Dodgson's Rule Approximations and Absurdity

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Background

Dodgson Rule:

- NP-Hard (Bartholdi et al., 1989)
- Θ_2^p-Complete (Hemaspaandra et al., 1997)
- "Efficient for fixed #alternatives m"$\sim f(m!^{m!} \ln n)$ (M^c^Cabe-Dansted, 2006)
- Impartial Culture (votes independent, equally likely)
  - Tideman rule: Converges as $n \to \infty$ (McCabe-Dansted et al., 2006)
  - Dodgson Quick: exponentially fast (McCabe-Dansted et al., 2006)
  - Greedy Winner: exponentially fast (Homan and Hemaspaandra, 2005)

Impartial Culture

Impartial Culture is implausible

- Voters are not independent
  - E.g. “How to vote cards”
- Votes not equally likely
  - Left > Right > Centre?

Important to test against other assumptions

Impartial Anonymous Culture

A “Voting Situation”:

- Represents number of voters who voted which way.
- Does not store who voted what.

IAC: Each voting situation equally likely

- 9:1 victory as likely as 6:4 (for two alternatives)

Without Independence

We show previous approximations do not converge.
We show the following do converge:

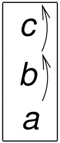
- Dodgson Relaxed and Rounded (new)
- Dodgson Relaxed (new)
- Dodgson Clone
  - Young: Fixes an Absurdity
  - Rothe et al. 2003: Polynomial

Improvements over original.

- Which was not actual proposed by Dodgson

Dodgson's Rule

- Picks candidate closest to being a Condorcet winner
- We swap neighbouring alternatives in votes to produce a Condorcet winner
- Dodgson score ($\mathrm{Sc}_{\mathbf{D}}$) is # of such swaps required
- Alternative with lowest Dodgson score is Winner
- E.g. single voter $\{cba\} \implies \mathrm{Sc}_{\mathbf{D}}(a) = 2$



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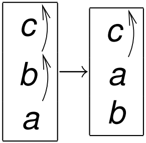

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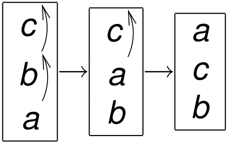

New Approximations

Can define Dodgson Clone in terms of cloning electorate.
ILP for Dodgson Score (Bartholdi et al., 1989)

- Relax integer constraints?
- Linear Program $\implies$ Polynomial time.

Fractional votes:

- Condorcet tie winner if switch *a* over *c* in 0.5 votes
- Dodgson Clone score is (0.5)(2).
- Dodgson Relaxed (DR): must switch $\lceil 0.5 \rceil$ times: score is (1)(2)
- Dodgson Relaxed and Rounded (D&): Round up DR score: score is $\lceil (1)(2) \rceil$.

Linear Programs

WLOG, all swaps swap d up profile.

min $\sum_i \sum_{j>0} y_{ij}$ subject to

$y_{i0} = N_i$ (for each type of vote i)

$\sum_{ij}(e_{ijk} - e_{i(j-1)k})y_{ij} \geq D_k$ (for each alternative k)

$y_{ij} \leq y_{i(j-1)}$ (for each i and $j > 0$)

$y_{ij} \geq 0$, and each y_{ij} must be integer.

- For each i and j variable y_{ij} represents the number of times that the candidate d is swapped up *at least j* positions in votes of the i^{th} type.
- e_{ijk} is 1 if swapping d up j positions in votes of the i^{th} i swaps d over k. (0 otherwise).
- D_k is number of times d must be swapped over k.
  - $\lceil \text{adv}(k, d)/2 \rceil$ [DR] or $\text{adv}(k, d)/2$ [DC]

Bounds

Note that:

1. A solution to an ILP is a solution to LP.
   - $\therefore \mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$
2. Rounding up variables to LP gives solution to ILP.
   - (for our LP)
   - $m!e$ variables $e = 2.71\ldots$
   - $\therefore \mathrm{Sc}_{\mathbf{D}}(d) - m!e < \mathrm{Sc}_{\mathbf{C}}(d)$
3. Every solution for DC LP is solution to DR LP.

$$\mathrm{Sc}_{\mathbf{D}}(d) - m!e < \mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{R}}(d) \leq \mathrm{Sc}_{\&}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$$

Convergence

$$\mathrm{Sc}_{\mathbf{D}}(d) - m!e < \mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{R}}(d) \leq \mathrm{Sc}_{\&}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$$

- Informally: Even neck-and-neck elections won by thousands or millions of votes.
- Converge under any reasonable assumption.

Convergence: IAC

$$\mathrm{Sc}_{\mathbf{D}}(d) - m!e < \mathrm{Sc}_{\mathbf{C}}(d) \leq \mathrm{Sc}_{\mathbf{R}}(d) \leq \mathrm{Sc}_{\&}(d) \leq \mathrm{Sc}_{\mathbf{D}}(d)$$

Let $\mathbf{v} = ab\ldots z$ and $\bar{\mathbf{v}} = z\ldots ba$
Group voting situations, differ only in $\#(\mathbf{v})$ and $\#(\bar{\mathbf{v}})$.

