Confluence Operators
– Negotiation as Pointwise Merging –

Sébastien Konieczny   Ramón Pino Pérez

CNRS - Centre de Recherche en Informatique de Lens (CRIL)
Université d'Artois, Lens, France
konieczny@cril.fr

Departamento de Matemáticas, Facultad de ciencias
Universidad de Los Andes, Mérida, Venezuela
pino@ula.ve

Belief change operators

Revision Belief revision is the process of accomodating a new piece of evidence that is more reliable than the current beliefs of the agent. In belief revision the world is static, it is the beliefs of the agents that evolve.

Update In belief update the new piece of evidence denotes a change in the world. The world is dynamic, and these (observed) changes modify the beliefs of the agent.

Merging Belief merging is the process of defining the beliefs of a group of agents. So the question is: Given a set of agents that have their own beliefs, what can be considered as the beliefs of the group?

Motivation

Revision

Update

Merging

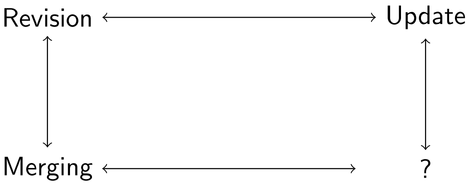
Motivation

Revision ⟷ Update

Merging

Motivation

Motivation



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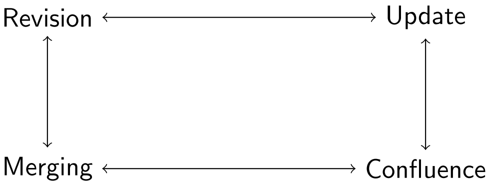

Framework

- Propositional logic:
  - A formula φ is build from
    - A set $\mathcal{P}$ of propositional symbols $(a, b, \ldots)$
    - And logical connectives $(\neg, \wedge, \vee, \rightarrow, \ldots)$
  - An interpretation ω is a function from $\mathcal{P}$ to $\{0, 1\}$
  - $mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$
  - A formula is complete if it has a unique model
- A base φ is a (finite set of) propositional formula
- A profile Ψ is a multi-set of bases : $\Psi = \{\varphi_1, \ldots, \varphi_n\}$

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- Primacy of update
- Coherence
- Minimal change

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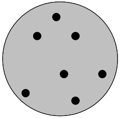

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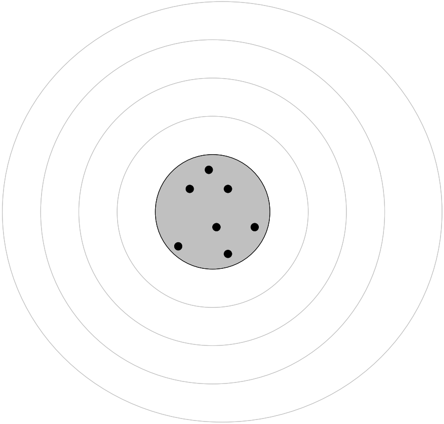

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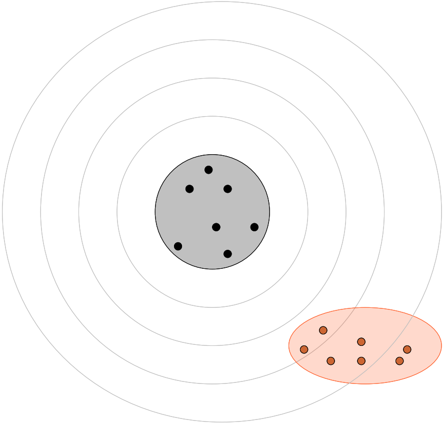

