$, as we have shown in the first part of the proof. If $c = c_i$, we are done. Otherwise, we can apply (1) to obtain $c_i \rightarrow_B c' \rightarrow_B^* b$ and hence $c_i \rightarrow_B^* b$. $\square$

We are now ready to state the main theorem of this paper.

**Theorem 2** *Deciding whether a particular alternative is in the tournament equilibrium set of a tournament is NP-hard.*

*Proof:* By reduction from *3SAT*. Consider an arbitrary *3CNF* $\varphi$ and construct the tournament $T_\varphi^{\mathit{TEQ}} = (C \cup U, \succ)$. This can be done in polynomial time. We show that

$$\varphi \text{ is satisfiable if and only if } d \in \mathit{TEQ}(T_\varphi^{\mathit{TEQ}}).$$

For the direction from left to right, observe that by an argument analogous to the proof of Theorem 1 it can be shown that $\varphi$ is satisfiable if and only if $d \in \mathit{BA}(T_\varphi^{\mathit{TEQ}})$. So assuming that $\varphi$ is not satisfiable yields $d \notin \mathit{BA}(T_\varphi^{\mathit{TEQ}})$. By the inclusion of TEQ in the Banks set (Proposition 1), it follows that $d \notin \mathit{TEQ}(T_\varphi^{\mathit{TEQ}})$.

For the opposite direction, assume that $\varphi$ is satisfiable. Then there is a choice set $W$ for $\{X_i\}_{1 \leq i \leq m}$ such that $x' = \bar{x}$ for no $x, x' \in W$. Obviously $W \cup \{y_1, \ldots, y_{m-1}\} \cup \{z_i^j \in Z \colon x_i^j \in W\} = \{u_1, \ldots, u_n\}$ contains no cycles and thus is transitive. Without loss of generality we may assume that $u_i \in U_i$ for all $1 \leq i \leq n$. For each $1 \leq i \leq n+1$, define a subset $\overline{D}_i$ of alternatives as follows. Set $\overline{D}_{n+1} = A$ and $\overline{D}_i = \bigcap_{i \leq j \leq n+1} \overline{D}(u_j)$ for each $1 \leq i \leq n$. Hence, $\overline{D}_1 \subsetneq \cdots \subsetneq \overline{D}_{n+1}$. In an effort to simplify notation, we write $\rightarrow_i$ and $\overline{D}_i(x)$ for $\rightarrow_{\overline{D}_i}$ and $\overline{D}_{\overline{D}_i}(x)$, respectively. It then suffices to prove that

$$d \in \mathit{TEQ}(\overline{D}_k), \text{ for all } 1 \leq k \leq n+1. \tag{2}$$

The theorem then follows as the special case in which $k = n+1$. We first make the following observations concerning the TEQ relation $\rightarrow_i$ in each $\overline{D}_i$, for each $1 \leq i, j \leq n+1$:

(*i*) $u_j \in \overline{D}_i$ if and only if $j < i$,

(*ii*) $c_j \in \overline{D}_i$ if and only if $j < i$,

(*iii*) $c_i \rightarrow_{i+1} c_j$ if $j < i \leq n$,

(*iv*) $u_i \rightarrow_{i+1} c_i$, if $i \leq n$.

For (*i*), observe that if $j < i$, $u_j \in \overline{D}(u_i)$ by transitivity of the set $\{u_1, \ldots, u_n\}$. Hence, $u_j \in \overline{D}_i$. If on the other hand $j \geq i$, then $u_j \notin \overline{D}(u_j)$ and thus $u_j \notin \overline{D}_i$. For (*ii*), observe that $c_j \in \overline{D}(u_i)$ for all $i \neq j$ and thus $c_j \in \overline{D}_i$ if $j < i$. However, $c_j \notin \overline{D}(u_j)$ and hence $c_j \notin \overline{D}_i$ if $j \geq i$. For (*iii*), merely observe that $c_i$ is the Condorcet winner in $\overline{D}_{i+1}(c_j)$, if $j < i \leq n$. To appreciate (*iv*), observe that by construction $\overline{D}_{i+1}(c_i)$ has to be either a singleton $\{u_i\}$ for some $u_i \in U_i$, or $U_i$ itself. The former is the case if $U_i \subseteq Y$, or if $U_i \subseteq X$ and $i \neq n$. The latter holds if $U_i = U_n$ or if $U_i \subseteq Z$. In either case, $\mathit{TEQ}(\overline{D}_{i+1}(c_i)) = \overline{D}_{i+1}(c_i)$ and $u_i \rightarrow_{i+1} c_i$ holds. For the case in which $U_i \subseteq X$ with $i \neq n$, let $U_i = \{u_i, u_i', u_i''\}$. By construction, $U_{i+2} \subseteq Z$ and $u_i', u_i'' \notin \overline{D}(u_{i+2})$. Accordingly, $u_i', u_i'' \notin \overline{D}_{i+1}$. From $u_i \in \overline{D}_{i+1}$ it then follows that $\overline{D}_{i+1} \cap U_i = \{u_i\}$.

