Three-sided stable matchings with cyclic preferences and the kidney exchange problem

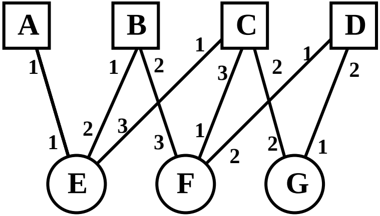
Péter Biró and Eric McDermid
Department of Computing Science
University of Glasgow
{pbiro,mcdermid}@dcs.gla.ac.uk

COMSOC 2008
Liverpool
5 September 2008

University of Glasgow

Stable marriage problem by Gale and Shapley [1962]

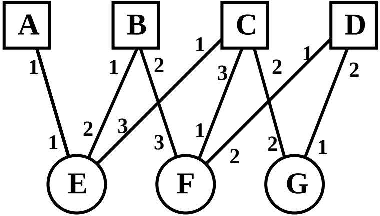
"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

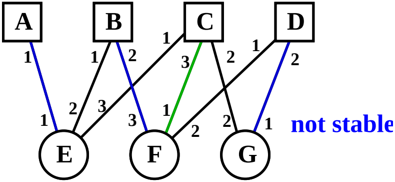
Stable marriage problem by Gale and Shapley [1962]

"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

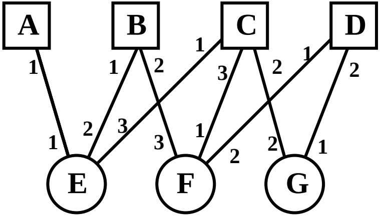
A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



(C,F) blocking pair

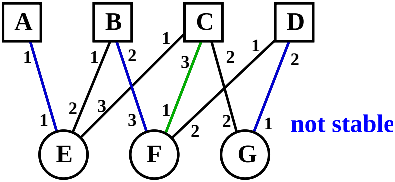
Stable marriage problem by Gale and Shapley [1962]

“College admission and the stability of marriage”



“Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner.”

A set of marriages is stable, if there is no “blocking pair”: a man and a woman who are not married to each other but prefer each other to their actual mates.

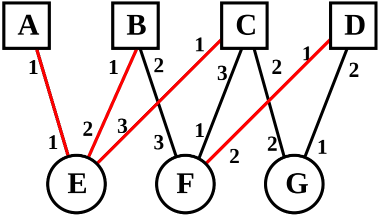


(C,F) blocking pair

Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time.

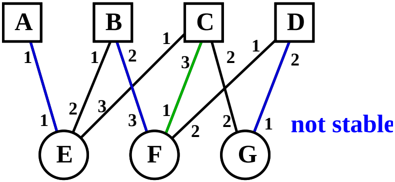
Stable marriage problem by Gale and Shapley [1962]

“College admission and the stability of marriage”



“Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner.”

A set of marriages is stable, if there is no “blocking pair”: a man and a woman who are not married to each other but prefer each other to their actual mates.

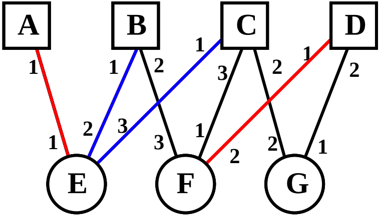


(C,F) blocking pair

Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time.

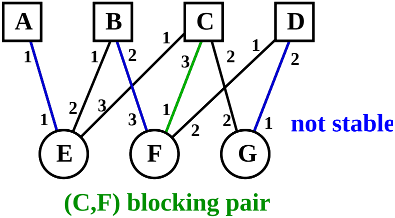
Stable marriage problem by Gale and Shapley [1962]

"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

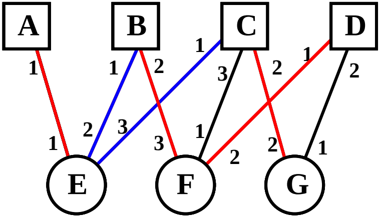
A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time.

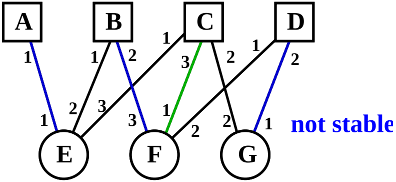
Stable marriage problem by Gale and Shapley [1962]

"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.