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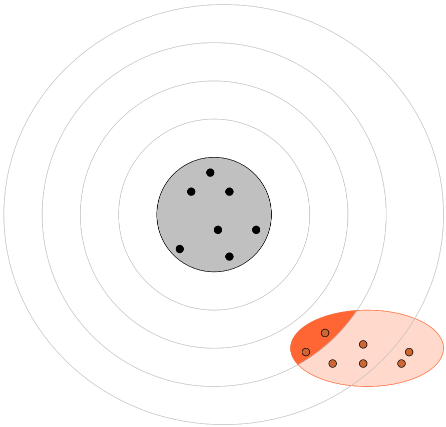

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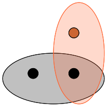
Revision

$\varphi = (b \wedge \neg m) \vee (\neg b \wedge m)$
$\mu = b$

Revision

$\varphi = (b \land \neg m) \lor (\neg b \land m)$ $\quad mod(\varphi) = \{10, 01\}$
$\mu = b$ $\quad mod(\mu) = \{10, 11\}$



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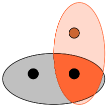
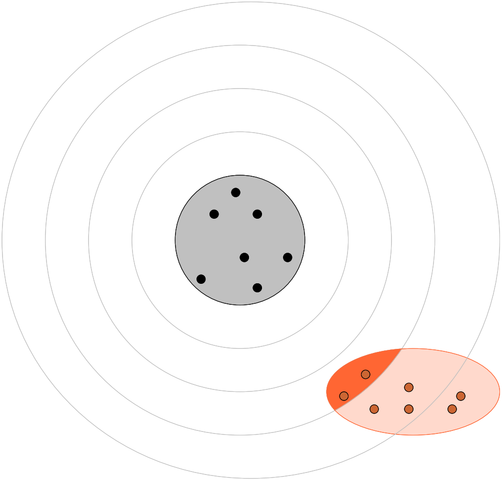


$\varphi \circ \mu = b \wedge \neg m$

Update

Update In belief update the new piece of evidence denotes a change in the world. The world is dynamic, and these (observed) changes modify the beliefs of the agent.



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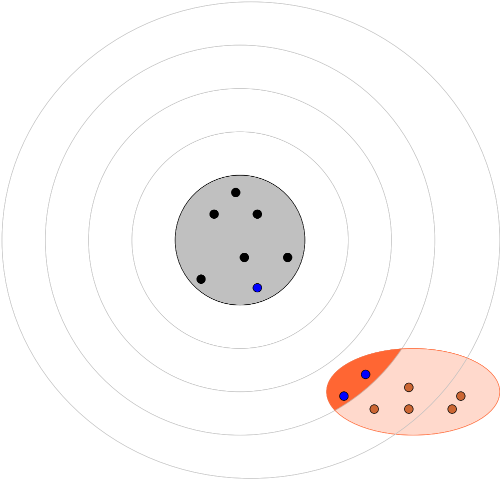

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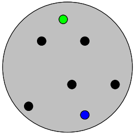

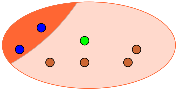

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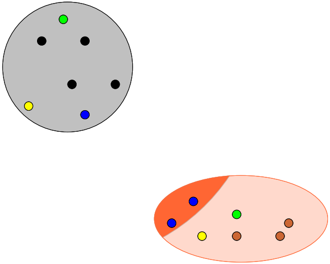

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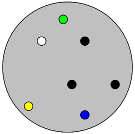

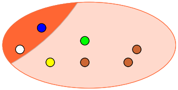

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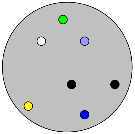

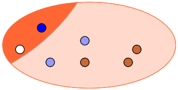

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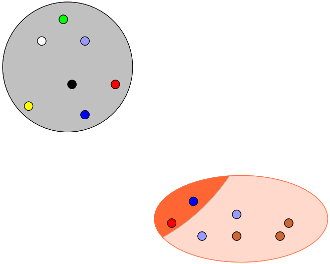

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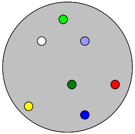