**Algorithm 1** Tournament Equilibrium Set

**procedure** $\texttt{TEQ}(X)$
$R \leftarrow \emptyset$
$B \leftarrow C \leftarrow \arg\min_{a\in X} |\overline{D}(a)|$
**loop**
$R \leftarrow R \cup \{(b,a) \colon a \in C \wedge b \in \texttt{TEQ}(\overline{D}(a))\}$
$D \leftarrow \bigcup_{a\in C} \texttt{TEQ}(\overline{D}(a))$
**if** $D \subseteq B$ **then return** $TC_B(R)$ **end if**
$C \leftarrow D$
$B \leftarrow B \cup C$
**end loop**

We are now in a position to prove (2) by induction on $k$. For $k = 1$, observe that $d$ is a Condorcet winner in $\overline{D}_1$ and thus $d \in \mathit{TEQ}(\overline{D}_1)$. For the induction step, let $k = i + 1$. With observation (*i*) we know that $u_i \in \overline{D}_{i+1}$ and, in virtue of the induction hypothesis, also that $d \in \mathit{TEQ}(\overline{D}_i)$. Hence, $d \rightarrow_{i+1} u_i$. Moreover, by observations (*iii*) and (*iv*), $c_i \rightarrow_{i+1} d \rightarrow_{i+1} u_i \rightarrow_{i+1} c_i$, *i.e.*, $c_i$, $d$ and $u_i$ constitute a $\rightarrow_{i+1}$-cycle. In virtue of Lemma 1 and observation (*ii*), we may conclude that $c_i \rightarrow^*_{i+1} a$ for all $a \in \overline{D}_{i+1}$. Accordingly, $\{c_i, d, u_i\} \subseteq TC_{\overline{D}_{i+1}}(\rightarrow_{i+1})$ and $d \in \mathit{TEQ}(\overline{D}_{i+1})$, which concludes the proof. □

# 5 A Heuristic for Computing TEQ

Computational intractability of the TEQ *membership* problem implies that TEQ cannot be computed efficiently either. Nevertheless, the running time of the naive algorithm, which straightforwardly implements the recursive definition of TEQ, can be greatly reduced when assuming that TEQ satisfies CTC. This assumption can fairly be made. For if CTC were not to hold, TEQ would be non-monotonic and thus compromised as a solution concept, the issue of computing it moot.

Algorithm 1 computes TEQ by starting with the set $B$ of all alternatives that have dominator sets of minimal size (*i.e.*, the so-called Copeland winners). These alternatives are good candidates to be included in TEQ and the small size of their dominator sets speeds up the computation of their TEQ-dominators. Then, all alternatives that TEQ-dominate any alternative in $B$ are iteratively added to $B$ until no more such alternatives can be found, in which case the algorithm returns the top cycle of $\rightarrow_B$. Of course, the *worst-case* running time of this algorithm is still exponential, but experimental results suggest that it outperforms the naive algorithm by a factor of about five in uniform random tournaments with up to 150 vertices (see Table 1). We implemented two versions of the naive algorithm, which differ in the subroutine that determines the top cycle. The first one uses the Floyd-Warshall algorithm with an asymptotic complexity of $O(n^3)$, whereas the second one employs Kosaraju's algorithm with a complexity of $O(n^2)$ (see, *e.g.*, Cormen et al., 2001). Surprisingly, the variant relying on Floyd-Warshall performs slightly better on moderately sized instances due to factors hidden in the asymptotic notation that are amplified as a consequence of TEQ's recursive definition.

While choosing tournaments uniformly at random might be useful for benchmarking algorithms, it raises a number of conceptual problems. First, in voting and most other applications uniform random tournaments do not represent a reasonably realistic model of social preferences. Secondly, these tournaments are "almost" regular and tournament solutions almost always select all alternatives in regular tournaments. One model of random tournaments that have more structure can be obtained by defining an arbitrary linear order

| \|A\| | Floyd-Warshall | Kosaraju | Algorithm 1 |
|---|---|---|---|
| Uniform random tournaments ($p = 0.5$) | | | |
| 50 | 0.48 s | 0.59 s | 0.09 s |
| 100 | 53.33 s | 65.73 s | 9.57 s |
| 150 | 1 166 s | 1 429 s | 210 s |
| Structured random tournaments ($p = 0.8$) | | | |
| 50 | 13.87 s | 16.56 s | 0.01 s |
| 100 | 18 416 s | 21 382 s | 8.46 s |
| 150 | — | — | 1273 s |