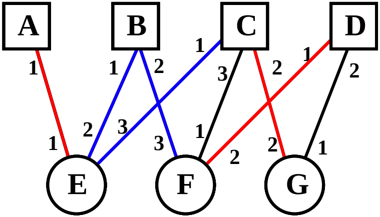


(C,F) blocking pair

Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time.

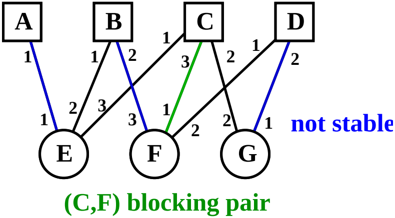
Stable marriage problem by Gale and Shapley [1962]

"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

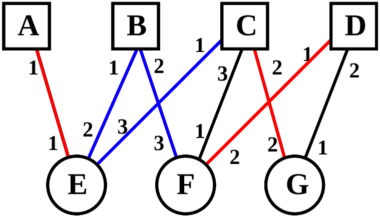
A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time.

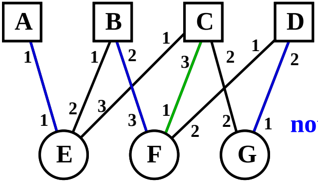
Stable marriage problem by Gale and Shapley [1962]

"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



not stable

(C,F) blocking pair

Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

Example for computational issues 1.: couples

National Residence Matching Program
to allocate junior doctors to hospitals in the U.S. since 1952.

Couples can submit joint preference lists...

Example for computational issues 1.: couples

National Residence Matching Program
to allocate junior doctors to hospitals in the U.S. since 1952.

Couples can submit joint preference lists...

Roth (1984): Stable matching may not exist.

Example for computational issues 1.: couples

National Residence Matching Program
to allocate junior doctors to hospitals in the U.S. since 1952.

Couples can submit joint preference lists...

Roth (1984): Stable matching may not exist.

Ronn (1990): The related decision problem is NP-hard.

Example for computational issues 1.: couples

National Residence Matching Program
to allocate junior doctors to hospitals in the U.S. since 1952.

Couples can submit joint preference lists...

Roth (1984): Stable matching may not exist.

Ronn (1990): The related decision problem is NP-hard.

McDermid (2008): It is NP-hard even for "consistent" couples.

Example for computational issues 1.: couples

National Residence Matching Program
to allocate junior doctors to hospitals in the U.S. since 1952.

Couples can submit joint preference lists...

Roth (1984): Stable matching may not exist.

Ronn (1990): The related decision problem is NP-hard.

McDermid (2008): It is NP-hard even for "consistent" couples.

A heuristic is used in the application.

Example for computational issues 2.: lower quotas

Higher education admission in Hungary since 1985

The colleges (studies) can have lower quota as well...

Example for computational issues 2.: lower quotas

Higher education admission in Hungary since 1985

The colleges (studies) can have lower quota as well...

B.-Fleiner-Irving-Manlove (2008): A stable matching may not exist, and the related problem is NP-complete.

Example for computational issues 2.: lower quotas

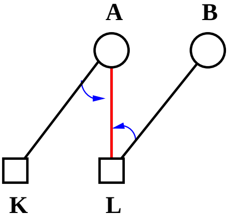
Higher education admission in Hungary since 1985

The colleges (studies) can have lower quota as well...

B.-Fleiner-Irving-Manlove (2008): A stable matching may not exist, and the related problem is NP-complete.

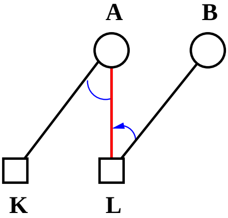
A natural heuristic is used in the application.

Example for computational issues 3.: ties, maximum size



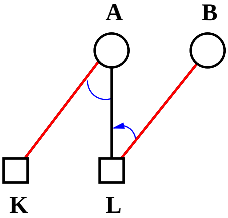
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.

Example for computational issues 3.: ties, maximum size



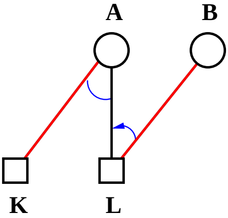
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

Example for computational issues 3.: ties, maximum size



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

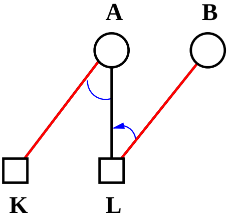
Example for computational issues 3.: ties, maximum size



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching. (PERFECT SMTI: same problem for perfect matching.)