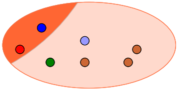

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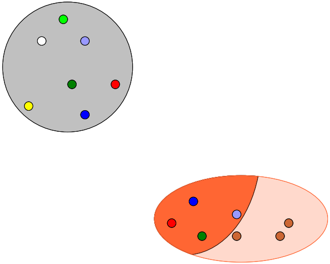

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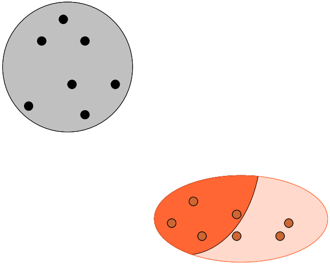

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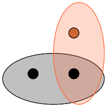

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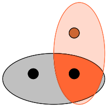

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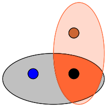

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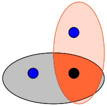

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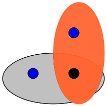

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Confluence

g: german car
e: expensive car
s: sport car

$$\varphi_1 = \neg g \land \neg e \land s \qquad mod(\varphi_1) = \{001\}$$
$$\varphi_2 = (g \land e \land s) \lor (\neg g \land \neg e \land s) \qquad mod(\varphi_2) = \{001, 111\}$$

$$\mu = \neg(g \land \neg e \land s) \qquad mod(\mu) = \mathcal{W} \setminus \{101\}$$

- Belief/Goal Merging:
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- Confluence:
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  - Unperfectly known goals
  - Potential evolution

Revision [Alchourrón-Gärdenfors-Makinson 85]

(R1) $\varphi \circ \mu \vdash \mu$
(R2) If $\varphi \wedge \mu \nvdash \bot$ then $\varphi \circ \mu \equiv \varphi \wedge \mu$
(R3) If $\mu \nvdash \bot$ then $\varphi \circ \mu \nvdash \bot$
(R4) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$ then $\varphi_1 \circ \mu_1 \equiv \varphi_2 \circ \mu_2$
(R5) $(\varphi \circ \mu) \wedge \phi \vdash \varphi \circ (\mu \wedge \phi)$
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A faithful assignment is a function mapping each base φ to a pre-order $\leq_\varphi$ over interpretations such that:

- If $\omega \models \varphi$ and $\omega' \models \varphi$, then $\omega \simeq_\varphi \omega'$
- If $\omega \models \varphi$ and $\omega' \not\models \varphi$, then $\omega <_\varphi \omega'$
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Theorem (Katsuno-Mendelzon 91a)

An operator $\circ$ is a revision operator (ie. it satisfies (R1)-(R6)) if and only if there exists a faithful assignment that maps each base φ to a total pre-order $\leq_\varphi$ such that

$$mod(\varphi \circ \mu) = \min(mod(\mu), \leq_\varphi).$$

Update [Kastuno-Mendelzon 91b]

(U1) $\varphi \diamond \mu \vdash \mu$
(U2) If $\varphi \vdash \mu$, then $\varphi \diamond \mu \equiv \varphi$
(U3) If $\varphi \nvdash \bot$ and $\mu \nvdash \bot$ then $\varphi \diamond \mu \nvdash \bot$
(U4) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$ then $\varphi_1 \diamond \mu_1 \equiv \varphi_2 \diamond \mu_2$
(U5) $(\varphi \diamond \mu) \wedge \phi \vdash \varphi \diamond (\mu \wedge \phi)$
(U6) If $\varphi \diamond \mu_1 \vdash \mu_2$ and $\varphi \diamond \mu_2 \vdash \mu_1$, then $\varphi \diamond \mu_1 \equiv \varphi \diamond \mu_2$
(U7) If φ is a complete formula, then $(\varphi \diamond \mu_1) \wedge (\varphi \diamond \mu_2) \vdash \varphi \diamond (\mu_1 \vee \mu_2)$
(U8) $(\varphi_1 \vee \varphi_2) \diamond \mu \equiv (\varphi_1 \diamond \mu) \vee (\varphi_2 \diamond \mu)$
(U9) If φ is a complete formula and $(\varphi \diamond \mu) \wedge \phi \nvdash \bot$, then $\varphi \diamond (\mu \wedge \phi) \vdash (\varphi \diamond \mu) \wedge \phi$

