Informational requirements of social choice rules

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Abstract

The needed amount of information to make a social choice is the cost of information processing, and it is a practically important feature of social choice rules. We introduce informational aspects into the analysis of social choice rules and prove that (i) if an anonymous, neutral, and monotonic social choice rule operates on minimal informational requirements, then it is a supercorrespondence of either the plurality rule or the antiplurality rule, and (ii) if the social choice rule is furthermore Pareto efficient, then it is a supercorrespondence of the plurality rule.

Keywords: antiplurality rule, minimal informational requirement, plurality rule, social choice rule.

1 Introduction

Each social choice rule utilizes information on the agents' preferences at different levels. For example, it is intuitively clear that dictatorship needs much less information than the Borda rule; under dictatorship, we need to know only the most preferred alternative of a dictator, while under the Borda rule, we need to know the whole preferences of all agents. Without some electronic device (this is the case in most situations where collective choice is to be made)[2], processing a large amount of information is not an easy task. The required amount of information can be considered as the cost of information processing; the larger the amount of information to process, the more time and human resources are needed and the more risk of making errors is involved.

Therefore, the informational requirement is a practically important feature of each social choice rule. That is, when the information processing cost is high, informational requirements should be one of the most important criteria of social choice rules in evaluating them. Therefore, in this paper, we incorporate the informational aspects into social choice. The fundamental problem we are to deal with is the following, "Given a group of social choice rules satisfying some "reasonable" properties, which of them operates on the smallest amount of information?" In other words, we incorporate minimal informational requirements into the axiomatic analysis of social choice rules.

Our main results are (i) if an anonymous, neutral, and monotonic social choice rule operates on minimal informational requirements, then it utilizes only information about either the top ranked alternatives or the bottom ranked alternatives by the agents and it is a supercorrespondence of either the plurality rule or the antiplurality rule, and (ii) if the social choice rule is furthermore efficient, then it utilizes only information about the top ranked alternatives by the agents and it is a supercorrespondence of the plurality rule. Thus, the plurality rule and the antiplurality rule are characterized as the most selective social choice rules among anonymous, neutral, and monotonic social choice rules which operate on minimal informational requirements, and the plurality rule can be characterized as the most selective social choice rule among anonymous, neutral, monotonic, and efficient social choice rules which operate on minimal informational requirements.

This last result is the easiest one to interpret. The plurality rule is widely used in our daily lives, and many people would agree that, compared with other "reasonable" social choice rules, the main

[1] I am grateful to three anonymous referees of this conference for helpful comments, especially, for letting me notice the literature on communication complexity.

[2] In some area of the world, electronic voting systems are adopted in some "big" elections. However, it is very unlikely that all social choices ranging from national elections to the choice of restaurant for a dinner are made with a electronic device, at least in the near future. Major obstacles for electronic voting systems are the cost of introducing the system and the reliability of hardware and software. Actually, in Japan, the result of the election in Kani city in 2003 was cancelled due to a hardware problem, and in Aki ward of Hiroshima city, electronic voting is abandoned in 2006 due to the financial constraint.

advantage of the plurality rule lies in its simplicity and selectivity (i.e., the set of "winners" is small). Our last result theoretically supports this common sense.

Let us mention some related literature. Conitzer and Sandholm (2005) present *communication complexities* of eleven major voting rules.[3] (See Kushilevitz and Nisan (1997) for a survey on the literature on communication complexity. A seminal work is Yao (1979).) In their model, each agent sends a bit of his private information necessary to make a social choice to the others sequentially. That is, the agents "communicate" to compute the value of a voting rule. Communication complexity of a voting rule is defined as the worst-case number of bits in the best protocol to compute the value of the voting rule. Communication complexity can be considered as a kind of informational size of a voting rule. Among many differences, the most significant and essential one between my approach and Kushilevitz and Nisan (1997) is that I introduce a minimal informational requirement as an "axiom" and hence measuring the informational size of some specific social choice rules is not my objective while it is in Kushilevitz and Nisan (1997).

Many social choice rules are proposed and axiomatically characterized in the theory of social choice.[4] Being prevalent in the real world, the plurality rule is axiomatically characterized by Richelson (1978); Roberts (1991); Ching (1996); Yeh (2008), among others. Our contribution to this literature is to characterize the plurality rule (and the antiplurality rule) based on minimal informational requirements and selectivity.

Some researchers consider social choice rules which rely on limited information on preferences. (For example, Moulin (1980); Roberts (1991); Yeh (2008), among others.) However, in their analyses, such restrictions are put as assumptions and do not intend to study the amount of necessary information to make a social choice under each social choice rule.

In sum, analyses in this paper such as investigation of the minimal informational size needed to be a "reasonable" social choice rule and characterizations based on minimal informational requirements seem to be novel in the literature, and would give useful insights in the evaluation of social choice rules.

In Section 2, we give basic notation and definitions. In Section 3, a series of results are presented. Proofs are collected in Section 4.

2 Basic notation and definitions

Let $N = \{1, \dots, n\}$ be a finite set of agents and let X be a finite set of alternatives with $|X| = m \geq 2$. Let $\mathcal{L}$ denote the set of all linear orders (complete, transitive, and antisymmetric binary relations) on X. An element $R_N = (R_1, \dots, R_n)$ of $\mathcal{L}^N$ is called a preference profile. A linear order R_i in a preference profile R_N is agent i's preference, and P_i is the strict part of R_i. For each preference $R \in \mathcal{L}$ and for each integer k with $1 \leq k \leq m$, let $r_k(R)$ denote the kth ranked alternative with respect to R. For each $i \in N$, a function φ_i of $\mathcal{L}$ onto a finite set $\mathcal{K}_i$ is called a *message function* and a set $\mathcal{K}_i$ is called a *message space*. A triple $(\varphi_N, \mathcal{K}_N, f)$ is called a *rule*, where φ_N is a profile of message functions $(\varphi_1, \dots, \varphi_n)$, $\mathcal{K}_N$ is the Cartesian product of message spaces $\mathcal{K}_i$, and f is a correspondence of $\mathcal{K}_N$ into X. When the agents have a preference profile R_N, then they report a message profile $\varphi_N(R_N) = (\varphi_1(R_1), \dots, \varphi_n(R_n)) \in \mathcal{K}_N$ and f makes a choice based on the received message $\varphi_N(R_N)$.

For our purpose, the labels or the names of messages are inessential and we restrict the form of message spaces (and message functions) to a specific form without loss of generality. This can be done as follows; let $(\varphi_N, \mathcal{K}_N, f)$ be a rule with a general form.