Example for computational issues 3.: ties, maximum size

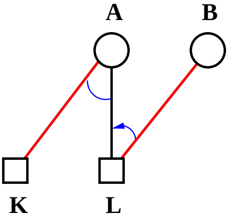


- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching. (PERFECT SMTI: same problem for perfect matching.)

Manlove *et al.* (2002): PERFECT SMTI is NP-complete.

Example for computational issues 3.: ties, maximum size



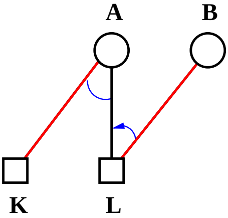
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching. (PERFECT SMTI: same problem for perfect matching.)

Manlove *et al.* (2002): PERFECT SMTI is NP-complete.

Király (2008): Polynomial-time $\frac{5}{3}$-approximation. (ESA best paper)

Example for computational issues 3.: ties, maximum size



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

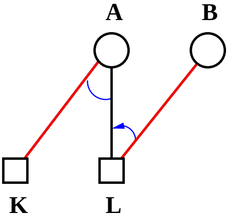
MAX SMTI: The problem of finding a maximum size weakly stable matching. (PERFECT SMTI: same problem for perfect matching.)

Manlove *et al.* (2002): PERFECT SMTI is NP-complete.

Király (2008): Polynomial-time $\frac{5}{3}$-approximation. (ESA best paper)

McDermid (2008): Polynomial-time $\frac{3}{2}$-approximation.

Example for computational issues 3.: ties, maximum size



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching. (PERFECT SMTI: same problem for perfect matching.)

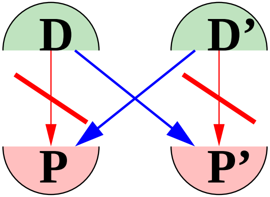
Manlove *et al.* (2002): PERFECT SMTI is NP-complete.

Király (2008): Polynomial-time $\frac{5}{3}$-approximation. (ESA best paper)

McDermid (2008): Polynomial-time $\frac{3}{2}$-approximation.

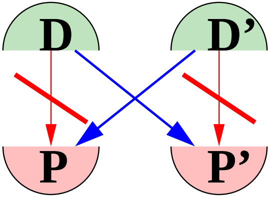
Application: Scottish Foundation Allocation Scheme (SFAS)
2006-2007: 781 residents, 53 hospitals, total capacity 789.
Maximum size weakly stable matching found was of size 744.

Another application: Kidney exchange problem



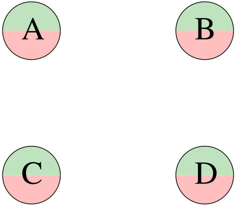
Given two incompatible patient-donor pairs (blood-type or tissue-type incompatibility). If they are compatible across, then a pairwise exchange is possible between them.

Another application: Kidney exchange problem

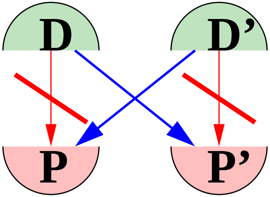


Given two incompatible patient-donor pairs (blood-type or tissue-type incompatibility). If they are compatible across, then a pairwise exchange is possible between them.

Let these pairs be the vertices of a nonbipartite graph.



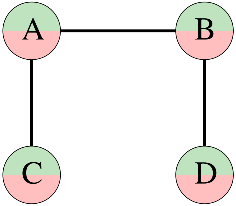
Another application: Kidney exchange problem



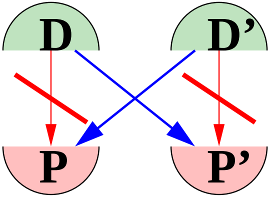
Given two incompatible patient-donor pairs (blood-type or tissue-type incompatibility). If they are compatible across, then a pairwise exchange is possible between them.

Let these pairs be the vertices of a nonbipartite graph.

Where two nodes are linked if the exchange is possible between the corresponding pairs.



Another application: Kidney exchange problem

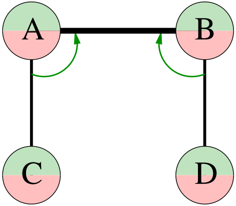


Given two incompatible patient-donor pairs (blood-type or tissue-type incompatibility). If they are compatible across, then a pairwise exchange is possible between them.