Theorem

An update operator $\diamond$ satisfies (U1)-(U8) if and only if there exists a faithful assignment that maps each interpretation ω to a partial pre-order $\leq_\omega$ such that

$$mod(\varphi \diamond \mu) = \bigcup_{\omega \models \varphi} \min(mod(\mu), \leq_\omega)$$

Update [Kastuno-Mendelzon 91b]

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(U2) If $\varphi \vdash \mu$, then $\varphi \diamond \mu \equiv \varphi$
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(U7) If φ is a complete formula, then $(\varphi \diamond \mu_1) \wedge (\varphi \diamond \mu_2) \vdash \varphi \diamond (\mu_1 \vee \mu_2)$
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(U9) If φ is a complete formula and $(\varphi \diamond \mu) \wedge \phi \nvdash \bot$, then $\varphi \diamond (\mu \wedge \phi) \vdash (\varphi \diamond \mu) \wedge \phi$

Theorem

An update operator $\diamond$ satisfies (U1)-(U5), (U8) and (U9) if and only if there exists a faithful assignment that maps each interpretation ω to a total pre-order $\leq_\omega$ such that

$$mod(\varphi \diamond \mu) = \bigcup_{\omega \models \varphi} \min(mod(\mu), \leq_\omega)$$

Merging [Konieczny-Pino-Pérez 99]

(IC0) $\triangle_\mu(\Psi) \vdash \mu$

(IC1) If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent

(IC2) If $\bigwedge \Psi$ is consistent with μ, then $\triangle_\mu(\Psi) \equiv \bigwedge \Psi \wedge \mu$

(IC3) If $\Psi_1 \equiv \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(\Psi_1) \equiv \triangle_{\mu_2}(\Psi_2)$

(IC4) If $\varphi_1 \vdash \mu$ and $\varphi_2 \vdash \mu$, then $\triangle_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_1$ is consistent if and only if $\triangle_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_2$ is consistent

(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \vdash \triangle_\mu(\Psi_1 \sqcup \Psi_2)$

(IC6) If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$

(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Psi)$

(IC8) If $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \vdash \triangle_{\mu_1}(\Psi)$

Merging [Konieczny-Pino-Pérez 99]

A *syncretic assignment* is a function mapping each profile Ψ to a total pre-order $\leq_\Psi$ over interpretations such that:

- If $\omega \models \Psi$ and $\omega' \models \Psi$, then $\omega \simeq_\Psi \omega'$
- If $\omega \models \Psi$ and $\omega' \not\models \Psi$, then $\omega <_\Psi \omega'$
- If $\Psi_1 \equiv \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$
- $\forall \omega \models \varphi \; \exists \omega' \models \varphi' \; \omega' \leq_{\{\varphi\} \sqcup \{\varphi'\}} \omega$
- If $\omega \leq_{\Psi_1} \omega'$ and $\omega \leq_{\Psi_2} \omega'$, then $\omega \leq_{\Psi_1 \sqcup \Psi_2} \omega'$
- If $\omega <_{\Psi_1} \omega'$ and $\omega \leq_{\Psi_2} \omega'$, then $\omega <_{\Psi_1 \sqcup \Psi_2} \omega'$

Theorem

An operator $\triangle$ is an IC merging operator if and only if there exists a syncretic assignment that maps each profile Ψ to a total pre-order $\leq_\Psi$ such that

$$mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$$

Revision vs Update

Proposition

If $\circ$ is a revision operator (i.e. it satisfies (R1)-(R6)), then the operator $\diamond$ defined by:

$$\varphi \ \diamond \ \mu = \bigvee_{\omega \models \varphi} \varphi_\omega \ \circ \ \mu$$

is an update operator that satisfies (U1)-(U9).
Moreover, for each update operator $\diamond$, there exists a revision operator $\circ$ such that the previous equation holds.