Table 1: Experimental evaluation of algorithms that compute TEQ. Average running time for ten instances on a 3.2GHz Core2Duo machine. Both versions of the naive algorithm did not terminate within 24 hours when run on structured random tournaments with 150 vertices.

on the alternatives $a_1, \ldots, a_m$ and letting $a_i \succ a_j$ for $i < j$ with probability $p > 0.5$. Letting $p = 1$ yields a "completely structured" transitive tournament. The more structure a tournament possesses, the more Algorithm 1 outperforms the naive algorithm, due to the increasing number of large dominator sets that have to be analyzed by the latter at every level of the recursion. In large structured tournaments, the performance gap becomes rather impressive (see Table 1). For example, the naive algorithm requires more than five hours to compute the TEQ of a structured random tournament with 100 vertices whereas it takes Algorithm 1 about eight seconds.[1]

We have further used the naive algorithm to try to disprove CTC (and thus TEQ's monotonicity), but failed to find a counterexample by an exhaustive search in all tournaments with up to ten vertices (roughly ten million non-isomorphic tournaments), all regular tournaments with up to 13 vertices, and all locally transitive tournaments with up to 20 vertices. We also investigated a fairly large number of uniform and structured random tournaments, again to no avail. This can be considered mild evidence that TEQ is indeed monotonic.

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## References

N. Alon. Ranking tournaments. *SIAM Journal of Discrete Mathematics*, 20(1):137–142, 2006.

D. Bouyssou, T. Marchant, M. Pirlot, A. Tsoukiàs, and P. Vincke. *Evaluation and Decision Models: Stepping Stones for the Analyst.* Springer-Verlag, 2006.

F. Brandt and F. Fischer. Computing the minimal covering set. *Mathematical Social Sciences*, 2008. To Appear.

[1] We also had some limited success with algorithms that make use of the easy-to-prove fact that $\mathit{TEQ}((A, \succ)) = \mathit{TEQ}(\mathit{TC}(\succ))$. Assuming CTC, a similar preprocessing step that first computes the minimal covering set of the tournament at hand is possible.

F. Brandt and P. Harrenstein. Dominance in social choice and coalitional game theory. In G. Bonanno, B. Löwe, and W. van der Hoek, editors, *Proceedings of the 8th Conference on Logic and the Foundations of Game and Decision Theory (LOFT)*, 2008.

F. Brandt, F. Fischer, and P. Harrenstein. The computational complexity of choice sets. In D. Samet, editor, *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK)*, pages 82–91. ACM Press, 2007.

V. Conitzer. Computing Slater rankings using similarities among candidates. In *Proceedings of the 21st National Conference on Artificial Intelligence (AAAI)*, pages 613–619. AAAI Press, 2006.

T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms*. MIT Press, 2nd edition, 2001.

Marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilité de décisions rendues à la pluralité de voix*. Imprimerie Royal, 1785. Facsimile published in 1972 by Chelsea Publishing Company, New York.

J. Duggan and M. Le Breton. Dutta's minimal covering set and Shapley's saddles. *Journal of Economic Theory*, 70:257–265, 1996.

P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence*, 77:321–357, 1995.

P. E. Dunne. Computational properties of argumentation systems satisfying graph-theoretic constraints. *Artificial Intelligence*, 171(10-15):701–729, 2007.

B. Dutta. On the tournament equilibrium set. *Social Choice and Welfare*, 7(4):381–383, 1990.

B. Dutta and J.-F. Laslier. Comparison functions and choice correspondences. *Social Choice and Welfare*, 16(4):513–532, 1999.

P. C. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

D. B. Gillies. Solutions to general non-zero-sum games. In A. W. Tucker and R. D. Luce, editors, *Contributions to the Theory of Games IV*, volume 40 of *Annals of Mathematics Studies*, pages 47–85. Princeton University Press, 1959.

G. Laffond, J.-F. Laslier, and M. Le Breton. More on the tournament equilibrium set. *Mathématiques et sciences humaines*, 31(123):37–44, 1993.

J.-F. Laslier. *Tournament Solutions and Majority Voting*. Springer, 1997.

C. H. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.

T. Schwartz. Cyclic tournaments and cooperative majority voting: A solution. *Social Choice and Welfare*, 7:19–29, 1990.

G. J. Woeginger. Banks winners in tournaments are difficult to recognize. *Social Choice and Welfare*, 20:523–528, 2003.

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