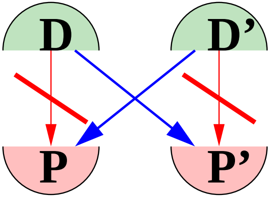
Let these pairs be the vertices of a nonbipartite graph.

Where two nodes are linked if the exchange is possible between the corresponding pairs.

The **weights** of the edges and the preferences come from the immunological factors.



Another application: Kidney exchange problem

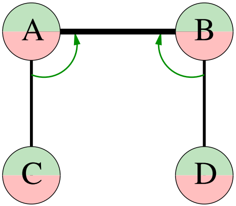


Given two incompatible patient-donor pairs (blood-type or tissue-type incompatibility). If they are compatible across, then a pairwise exchange is possible between them.

Let these pairs be the vertices of a nonbipartite graph.

Where two nodes are linked if the exchange is possible between the corresponding pairs.

The **weights** of the edges and the preferences come from the immunological factors.



What is the criteria of a matching to be “good”?

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	**yes**		
	hard to find?			
stable	does exist?			
	hard to find?			

Complexity of exchange problems

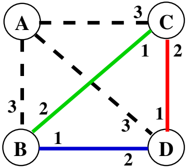
		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	**P**		
stable	does exist?			
	hard to find?			

Edmonds (1967): Polynomial time algorithms for maximum size / maximum weight matching problem.

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	**may not**		
	hard to find?			

stable pairwise exchange = stable roommates



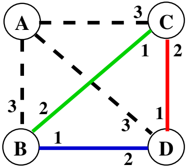
Gale and Shapley (1962):

Stable matching may not exist!

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	**P**		

stable pairwise exchange = stable roommates

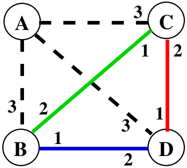


Gale and Shapley (1962):
Stable matching may not exist!
Irving (1985): A stable matching can be found in linear time, if one exists.

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	P		

stable pairwise exchange = stable roommates



Gale and Shapley (1962): Stable matching may not exist!
Irving (1985): A stable matching can be found in linear time, if one exists.

Abraham-B.-Manlove (2006): The problem of minimising the number of blocking pairs is NP-hard (and not apprimable within $n^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$, unless P=NP).

Complexity of exchange problems

		exchanges		
		pairwise	**2-3-way**	
maximum size/weight	does exist?	yes	**yes**	
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	P		

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	
maximum size/weight	does exist?	yes	yes	
	hard to find?	P	**NPc**	
stable	does exist?	may not		
	hard to find?	P		

Abraham et al.; B.-Manlove-Rizzi: The problem of finding a maximum size/weight 2-3-way exchange is NP-complete (APX-hard).

B.-Manlove-Rizzi: An $O(2^{\frac{m}{2}})$-time exact algorithm.

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	
maximum size/weight	does exist?	yes	yes	
	hard to find?	P	NPc	
stable	does exist?	may not	**may not**	
	hard to find?	P	**NPc**	

Abraham et al.; B.-Manlove-Rizzi: The problem of finding a maximum size/weight 2-3-way exchange is NP-complete (APX-hard).

B.-Manlove-Rizzi: An $O(2^{\frac{m}{2}})$-time exact algorithm.

B.-McDermid (2008): Stable 2-3-way exchange may not exist, and the related problem is NP-complete, even for tripartite graphs. (equivalent to stable 3D matching with cyclic preferences!)

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	**unbounded**
maximum size/weight	does exist?	yes	yes	**yes**
	hard to find?	P	NPc	
stable	does exist?	may not	may not	
	hard to find?	P	NPc	

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	unbounded
maximum size/weight	does exist?	yes	yes	yes
	hard to find?	P	NPc	**P**
stable	does exist?	may not	may not	
	hard to find?	P	NPc	

e.g. Abraham et al.; B.-Manlove-Rizzi: The problem of finding a maximum size/weight (unbounded) exchange is P-time solvable.

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	unbounded
maximum size/weight	does exist?	yes	yes	yes
	hard to find?	P	NPc	P
stable	does exist?	may not	may not	**yes**
	hard to find?	P	NPc	**P**

e.g. Abraham et al.; B.-Manlove-Rizzi: The problem of finding a maximum size/weight (unbounded) exchange is P-time solvable.

Scarf-Shapley (1972): Stable exchange always exists ("the core of a houseswapping game is nonempty"). A stable solution can be found by the Top Trading Cycle algorithm of Gale.

Thank you for your attention!

Further information at

www.optimalmatching.com