Revision vs Merging

Proposition

If $\triangle$ is an IC merging operator (it satisfies (IC0-IC8)), then the operator $\circ$, defined as $\varphi \circ \mu = \triangle_\mu(\varphi)$, is an AGM revision operator (it satisfies (R1-R6)).

Confluence

Confluence

Confluence

An operator $\Diamond$ is a confluence operator if it satisfies the following properties:

(UC0) $\Diamond_\mu(\Psi) \vdash \mu$

(UC1) If μ is consistent and Ψ is p-consistent, then $\Diamond_\mu(\Psi)$ is consistent

(UC2) If Ψ is complete, Ψ is consistent and $\bigwedge \Psi \vdash \mu$, then $\Diamond_\mu(\Psi) \equiv \bigwedge \Psi$

(UC3) If $\Psi_1 \equiv \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Diamond_{\mu_1}(\Psi_1) \equiv \Diamond_{\mu_2}(\Psi_2)$

(UC4) If φ_1 and φ_2 are complete formulae and $\varphi_1 \vdash \mu$, $\varphi_2 \vdash \mu$, then $\Diamond_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_1$ is consistent if and only $\Diamond_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_2$ is consistent

(UC5) $\Diamond_\mu(\Psi_1) \wedge \Diamond_\mu(\Psi_2) \vdash \Diamond_\mu(\Psi_1 \sqcup \Psi_2)$

(UC6) If Ψ_1 and Ψ_2 are complete profiles and $\Diamond_\mu(\Psi_1) \wedge \Diamond_\mu(\Psi_2)$ is consistent, then $\Diamond_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Diamond_\mu(\Psi_1) \wedge \Diamond_\mu(\Psi_2)$

(UC7) $\Diamond_{\mu_1}(\Psi) \wedge \mu_2 \vdash \Diamond_{\mu_1 \wedge \mu_2}(\Psi)$

(UC8) If Ψ is a complete profile and if $\Diamond_{\mu_1}(\Psi) \wedge \mu_2$ is consistent then $\Diamond_{\mu_1 \wedge \mu_2}(\Psi) \vdash \Diamond_{\mu_1}(\Psi) \wedge \mu_2$

(UC9) $\Diamond_\mu(\Psi \sqcup \{\varphi \vee \varphi'\}) \equiv \Diamond_\mu(\Psi \sqcup \{\varphi\}) \vee \Diamond_\mu(\Psi \sqcup \{\varphi'\})$

State

Definition

- A multi-set of interpretations e will be called a *state*.
- If $\Psi = \{\varphi_1, \ldots, \varphi_n\}$ is a profile and $e = \{\omega_1, \ldots, \omega_n\}$ is a state such that $\omega_i \models \varphi_i$ for each i, we say that e is a state of the profile Ψ, that will be denoted by $e \models \Psi$.
- If $e = \{\omega_1, \ldots, \omega_n\}$ is a state, we define the profile Ψ_e by putting $\Psi_e = \{\varphi_{\{\omega_1\}}, \ldots, \varphi_{\{\omega_n\}}\}$

Lemma

If $\Diamond$ satisfies (UC3) and (UC9) then $\Diamond$ satisfies the following

$$\Diamond_\mu(\Psi) \equiv \bigvee_{e \models \Psi} \Diamond_\mu(\Psi_e)$$

Representation theorem

A *distributed assignment* is a function mapping each state e to a total pre-order $\leq_e$ over interpretations such that:

- $\omega <_{\{\omega,\ldots,\omega\}} \omega'$ if $\omega' \neq \omega$
- $\omega \simeq_{\{\omega,\omega'\}} \omega'$
- If $\omega \leq_{e_1} \omega'$ and $\omega \leq_{e_2} \omega'$, then $\omega \leq_{e_1 \sqcup e_2} \omega'$
- If $\omega <_{e_1} \omega'$ and $\omega \leq_{e_2} \omega'$, then $\omega <_{e_1 \sqcup e_2} \omega'$