[3] More precisely, Conitzer and Sandholm (2005) present the asymptotic lower and upper bounds of communication complexities of voting rules. (For example, the plurality rule belongs to $\Theta(n \log_2 m)$.) This is a standard way to measure efficiency of an algorithm in computer science.

[4] See Sen (1986) and Moulin (1988), among others, for surveys of the literature.

- (Message spaces) For each $i \in N$, we can define the partition $\mathcal{M}_i$ of $\mathcal{L}$ induced by φ_i^{-1}. Formally, $\mathcal{M}_i = \{\varphi_i^{-1}(k_i) \mid k_i \in \mathcal{K}_i\}$. Then, we can regard this $\mathcal{M}_i$ as a message space equivalent to $\mathcal{K}_i$ in the sense that there exists a natural bijection between $\mathcal{K}_i$ and $\mathcal{M}_i$; let τ_i be the bijection between $\mathcal{K}_i$ and $\mathcal{M}_i$ defined by $\tau_i(k_i) = \varphi_i^{-1}(k_i)$.
- (Message functions) For each $i \in N$ and for each $R_i \in \mathcal{L}$, let $\varphi'_i(R_i)$ be the element of $\mathcal{M}_i$ such that $R_i \in \varphi'_i(R_i)$. Note that if $\varphi_i(R_i) = k$, then $\tau_i(k) = \varphi'_i(R_i)$. Thus, under φ'_i, agent i reports $\varphi'_i(R_i)$, which is a message corresponding to $\varphi(R_i)$. Formally, $\varphi'_i = \tau_i \circ \varphi_i$.
- (Social choice rule) For each $M_N \in \prod_{i \in N} \mathcal{M}_i = \mathcal{M}_N$, let $f'(M_N)$ be $f(k_N)$, where $k_N \in \mathcal{K}_N$ is the message profile corresponding to M_N in the sense that $\tau_N(k_N) = (\tau_1(k_1), \ldots, \tau_n(k_n)) = M_N$. Formally, $f' = f \circ \tau_N^{-1}$.

Now, we have a new rule $(\varphi'_N, \mathcal{M}_N, f')$ which is equivalent to $(\varphi_N, \mathcal{K}_N, f)$ in the sense that the only difference is the labels or the names of messages. In $(\varphi_N, \mathcal{K}_N, f)$, agent i reports $\varphi_i(R_i)$. When we just relabel this message $\varphi_i(R_i)$ as $\tau_i(\varphi_i(R_i))$, then we have a rule $(\varphi'_N, \mathcal{M}_N, f')$.

Thus, without loss of generality, we can restrict our attention to the rules such that message spaces are partitions of $\mathcal{L}$ and message functions assign each preference the set in the partition to which that preference belongs. In the following, unless otherwise stated, we assume that every rule takes this restricted form.

In the restricted form of rules, a profile of message functions φ_N is uniquely determined by a profile $\mathcal{M}_N$ of message spaces (partitions of $\mathcal{L}$). Thus, in the following, we drop the message functions and write $(\mathcal{M}_N, f)$ for a rule. Given a rule $(\mathcal{M}_N, f)$, when we speak of φ_N, then it should be always understood to be the profile of message functions such that $\varphi_i(R_i) = M_i \in \mathcal{M}_i$ with $R_i \in M_i$. In sum, given a rule $(\mathcal{M}_N, f)$, agents are required to report a profile of sets of linear orders $M_N \in \mathcal{M}_N$ such that the profile of their preferences R_N belongs to M_N, and f makes a choice based on M_N.

It is worth noting that a profile of message spaces $\mathcal{M}_N$ (and hence a profile of message functions φ_N) as well as f is set by the social choice rule designer, and not the variable determined by the agents. We introduce message spaces to clarify what information a social decision requires and to define the informational size of each social choice rule.

Next, we define the informational size of a rule, which is a core concept of this paper.

Definition 2.1 For each rule $(\mathcal{M}_N, f)$, the sum of the numbers of possible messages $\sum_{i \in N} |\mathcal{M}_i|$ is called the *informational size of* $(\mathcal{M}_N, f)$.

Definition 2.2 (The plurality rule) The plurality rule chooses the alternatives ranked as the top by the largest number of agents. In our model, this rule can be written as follows. For each $x \in X$, let $M(x) = \{R \in \mathcal{L} \mid r_1(R) = x\}$. (The set of preferences which rank x at the top.) For each $i \in N$, let $\mathcal{M}_i^p = \{M(x) \mid x \in X\}$. Then, $\mathcal{M}_i^p$ is a partition of $\mathcal{L}$. For each message profile $M_N \in \prod_{i \in N} \mathcal{M}_i^p = \mathcal{M}_N^p$ and for each $x \in X$, let $N_x(M_N) = |\{i \in N \mid M_i = M(x)\}|$. (The number of agents whose message is $M(x)$.) Finally, for each message profile M_N, let $f^p(M_N) = \{x \in X \mid N_x(M_N) \geq N_y(M_N) \ \forall y \in X\}$. Then, $(\mathcal{M}_N^p, f^p)$ is called the *plurality rule*. Its informational size is nm. (Remember that n is the number of the agents and m is the number of alternatives.)

In the plurality rule, f^p makes choice based on information contained in a message profile M_N in $\mathcal{M}_N^p$. From the viewpoint of f^p, it is known that the agent i's preference is in a reported message M_i, but it is not known which is the agent i's preference in M_i. However, the plurality rule can be defined based on this restricted information, because each $M_i \in \mathcal{M}_i^p$ tells what is the alternative ranked as the top.

Thus, in our model, by introducing message spaces between a choice rule and preferences, we can measure the amount of needed information to make a social choice.

Definition 2.3 (The antiplurality rule) The antiplurality rule chooses the alternatives ranked as the bottom by the smallest number of agents. For each $x \in X$, let $M(x) = \{R \in \mathcal{L} \mid r_m(R) = x\}$. (The set of preferences which rank x at the bottom.) For each $i \in N$, let $\mathcal{M}_i^a = \{M(x) \mid x \in X\}$. Then, $\mathcal{M}_i^a$ is a partition of $\mathcal{L}$. For each message profile $M_N \in \prod_{i \in N} \mathcal{M}_i^a = \mathcal{M}_N^a$ and for each $x \in X$, let $N_x(M_N) = |\{i \in N \mid M_i = M(x)\}|$. (The number of agents whose message is $M(x)$.) Finally, for each message profile M_N, let $f^a(M_N) = \{x \in X \mid N_x(M_N) \leq N_y(M_N) \ \forall y \in X\}$. Then, $(\mathcal{M}_N^a, f^a)$ is called the *antiplurality rule*. Its informational size is nm.