- Replacing $\mathbf{v}$ with $\bar{\mathbf{v}}$ will improve relative score of z over a by ≥ 1
  - less than $m!e$ members s.t. DC winner differs

#Groups increase slower than #voting situations.
$\therefore$ converges.

Accuracy of Tideman's Rule Under IC

Frequency that Tideman winner is Dodgson winner

	3	5	7	9	15	25	85
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9984	0.9974	0.9961	0.9972	0.9936	0.9917	0.9930
7	0.9902	0.9864	0.9852	0.9868	0.9845	0.9805	0.9847
9	0.9792	0.9730	0.9724	0.9731	0.9718	0.9760	0.9815
15	0.9468	0.9292	0.9263	0.9273	0.9379	0.9485	0.9649
25	0.8997	0.8691	0.8620	0.8625	0.8833	0.9113	0.9534

x: number of voters

y: number of alternatives

D& winner differs only once at (5,25)

A "bad" voting ratio

We say a voting ratio is bad if every even profile $\mathcal{P}$ that reduces to it has different DQ and Dodgson winners.

$$g(\mathbf{v}) = \begin{cases} 7/18 \text{ if} & \mathbf{v} = abcde \\ 6/18 \text{ if} & \mathbf{v} = cdabe \\ 5/18 \text{ if} & \mathbf{v} = bcead \\ 0 & \text{otherwise} \end{cases}$$

Recall: DQ score $\mathrm{Sc}_{\mathbf{Q}}(x)$ of x is $\sum_y \lceil \mathrm{adv}(y,x)/2 \rceil$
For $18n$ agents:

- DQ and Dodgson score of c will be $3n$
- the DQ score of a will be $2n$ and the Dodgson score of a will be $4n$.
- Hence a is DQ winner but c is Dodgson winner.

Proof of Non-Convergence

We have a bad voting ratio.

- Has neighbourhood S of “bad” voting ratios.

IAC: every voting situation equally likely

- Probably of falling in S does not converge to 0 as $n \to \infty$.

Tideman based rules converge to DQ, not Dodgson.

Overview

	IAC Converges	IC: fast	Split-ties	Non-absurd
Tideman	No	No	N/A	(Yes)
Dodgson Quick	No	Yes	N/A	(No)
Dodgson Clone	Yes	(No)	N/A	Yes
DR	**Yes**	**Yes**	**Yes**	**(No)**
D&	**Yes**	**Yes**	**No**	**(No)**
Dodgson	+	+	No	No

(X): X “obvious” but not proven.

Conclusion

Old Approximations (DQ etc.)

- Do not converge under IAC.

New Approximations:

- Do converge.
  - D& picked Dodgson Winner in all but one of 43 million simulations (McCabe-Dansted, 2006)
- Can sacrifice accuracy for
  - Splitting ties
  - Invulnerability to cloning the electorate
- For many purposes better.

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Analysis: Background

Swapping Neighbouring Candidates a natural measure of distance

- Kemeny uses similar measure, compares difference to entire rankings.
- To use this measure implies Dodgson rule.

Dodgson's rule has flaws, particularly

- Hard to compute
  - NP-hard
  - $\mathcal{O}(f(m) \ln n)$, but $f(m) \sim m!^{m!}$
- Cloning electorate changes winner.

Minor modification (DC) fixes both of above.

Analysis: New Convergence Result

Stronger:

- Does not require IC

Weaker:

- not exponentially fast.
- Fixed m?
  - $n \gg m!$ vs $n \gg m^2$
    - (Actual convergence better)
    - 43 million, only one D& $\neq$ Dodgson Winner (McCabe-Dansted, 2006)

Number of Variables

Alternative d is the alternative we are computing Dodgson score of.
#Linear orders with d ranked in i^{th} position = $(m-1)!$
#Vote types with d ranked in i^{th} position $= \frac{(m-1)!}{(m-i)!}$
#Vote types
$= \sum_i \frac{(m-1)!}{(m-i)!} < (m-1)!\left(\frac{1}{0!} + \frac{1}{1!} + \cdots\right) = (m-1)!e$
$(e = 2.71\ldots)$
Less than m variables y_{ij} per vote type $\implies$ less than $m!e$ variables

Tideman-like Approximations

- We define each approximation in terms of the score (lowest score wins)
- We can compute these scores from the "advantages"
- n_{ba} : Number of voters who prefer b to a
- $\text{adv}(b, a) = \max(0, n_{ba} - n_{ab})$: Advantage of b over a
  - Also called "margin of defeat"
- Dodgson Quick (DQ) score: $\text{Sc}_{\mathbf{Q}}(a) = \sum_{b \neq a} \left\lceil \frac{\text{adv}(b,a)}{2} \right\rceil$
  - (this is our new approximation)
- Tideman score: $\text{Sc}_{\mathbf{T}}(a) = \sum_{b \neq a} \text{adv}(b, a)$