Theorem

An operator $\Diamond$ is a confluence operator if and only if there exists a distributed assignment that maps each state e to a total pre-order $\leq_e$ such that

$$mod(\Diamond_\mu(\Psi)) = \bigcup_{e \models \Psi} \min(mod(\mu), \leq_e) \tag{1}$$

Confluence vs Update and Merging

Proposition

If $\Diamond$ is a confluence operator (i.e. it satisfies (UC0-UC9)), then the operator $\diamond$, defined as $\varphi \diamond \mu = \Diamond_\mu(\varphi)$, is an update operator (i.e. it satisfies (U1-U9)).

Proposition

If $\triangle$ is an IC merging operator (i.e. it satisfies (IC0-IC8)) then the operator $\Diamond$ defined by

$$\Diamond_\mu(\Psi) = \bigvee_{e \models \Psi} \triangle_\mu(\Psi_e)$$

is a confluence operator (i.e. it satisfies (UC0-UC9)).
Moreover, for each confluence operator $\Diamond$, there exists a merging operator $\triangle$ such that the previous equation holds.

Example of confluence operators

- A distance d between interpretations
  - Drastic distance, Hamming (Dalal) distance, ...
- An aggregation function f
  - sum, leximax, ...
- $\omega \leq_e \omega'$ if and only if $d(\omega, e) \leq d(\omega', e)$, where ($e = \{\omega_1, \ldots, \omega_n\}$):

$$d(\omega, e) = f(d(\omega, \omega_1) \ldots, d(\omega, \omega_n))$$

- $mod(\Diamond_\mu(\Psi)) = \bigcup_{e \models \Psi} \min(mod(\mu), \leq_e)$

Example

Let $\Psi = \{\varphi_1, \varphi_2\}$ and μ :

$mod(\mu) = \mathcal{W} \setminus \{101\}$
$mod(\varphi_1) = \{001\}$
$mod(\varphi_2) = \{001, 111\}$

The corresponding states are:

$e_1 = \{001, 001\}$
$e_2 = \{001, 111\}$

$\mathcal{W}$	001	111	e_1		e_2		$\Diamond_\mu^{d_H,\Sigma}$	$\Diamond_\mu^{d_H,\mathrm{Gmax}}$
			Σ	Gmax	Σ	Gmax		
000	1	3	2	11	4	31		
001	0	2	**0**	**00**	**2**	20	$\times$	$\times$
010	2	2	4	22	4	22		
011	1	1	2	11	**2**	**11**	$\times$	$\times$
100	2	2	4	22	4	22		
101	1	1	2	11	2	11		
110	3	1	6	33	4	31		
111	2	0	4	22	**2**	20	$\times$	

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100	2	2	4	22	4	22		
101	1	1	2	11	2	11		
110	3	1	6	33	4	31		
111	2	0	4	22	**2**	20	$\times$	

$$mod(\Diamond_\mu^{d_H,\Sigma}(\Psi)) = \{001, 011, 111\}$$

Example

Let $\Psi = \{\varphi_1, \varphi_2\}$ and μ :

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			Σ	Gmax	Σ	Gmax		
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001	0	2	**0**	**00**	**2**	20	$\times$	$\times$
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011	1	1	2	11	**2**	**11**	$\times$	$\times$
100	2	2	4	22	4	22		
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110	3	1	6	33	4	31		
111	2	0	4	22	**2**	20	$\times$	

$$mod(\Diamond_{\mu}^{d_H,\Sigma}(\Psi)) = \{001, 011, 111\}$$

$$mod(\Diamond_{\mu}^{d_H,\text{Gmax}}(\Psi)) = \{001, 011\}$$

Conclusion

- Confluence operators
- Pointwise merging
- Negotiation
- Belief vs Goal aggregation