At this point, several remarks are in order. First, the reader would notice that to define the informational size and to describe the procedure of making a social choice, it suffices to consider a correspondence which assigns a social outcome to each message profile. For example, in defining the plurality rule, we could define g^p as a correspondence of X^N to X such that for each message profile $x_N = (x_1, \ldots, x_n) \in X^N$, $g^p(x_N)$ is the set of alternatives which are reported by the largest number of agents. If we defined this g^p as the plurality rule, then there would be no agents' "preferences" in the model. The reason to incorporate preferences into our model is that our objective is to find the rules which operate on the minimal information requirements among the rules satisfying some plausible properties such as (weak) monotonicity and efficiency, and these properties refer to agents' preferences. (If our objective were to find the social choice rule which operates on minimal informational requirements without any restriction, then the answer would be constant social choice rules, or "custom", which needs no information to make a social choice.)

Next, although we call φ_i (derived from $\mathcal{M}_i$) a message function and use the word "report", we do not need to interpret them literally. The only role of φ_i is to specify what kind of information is necessary to make a social choice. Thus, we could consider the following model; agents report a preference profile R_N and the central institution which is responsible to make a social decision would take two steps to make a decision. At the first stage, pick up necessary information according to φ_N from R_N, and at the second stage, process information $\varphi_N(R_N) = M_N \in \mathcal{M}_N$ and make a social decision.

Thirdly, we model f as a correspondence and not a function. There are two reasons for this. First, we do not exclude the cases where the society is to choose a set of "satisfactory" alternatives (not necessarily the "best" alternatives). In this case, the social outcome is naturally formulated as sets of alternatives. Second, even when the society is to choose the "best" alternatives, almost all practically important rules such as the plurality rule, the Borda rule, the Copeland rule, and the Simpson rule (See Moulin, 1988), are formulated as correspondences. When we ultimately need to choose a single outcome whereas f can choose multiple alternatives, then it is done by some tie-breaking rule, but this is outside the scope of our analysis.

We define several properties of a rule.

Definition 2.4 A rule $(\mathcal{M}_N, f)$ is said to satisfy

- *anonymity* if for every permutation σ of N and for every $R_N \in \mathcal{L}^N$, $(f \circ \varphi_N)(R_N) = (f \circ \varphi_N)(R_N^\sigma)$, where R_N^σ is defined by for each $i \in N$, $R_i^\sigma = R_{\sigma(i)}$.
- *neutrality* if for every permutation ρ of X and for every $R_N \in \mathcal{L}^N$, $\rho[(f \circ \varphi_N)(R_N)] = (f \circ \varphi_N)[\rho(R_N)]$, where $\rho(R_N) = (\rho(R_1), \ldots, \rho(R_N))$ is defined by for each $i \in N$, $\rho(R_i) = \{(x, y) \in X^2 \mid (\rho^{-1}(x), \rho^{-1}(y)) \in R_i\}$.
- *(weak) monotonicity* if for any R_N and R'_N such that $x \in f(\varphi_N(R_N))$, R_N and R'_N coincide on $(X \setminus \{x\})^2$ and $x = r_k(R_i) = r_{k'}(R'_i)$ with $k' \leq k$ for all $i \in N$, we have $x \in f(\varphi_N(R'_N))$.
- *(Pareto) efficiency* if for any distinct $x, y \in X$ with xP_iy for all $i \in N$, $y \notin f(\varphi_N(R_N))$.

Anonymity requires symmetric treatment of the agents and neutrality requires symmetric treatment of alternatives. Monotonicity requires that when x is chosen at R_N and the position of x (weakly) improves through the change from R_N to R'_N while the relative comparison of any other pair of alternatives is unchanged, then x is still chosen at R'_N. In the literature, it is often called weak monotonicity to distinguish from the so called Maskin monotonicity which does not appear in this paper. Note that monotonicity (in the sense of this paper) is much weaker than the Maskin monotonicity because the relative rankings except x are fixed from the change from R_N to R'_N. Efficiency requires that when an alternative y is dominated by some alternative x, then y cannot belong to the social outcome. Although efficiency is one of the most standard axioms in social choice theory and in economic theory, its relevance depends on the context under consideration. For example, in this paper, as mentioned earlier, we do not exclude cases where the society is to choose a set of "satisfactory" alternatives. In such a case, the fact that y is dominated by x does not imply that y is not satisfactory, and hence y can belong to the social outcome. Based on this observation, we give results with and without efficiency in the next section.

Next, we define formally the minimality of informational requirements.

Definition 2.5 Given a set of rules $\mathcal{F}$, a rule $(\mathcal{M}_N, f)$ is said to *operate on minimal informational requirements in* $\mathcal{F}$ if the informational size of $(\mathcal{M}_N, f)$ is not larger than the informational size of any other rules in $\mathcal{F}$. In this case, the informational size of $(\mathcal{M}_N, f)$ is called the *minimal informational size in* $\mathcal{F}$.

3 Results

In this section, we give a series of results. Let $\mathcal{AN}$ denote the set of nonconstant[5] rules satisfying anonymity and neutrality, let $\mathcal{ANM}$ denote the set of nonconstant rules satisfying anonymity, neutrality, and monotonicity, and let $\mathcal{ANMP}$ denote the set of rules satisfying anonymity, neutrality, monotonicity, and efficiency. Throughout this section, assume $m \geq 2$. (When $m = 1$, then there is no room for "choice".)

Theorem 3.1 *If a rule $(\mathcal{M}_N, f)$ operates on minimal informational requirements in $\mathcal{AN}$, then*

(i) its informational size is nm, and more specifically,

(ii) there exists $h \in \{1, \dots, m\}$ such that for any $i \in N$, $\mathcal{M}_i = \{M_i(x) \mid x \in X\}$, where $M_i(x) = \{R_i \in \mathcal{L} \mid r_h(R_i) = x\}$.

The second statement of the theorem implies that we can associate each message with one alternative in X and that this relation is a bijection. Moreover, the statement explicitly specifies what information a rule $(\mathcal{M}_N, f)$ depends on; it relies on information what are the hth ranked alternatives in R_N. Consider that agent i with a preference R_i changes his preference to R'_i. Then, agent i sends the same message iff $r_h(R_i) = r_h(R'_i)$.

For example, the plurality rule and the antiplurality rule operate on minimal informational requirements in $\mathcal{AN}$ with $h = 1$ and $h = m$, respectively. Also, the rule $(\mathcal{M}_N, f)$ such that each $\mathcal{M}_i$ is the one defined in the second statement of the theorem with $h = 2$ and f chooses the alternatives second ranked by the largest number of agents also operates on minimal informational requirements in $\mathcal{AN}$.

Before the next theorem, we prepare the following terminology.

Definition 3.1 A rule $(\mathcal{M}_N, f)$ is said to be a *supercorrespondence* of a rule $(\mathcal{M}'_N, f')$ is for every preference profile R_N, $f(\varphi_N(R_N)) \supset f'(\varphi'_N(R_N))$ holds.

[5]A rule $(\mathcal{M}_N, f)$ is said to be *nonconstant* if the correspondence $f \circ \varphi_N$ is nonconstant on $\mathcal{L}^N$.

When a rule $(\mathcal{M}_N, f)$ is a supercorrespondence of a rule $(\mathcal{M}'_N, f')$, then, a rule $(\mathcal{M}_N, f)$ is less selective than $(\mathcal{M}'_N, f')$. Of course, selectivity is not always a plausible axiom. For example, when you want to chooses a set of *satisfactory* (and not necessarily the best) alternatives, then selectivity is not an appealing condition for rules. However, in many cases, we want to choose the socially best alternatives, and in such situations, we usually do not want to rely on a tie-breaking rule (usually, some random device) as much as possible. We want to determine a final social outcome by *preferences* as much as possible. For instance, in elections where we want to choose one winner, it is absurd to use a rule $(\mathcal{M}_N, f)$ such that each voter reports his most preferred candidate and f chooses the candidates who receive at least one vote. (The final outcome is determined by some random device, which is outside our model.)

Theorem 3.2 *If a rule $(\mathcal{M}_N, f)$ operates on minimal informational requirements in $\mathcal{ANM}$, then*

(i) h in Theorem 3.1 is either 1 or m, and

(ii) If $h = 1$, then $(\mathcal{M}_N, f)$ is a supercorrespondence of the plurality rule and if $h = m$, then $(\mathcal{M}_N, f)$ is a supercorrespondence of the antiplurality rule.

This theorem shows that if monotonicity is additionally required, then necessary information to make a social choice is either the top ranked alternatives or the bottom ranked alternatives by the agents. If the rule relies on information on the top ranked alternatives, then, the alternatives chosen by the plurality rule are contained in the value of the rule. If the rule relies on information on the bottom ranked alternatives, then the alternatives chosen by the antiplurality rule are contained in the value of the rule.

Because the antiplurality rule is not efficient[6] when $m \geq 3$ and it is equal to the plurality rule when $m = 2$, Theorem 3.2 readily implies the following theorem.

Theorem 3.3 *If a rule $(\mathcal{M}_N, f)$ operates on minimal informational requirements in $\mathcal{ANMP}$, then it is a supercorrespondence of the plurality rule.*

This theorem gives a new characterization of the plurality rule; it is the most selective rule among the rules operating on minimal informational requirements in $\mathcal{ANMP}$. When you want to choose the socially best alternatives, then it is natural to adopt a rule in $\mathcal{ANMP}$. Theorem 3.3 shows that if you care for the informational processing cost and selectivity, then the answer is the plurality rule.

We conclude this section with the following remark. We defined the informational size of $(\mathcal{M}_N, f)$ simply by $\sum_{i \in N} |\mathcal{M}_i|$. Our results do not depend on this specific way of defining the informational size. Let g be any strictly increasing function on the positive orthant of $\mathbb{R}^n$, the n-dimensional Euclidean space, and let us define $g(|\mathcal{M}_1|, \ldots, |\mathcal{M}_n|)$ to be the informational size of $(\mathcal{M}_N, f)$. Then, we can obtain the same results with this definition of the informational size.

4 Proofs

In this section, we introduce many permutations of N and X. For simplicity, when we describe a permutation, we do not specify the part on which the permutation is the identity function. For example, when we say that σ is the permutation of N exchanging i and j, then it should be understood that σ is the identity function on $N \setminus \{i, j\}$.

[6]For example, let $X = \{x, y, z\}$ and let R_N be a preference profile such that xR_iyR_iz for all $i \in N$. Then, the antiplurality rule chooses $\{x, y\}$ while y is dominated by x.

4.1 Proof of Theorem 3.1

We proceed to establish Theorem 3.1 through a series of lemmas. Let $(\mathcal{M}_N, f)$ be a rule which operates on minimal informational requirements in $\mathcal{AN}$. Because the plurality rule is in $\mathcal{AN}$, the informational size of $(\mathcal{M}_N, f)$ is not greater than nm.

Lemma 4.1 *$\mathcal{M}_i = \mathcal{M}_j$ for all $i, j \in N$.*

Proof. Suppose to the contrary that $\mathcal{M}_i \neq \mathcal{M}_j$ for some $i, j \in N$.
CLAIM 1: At least one of the following two statements holds:

(i) There exist $M_i^* \in \mathcal{M}_i$ and $M_j^1, M_j^2 \in \mathcal{M}_j$ such that $M_i^* \cap M_j^1 \neq \emptyset$ and $M_i^* \cap M_j^2 \neq \emptyset$.

(ii) There exist $M_i^1, M_i^2 \in \mathcal{M}_i$ and $M_j^* \in \mathcal{M}_j$ such that $M_i^1 \cap M_j^* \neq \emptyset$ and $M_i^2 \cap M_j^* \neq \emptyset$.

Proof of Claim 1. Suppose that neither of the statements holds. Because (i) does not hold, for any $M_i \in \mathcal{M}_i$, there exists M_j such that $M_i \subset M_j$. Because (ii) does not hold, for any $M_j \in \mathcal{M}_j$, there exists $M_i \in \mathcal{M}_i$ such that $M_j \subset M_i$. Thus, for any $M_i \in \mathcal{M}_i$, there exist $M_j \in \mathcal{M}_j$ and $M_i' \in \mathcal{M}_i$ such that $M_i \subset M_j \subset M_i'$. Because $\mathcal{M}_i$ is a partition of $\mathcal{L}$, this implies that $M_i = M_j$. Therefore, $\mathcal{M}_i = \mathcal{M}_j$, which is a contradiction. □

Without loss of generality, assume that statement (i) of Claim 1 holds.
CLAIM 2: $f(M_j^1, M_{-j}) = f(M_j^2, M_{-j})$ for all $M_{-j} \in \mathcal{M}_{-j}$.
Proof of Claim 2. Suppose to the contrary that $f(M_j^1, M_{-j}) \neq f(M_j^2, M_{-j})$ for some $M_{-j} \in \mathcal{M}_{-j}$. Let R_j and R_j' be such that $R_j \in M_i^* \cap M_j^1$ and $R_j' \in M_i^* \cap M_j^2$. Let R_{-j} be an element of M_{-j}. $f(\varphi_N(R_j, R_{-j})) \neq f(\varphi_N(R_j', R_{-j}))$. Now, interchange the preferences of agents i and j. (Let σ denote the permutation interchanging agents i and j.) Then, by anonymity, $f(\varphi_N([(R_j, R_{-j})^\sigma])) \neq f(\varphi_N([(R_j', R_{-j})^\sigma]))$. However, because $R_j, R_j' \in M_i^*$, $\varphi_N([(R_j, R_{-j})^\sigma]) = \varphi_N([(R_j', R_{-j})^\sigma])$, which is a contradiction. □

Claim 2 implies that distinct messages M_j^1 and M_j^2 can be integrated into one message without any essential change. Formally, let $\mathcal{M}_j' = \{M_j \mid M_j \in \mathcal{M}_j \setminus \{M_j^1, M_j^2\} \text{ or } M_j = M_j^1 \cup M_j^2\}$. For $i \in N \setminus \{j\}$, let $\mathcal{M}_i' = \mathcal{M}_i$. Let $\mathcal{M}_N' = \prod_{i \in N} \mathcal{M}_i'$. For each message profile $M_N \in \mathcal{M}_N'$, let $f'(M_N) = f(M_N)$ if $M_j \neq M_j^1 \cup M_j^2$ and $f'(M_N) = f(M_j^1, M_{-j})$ if $M_j = M_j^1 \cup M_j^2$. We claim that $f'(\varphi_N'(R_N)) = f(\varphi_N(R_N))$ for every preference profile R_N. If $R_j \notin M_j^1 \cup M_j^2$, then $f'(\varphi_N'(R_N)) = f'(M_N) = f(M_N) = f(\varphi_N(R_N))$. If $R_j \in M_j^1$, then $f'(\varphi_N'(R_N)) = f'(M_j^1 \cup M_j^2, M_{-j}) = f(M_j^1, M_{-j}) = f(\varphi_N(R_N))$. If $R_j \in M_j^2$, then $f'(\varphi_N'(R_N)) = f'(M_j^1 \cup M_j^2, M_{-j}) = f(M_j^1, M_{-j}) = f(M_j^2, M_{-j}) = f(\varphi_N(R_N))$. ($\varphi_N'$ is a profile of message functions associated with $\mathcal{M}_N'$.) Therefore, $(\mathcal{M}_N', f')$ is in $\mathcal{AN}$ whereas the informational size of $(\mathcal{M}_N', f')$ is less than that of $(f, \mathcal{M}_N)$, which is a contradiction to the fact that $(\mathcal{M}_N, f)$ attains the minimal informational size in $\mathcal{AN}$. ■

Consider the case $m = 2$. Let $X = \{x, y\}$, let R_i be the linear order such that $r_1(R_i) = x$ and $r_2(R_i) = y$ and let R_i' be the linear order such that $r_1(R_i') = y$ and $r_2(R_i') = x$. Then, by Lemma 4.1, either $\mathcal{M}_i = \{\{R_i, R_i'\}\}$ for all $i \in N$ or $\mathcal{M}_i = \{\{R_i\}, \{R_i'\}\}$ for all $i \in N$. In the former case holds, because there is only one possible message profile, $f \circ \varphi_N$ should be constant on $\mathcal{L}^N$, which is a contradiction. Thus, the latter case holds. Therefore, we complete the proof of Theorem 3.1 for the case $m = 2$. (h can be either 1 or 2.) In the following, we assume $m \geq 3$.

Lemma 4.2 *For any $i \in N$, for any permutation ρ of X, and for any $M \in \mathcal{M}_i$, $\rho(M) \in \mathcal{M}_i$.*

Proof. Suppose $\rho(M) \notin \mathcal{M}_i$ for some $M \in \mathcal{M}_i$. There are two cases to consider.
CASE 1: $\rho(M) \subsetneq M'$ for some $M' \in \mathcal{M}_i$. Because $M \subsetneq \rho^{-1}(M')$, there exists $M^* \in \mathcal{M}_i$ such that $M^* \neq M$ and $M^* \cap \rho^{-1}(M') \neq \emptyset$.
CLAIM: $f(M, M_{-i}) = f(M^*, M_{-i})$ for all $M_{-i} \in \mathcal{M}_{-i}$.

Proof of Claim. Suppose to the contrary that $f(M, M_{-i}) \neq f(M^*, M_{-i})$ for some $M_{-i} \in \mathcal{M}_{-i}$. Let R_i be any element of M, let R_{-i} be any element of M_{-i}, and let $\hat{R}_i$ be any element of $\rho^{-1}(M') \cap M^*$. Then, $(f \circ \varphi_N)(R_i, R_{-i}) \neq (f \circ \varphi_N)(\hat{R}_i, R_{-i})$. By neutrality, $(f \circ \varphi_N)[\rho(R_i), \rho(R_{-i})] \neq (f \circ \varphi_N)[\rho(\hat{R}_i), \rho(R_{-i})]$. However, because $\rho(R_i), \rho(\hat{R}_i) \in M'$, $\varphi_N[\rho(R_i), \rho(R_{-i})] = \varphi_N[\rho(\hat{R}_i), \rho(R_{-i})]$, which is a contradiction. $\square$

This claim shows that we can integrate distinct messages M and M^* into one message without any substantial change. See the argument following Claim 2 in the proof of Lemma 4.1. The same reasoning applies here, and we have a contradiction.

CASE 2: $\rho(M) \cap M^1 \neq \emptyset$ and $\rho(M) \cap M^2 \neq \emptyset$ for some $M^1, M^2 \in \mathcal{M}_i$. In this case, we claim $f(M^1, M_{-i}) = f(M^2, M_{-i})$ for all $M_{-i} \in \mathcal{M}_{-i}$. Suppose not. Then, for any $R_i^1 \in \rho(M) \cap M^1$ and for any $R_i^2 \in \rho(M) \cap M^2$, $(f \circ \varphi_N)(R_i^1, R_{-i}) \neq (f \circ \varphi_N)(R_i^2, R_{-i})$. By neutrality, $(f \circ \varphi_N)(\rho^{-1}(R_i^1), \rho^{-1}(R_{-i})) \neq (f \circ \varphi_N)(\rho^{-1}(R_i^2), \rho^{-1}(R_{-i}))$. However, because $\rho^{-1}(R_i^1), \rho^{-1}(R_i^2) \in M$, we have $\varphi_N(\rho^{-1}(R_i^1), \rho^{-1}(R_{-i})) = \varphi_N(\rho^{-1}(R_i^2), \rho^{-1}(R_{-i}))$, which is a contradiction.

Thus, $f(M^1, M_{-i}) = f(M^2, M_{-i})$ for all $M_{-i} \in \mathcal{M}_{-i}$. This implies that we can integrate M^1 and M^2 into one message without affecting any essential aspects of a rule $(\mathcal{M}_N, f)$. By the same argument as in the proof of Lemma 4.1, we have a contradiction. ■

Lemma 4.3 *For any $i \in N$, there exists $h \in \{1, \ldots, m\}$ such that for any $M \in \mathcal{M}_i$, $r_h(M) = \{x \in X \mid r_h(R_i) = x$ for some $R_i \in M\}$ is a singleton.*

Proof. Suppose to the contrary that for any $h \in \{1, \ldots, m\}$, there exists $M \in \mathcal{M}_i$ such that $r_h(M)$ is not a singleton. Let M' be any element of $\mathcal{M}_i$ and let R_i and R_i' be any elements of M and M', respectively. Let ρ be the permutation of X such that $\rho(R_i) = R_i'$. Then, $\rho(M) \cap M' \neq \emptyset$. By Lemma 4.2, $\rho(M) \in \mathcal{M}_i$. Because $\mathcal{M}_i$ is a partition of $\mathcal{L}$, $\rho(M) = M'$. This implies that $r_h(M')$ is not a singleton. This argument shows that for any $h \in \{1, \ldots, m\}$ and for any $M \in \mathcal{M}_i$, there exist $R, R' \in M$ such that $r_h(R) \neq r_h(R')$.
CLAIM 1: For any $h \in \{1, \ldots, m\}$, for any $M \in \mathcal{M}_i$, and for any $x \in X$, there exists $R \in M$ such that $r_h(R) = x$. In other words, $r_h(M) = X$ for all $h \in \{1, \ldots, m\}$ and $M \in \mathcal{M}_i$.
Proof of Claim 1. Suppose not. Then, there exist $h \in \{1, \ldots, m\}$ and $M \in \mathcal{M}_i$ such that $r_h(M) \neq X$. We claim that $|r_h(M)| = m - 1$.

Suppose $|r_h(M)| \leq m - 2$. Let $X \setminus r_h(M) = \{y_1, \ldots, y_{h_1}\}$ and let $r_h(M) = \{x_1, \ldots, x_{h_2}\}$. Because $|r_h(M)| \leq m - 2$, $h_1 \geq 2$. Because $r_h(M)$ is not a singleton, $h_2 \geq 2$. For each pair (ℓ_1, ℓ_2) such that $1 \leq \ell_1 \leq h_1$ and $1 \leq \ell_2 \leq h_2$, let $\rho_{\ell_1}^{\ell_2}$ be the permutation exchanging y_{ℓ_1} and x_{ℓ_2}. Then, $M \neq \rho_{\ell_1}^{\ell_2}(M) \neq \rho_{\ell_1'}^{\ell_2'}(M)$ for any $\ell_1, \ell_1', \ell_2, \ell_2'$ with $(\ell_1, \ell_2) \neq (\ell_1', \ell_2')$. By Lemma 4.2, $\rho_{\ell_1}^{\ell_2}(M) \in \mathcal{M}_i$ for all ℓ_1, ℓ_2. Thus, $|\mathcal{M}_i| \geq h_1 \cdot h_2 + 1 \geq 2 \cdot \max\{h_1, h_2\} + 1 \geq m + 1 > m$. Then, by Lemma 4.1, the informational size of $(\mathcal{M}_N, f)$ is greater than nm, which is a contradiction. Thus, $|r_h(M)| = m - 1$.

Let $\{x\} = X \setminus r_h(M)$. Let R be any element of M and let h' be such that $r_{h'}(R) = x$. Because $r_{h'}(M)$ is not a singleton (see the statement right above Claim 1), there exists $R' \in M$ such that $r_{h'}(R') \neq x$. Let h'' be such that $r_{h''}(R') = x$. Note that $h' \neq h''$ and $h', h'' \neq h$. Let y denote $r_{h''}(R)$ and let ρ be the permutation of X such that $\rho(R) = R'$. Then, because $\rho(M) \cap M \neq \emptyset$ and $\mathcal{M}_i$ is a partition of $\mathcal{L}$, $\rho(M) = M$. Note that $\rho(y) = x$. Because $y \in r_h(M)$, there exists $R'' \in M$ such that $r_h(R'') = y$. For such R'', $r_h(\rho(R'')) = x$, which is a contradiction to $\rho(R'') \in M$. $\square$
CLAIM 2: $f(M_N) = X$ for all $M_N \in \mathcal{M}_N$.
Proof of Claim 2. Suppose to the contrary that $f(M_N) \neq X$ for some $M_N \in \mathcal{M}_N$. Let R_N be any element of M_N. Then, $(f \circ \varphi_N)(R_N) \neq X$. Let x be an element of $X \setminus (f \circ \varphi_N)(R_N)$. By neutrality, there exists R_N' such that $x \in (f \circ \varphi_N)(R_N')$. Let M_N' be the element of $\mathcal{M}_N$ such that $R_N' \in M_N'$.

$r_h(R'_i)$	$r_h(R_i)$	Operation on R_i	Operation on R''_i
x	x	Do not change	Do not change
	Not x	Interchange $r_{h+1}(R_i)$ and x	Lift x to hth position
Not x (Let $y = r_h(R'_i)$)	y	Do not change	Do not change
	x	Interchange $r_{h-1}(R_i)$ and y	Lift x to the top
	Others	First, interchange x and $r_m(R_i)$ and next, interchange y and $r_{h-1}(R_i)$	Lift x to the top

Table 1: The profiles R''_N and R^*_N in the proof of Theorem 3.2

Let i be any agent and let h be such that $r_h(R_i) = x$. By Claim 1, $r_h(M'_i) = X$. Thus, in M'_i, we can find R''_i such that $r_h(R''_i) = x$. Let R''_N be a profile of such R''_i. Note that the positions of x in R''_N are the same as in R_N. Also, because R''_N belongs to M'_N, $\varphi'_N(R'_N) = \varphi'_N(R''_N)$. Thus, $x \in (f \circ \varphi_N)(R''_N)$. Let ρ_N be a profile of permutations such that $\rho_i(R''_i) = R_i$. Note that $\rho_i(x) = x$ for all $i \in N$. By neutrality, $x \in (f \circ \varphi_N)(\rho(R''_N)) = (f \circ \varphi_N)(R_N)$, which is a contradiction. □

Claim 2 implies that a rule (f, φ_N) is constant, which is a contradiction. ■

Lemma 4.3 shows that each $M_i \in \mathcal{M}_i$ is contained in $\{R_i \in \mathcal{L} \mid r_h(R_i) = x\}$ for some $x \in X$. Thus, if the hth ranked alternatives in two preferences R and R' are different, then R and R' belong to distinct M_i and M'_i in $\mathcal{M}_i$. This implies that there are at least m elements in $\mathcal{M}_i$. If $|\mathcal{M}_i| > m$, then by Lemma 4.1, the informational size of $(\mathcal{M}_N, f)$ is greater than nm, which is a contradiction. Thus, $|\mathcal{M}_i| = m$ for all $i \in N$, and the informational size of $(\mathcal{M}_N, f)$ is nm.

For each $M_i \in \mathcal{M}_i$, let $C(M_i)$ denote the element of X such that $M_i \subset \{R_i \in \mathcal{L} \mid r_h(R_i) = C(M_i)\}$. We show that this C is a bijection. Because $|\mathcal{M}_i| = m = |X|$, it suffices to show that C is onto. Let x be any element of X. Then, because $\{R_i \in \mathcal{L} \mid r_h(R_i) = x\}$ is not the empty set and $\mathcal{M}_i$ is a partition of $\mathcal{L}$, there exists $M_i \in \mathcal{M}_i$ such that $M_i \subset \{R_i \in \mathcal{L} \mid r_h(R_i) = x\}$, and hence $C(M_i) = x$. Thus, C is a bijection. This implies that for any $M_i \in \mathcal{M}_i$, for any $M'_i \in \mathcal{M}_i \setminus \{M_i\}$, and for any $R_i \in M'_i$, $r_h(R_i)$ is not $C(M_i)$. Thus, $M_i \subsetneq \{R_i \in \mathcal{L} \mid r_h(R_i) = C(M_i)\}$ leads to a contradiction to the fact that $\mathcal{M}_i$ is a partition of $\mathcal{L}$. Therefore, for each $M_i \in \mathcal{M}_i$, $M_i = \{R_i \in \mathcal{L} \mid r_h(R_i) = C(M_i)\}$. That is, each $M_i \in \mathcal{M}_i$ is associated with an alternative $C(M_i)$ in X and M_i consists of all preferences which rank $C(M_i)$ at the hth position. Because C is a bijection, we complete the proof of the Theorem 3.1.

4.2 Proof of Theorem 3.2

Let $(\mathcal{M}_N, f)$ be a rule which operates on minimal informational requirements in $\mathcal{ANM}$.

First, we prove the statement (i). If $m = 2$, then this statement is a direct consequence of Theorem 3.1. Thus, let $m \geq 3$. Suppose to the contrary that $1 < h < m$, and we claim that $(\mathcal{M}_N, f)$ is constant. Let R_N and R'_N be any preference profiles. We prove $(f \circ \varphi_N)(R_N) = (f \circ \varphi_N)(R'_N)$.

First, we show $(f \circ \varphi_N)(R_N) \subset (f \circ \varphi_N)(R'_N)$. Let x be any element of $(f \circ \varphi_N)(R_N)$. Now, make a new preference profile R''_N from R_N according to the third column of Table 7.1. (At this stage, see only the first three columns.) Depending on $r_h(R'_i)$ and $r_h(R_i)$, there are five possible cases as described in the first two columns of Table 7.1. The third column specifies the operation on R_i in each case. Note that these operations are feasible because neither $h = 1$ nor $h = m$.

Let R''_N denote the resulting preference profile. It can be seen that $r_h(R_i) = r_h(R''_i)$ for all $i \in N$. Thus, $(f \circ \varphi_N)(R_N) = (f \circ \varphi_N)(R''_N)$ and x is also in $(f \circ \varphi_N)(R''_N)$. Now, apply the operation on R''_N described in the forth column of Table 7.1, and let R^*_N denote the resulting preference profile. Then, by monotonicity, $x \in (f \circ \varphi_N)(R^*_N)$. It can be seen that $r_h(R'_i) = r_h(R^*_i)$ for all $i \in N$, and hence $(f \circ \varphi_N)(R'_N) = (f \circ \varphi_N)(R^*_N)$. Therefore, $x \in (f \circ \varphi_N)(R'_N)$, and we complete the proof of the relation $(f \circ \varphi_N)(R_N) \subset (f \circ \varphi_N)(R'_N)$.

By the symmetric argument, we can prove $(f \circ \varphi_N)(R_N) \supset (f \circ \varphi_N)(R'_N)$.

Because R_N and R'_N was arbitrary, we can conclude $(\mathcal{M}_N, f)$ is a constant rule, which is a contradiction. Thus, h should be either 1 or m.

Next, we prove the second statement of the theorem.

CASE 1: $h = 1$. By Theorem 3.1, $\mathcal{M}_N$ is equal to the domain of f^p. In this case, we prove $f^p(M_N) \subset f(M_N)$ for all $M_N \in \mathcal{M}_N$. Suppose $f^p(M_N) \not\subset f(M_N)$ for some $M_N \in \mathcal{M}_N$. Let x be an element of $f^p(M_N) \setminus f(M_N)$, and let R_N be such that $R_i \in M_i$ for all $i \in N$.

We claim that $f^p(M_N) \cap f(M_N) = \emptyset$. Suppose to the contrary that there exists $y \in f^p(M_N) \cap f(M_N)$. Then, $y \in (f^p \circ \varphi^p_N)(R_N) \cap (f \circ \varphi_N)(R_N)$. (Note that $\varphi^p_N = \varphi_N$.) Let σ be a permutation of N such that $\sigma(\{i \in N \mid r_1(R_i) = x\}) = \{i \in N \mid r_1(R_i) = y\}$ and $\sigma(\{i \in N \mid r_1(R_i) = y\}) = \{i \in N \mid r_1(R_i) = x\}$. By anonymity, $y \in (f \circ \varphi_N)(R^\sigma_N)$. Let ρ be the permutation of X exchanging x and y. By neutrality, $x \in (f \circ \varphi_N)(\rho(R^\sigma_N))$. Note that the two n-tuples of top ranked alternatives in R_N and $\rho(R^\sigma_N))$ are the same. Because $h = 1$, $(f \circ \varphi_N)(R_N) = (f \circ \varphi_N)(\rho(R^\sigma_N))$. Thus, $x \in (f \circ \varphi_N)(R_N) = f(M_N)$, which is a contradiction.

Let z be any element of $(f \circ \varphi_N)(R_N)$. By the above argument, $z \notin (f^p \circ \varphi^p_N)(R_N)$. Let N_x be a subset of $\{i \in N \mid r_1(R_i) = x\}$ such that $|N_x| = |\{i \in N \mid r_1(R_i) = z\}|$. Then, let σ' be the permutation such that $\sigma'(N_x) = \{i \in N \mid r_1(R_i) = z\}$ and $\sigma'(\{i \in N \mid r_1(R_i) = z\}) = N_x$. By anonymity, $z \in (f \circ \varphi_N)(R^{\sigma'}_N)$. Let ρ' be the permutation exchanging x and z. By neutrality, $x \in (f \circ \varphi_N)(\rho'(R^{\sigma'}_N))$. For each $i \in \{j \in N \mid r_1(R_j) = x\} \setminus N_x$, lift x to the top in $\rho'(R^{\sigma'}_i)$. Let R''_N denote the resulting preference profile. By monotonicity, $x \in (f \circ \varphi_N)(R''_N)$. It can be seen that the two n-tuples of top ranked alternatives in R_N and R''_N are the same, and hence $x \in (f \circ \varphi_N)(R_N) = f(M_N)$, which is a contradiction. Therefore, $f^p(M_N) \subset f(M_N)$ for all $M_N \in \mathcal{M}_N$.

CASE 2: $h = m$. Suppose $f^a(M_N) \not\subset f(M_N)$ for some $M_N \in \mathcal{M}_N$. Let x be an element of $f^a(M_N) \setminus f(M_N)$, and let R_N be such that $R_i \in M_i$ for all $i \in N$.

We claim that $f^a(M_N) \cap f(M_N) = \emptyset$. Suppose to the contrary that there exists $y \in f^a(M_N) \cap f(M_N)$. Then, $y \in (f^a \circ \varphi^a_N)(R_N) \cap (f \circ \varphi_N)(R_N)$. There are two cases to consider.

First, assume $\{i \in N \mid r_m(R_i) = y\} = \emptyset$. Then, $\{i \in N \mid r_m(R_i) = x\}$ is also the empty set. Let ρ be the permutation of X exchanging x and y. By neutrality, $x \in (f \circ \varphi_N)(\rho(R_N))$. Note that the two n-tuples of the bottom ranked alternatives in R_N and $\rho(R_N)$ are the same. Thus, $x \in (f \circ \varphi_N)(R_N) = f(M_N)$, which is a contradiction.

Next, assume $\{i \in N \mid r_m(R_i) = y\} \neq \emptyset$. Then, $|\{i \in N \mid r_m(R_i) = y\}| = |\{i \in N \mid r_m(R_i) = x\}| > 0$. Let σ be a permutation of N such that $\sigma(\{i \in N \mid r_m(R_i) = y\}) = \{i \in N \mid r_m(R_i) = x\}$ and $\sigma(\{i \in N \mid r_m(R_i) = x\}) = \{i \in N \mid r_m(R_i) = y\}$. By anonymity, $y \in (f \circ \varphi_N)(R^\sigma_N)$. Let ρ be the permutation of X exchanging x and y. By neutrality, $x \in (f \circ \varphi_N)(\rho(R^\sigma_N))$. Note that the two n-tuples of the bottom ranked alternatives in R_N and $\rho(R^\sigma_N)$ are the same. Thus, $x \in (f \circ \varphi_N)(R_N)$, which is a contradiction. Therefore, in any case, $f^a(M_N) \cap f(M_N) = \emptyset$.

Let z be any element of $(f \circ \varphi_N)(R_N)$. By the above argument, $z \notin (f^a \circ \varphi^a_N)(R_N)$, that is, $|\{i \in N \mid r_m(R_i) = z\}| > |\{i \in N \mid r_m(R_i) = x\}|$. Let N_z be a subset of $\{i \in N \mid r_m(R_i) = z\}$ such that $|N_z| = |\{i \in N \mid r_m(R_i) = x\}|$. Let σ' be a permutation of N such that $\sigma'(N_z) = \{i \in$

$N \mid r_m(R_i) = x\}$ and $\sigma'(\{i \in N \mid r_m(R_i) = x\}) = N_z$. By anonymity, $z \in (f \circ \varphi_N)(R_N^{\sigma'})$. Let ρ' be the permutation of X exchanging x and z. By neutrality, $x \in (f \circ \varphi_N)(\rho(R_N^{\sigma'}))$. Now, for each $i \in \{j \in N \mid r_m(R_j) = z\} \setminus N_z$, take z to the second place from the bottom at $\rho(R_i^{\sigma'})$. Let R'_N be the resulting preference profile. Note that the two n-tuples of bottom ranked alternatives in $\rho(R_N^{\sigma})$ and R'_N are the same, and hence $x \in (f \circ \varphi_N)(R'_N)$. Now, for each $i \in \{j \in N \mid r_m(R_j) = z\} \setminus N_z$, lift x to the top of his preference. Let R''_N denote resulting preference profile. By monotonicity, $x \in (f \circ \varphi_N)(R''_N)$. Then, it can be seen that the two n-tuples of bottom ranked alternatives in R''_N and R_N are the same. Thus, $x \in (f \circ \varphi_N)(R_N)$, which is a contradiction. Therefore, $f^a(M_N) \subset f(M_N)$ for all $M_N \in \mathcal{M}_N$.

References

Ching, Stephen (1996) A simple characterization of plurality rule. Journal of Economic Theory 71, 298–302.

Conitzer, Vincent; Sandholm, Tuomas (2005) Communication complexity of common voting rules. Proceedings of the ACM Conference on Electronic Commerce, 78–87.

Kushilevitz, Eyal; Nisan, Noam (1997) Communication complexity. Cambridge University Press, Cambridge.

Moulin, Hervé (1980) On strategy-proofness and single peakedness. Public Choice 35, 437–455.

Moulin, Hervé (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge.

Richelson, Jeffrey T (1978) A characterization result for the plurality rule. Journal of Economic Theory 19, 548–550.

Roberts, Fred S. (1991) Characterization of the plurality function. Mathematical Social Sciences 21, 101–127.

Sen, Amartya K. (1986) Social choice theory. In: Arrow, Kenneth J.; Intrigator, Michael D. (eds) Handbook of mathematical economics, vol 3. North-Holland, Amsterdam, pp 1073–1181.

Yao, Andrew Chi-Chih (1979) Some complexity questions related to distributed computing. Proceedings of the 11th ACM symposium on theory of computing, 209–213.

Yeh, Chun-Hsien (2008) An efficiency characterization of plurality rule in collective choice problems. Economic Theory 34, 575–583.

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