Strategy-proof rules for the choice of multiattribute alternatives

Salvador Barberà

UNIVERSITAT AUTONOMA DE BARCELONA
and
BARCELONA GRADUATE SCHOOL OF ECONOMICS

THE SETUP

- A set of alternatives
- $N = \{1, 2, ..., n\}$ is a set of agents
- Preferences will be always complete, reflexive, transitive binary relations on A
- R will stand for the set of all possible preferences on A
- D_i represents the set of preferences which are admissible for agent i
- A social choice function on the domain $\times_{i \in N} D_i \subset R^n$ is a function $f : \times_{i \in N} D_i \rightarrow A$
- Elements $\succeq_N \in \times_{i \in N} D_i$ are called preference profiles. Sometimes we will use the notation $\succeq_N = (\succeq_C, \succeq_{-C}) \in \times_{i \in N} D_i$ when we want to stress the role of a coalition $C \subset N$. Then $\succeq_C \in \times_{i \in C} D_i$ and $\succeq_{-C} \in \times_{i \in N \setminus C} D_i$ denote the preferences of agents in C and in $N \setminus C$, respectively.
- For any $x \in A$ and $\succeq_i \in D_i$, define the lower contour set of $\succeq_i$ at x as $L(x, \succeq_i) = \{y \in A : x \succeq_i y\}$. Let P_i be the strict part of $\succeq_i$. Then the strict lower contour set at x is $\bar{L}(x, \succeq_i) = \{y \in A : xP_iy\}$.

Manipulation and strategy-proofness I

Definition

A social choice function $f : \times_{i \in N} D_i \rightarrow A$ is manipulable iff there exists some preference profile $\succeq_N = (\succeq_1, ..., \succeq_n) \in \times_{i \in N} D_i$, and some preference $\succeq' \in D_i$, such that

$$f(\succeq_1, ..., \succeq_i', ... \succeq_n) \succ_i f(\succeq_1, ..., \succeq_i, ... \succeq_n)$$

The function f is **strategy-proof** iff it is not manipulable.

Manipulation and strategy-proofness II

Definition

A social choice function f is group manipulable on $\times_{i\in N} D_i$ at $\succeq_N \in \times_{i\in N} D_i$ if there exists a coalition C and $\succeq'_C \in \times_{i\in C} D_i$ ($\succeq'_i \neq \succeq_i$ for any $i \in C$) such that $f(\succeq'_C, \succeq_{-C}) P_i f(\succeq_N)$ for all $i \in C$. We say that f is individually manipulable if there exists a possible manipulation where coalition C is a singleton.

Definition

A social choice function f is group strategy-proof on $\times_{i\in N} D_i$ if f is not group manipulable for any $\succeq_N \in \times_{i\in N} D_i$. Similarly, f is strategy-proof if it is not individually manipulable.

Manipulation and strategy-proofness III

Gibbard-Satterhwaite Theorem

Let f be a voting scheme whose range contains more than two alternatives. Then f is either dictatorial or manipulable.

One way out: restricted preference domains
The case of linearly ordered sets of alternatives

- Finite set of alternatives linearly ordered according to some criterion.
- Preference of agents over alternatives is single-peaked.
  - Each agent has a single preferred alternative $\tau(\succeq_i)$
  - If alternative z is between x and $\tau(\succeq_i)$, then z is preferred to x
- Consider the case where the number of alternatives is finite, and identify them with the integers in an interval $[a, b] = \{a, a+1, ..., b\} \equiv A$ (*Moulin*(1980*a*)).

Option sets: an alternative definition of strategy-proofness

Definition

Given a social choice function $f : \times_{i \in N} D_i$, **the options of agent** i at profile $\succeq_N = (\succeq_1, ..., \succeq_i, ... \succeq_n) \in \times_{i \in N} D_i$ are defined to be the set of alternatives

$$\theta_{\times_{i \in N} D_i}(i, \succeq_N) = \left\{ x \in A | \exists \succeq_i' \in D_i \ \ s.t. \ \ f(\succeq_{-i}, \succeq_i') = x \right\}$$

Remark

f is strategy-proof on $\times_{i \in N} D_i$ iff, for all $\succeq_{\times_{i \in N} D_i}$, all i,
$f(\succeq_N) = C(\succeq_i, \theta_{\times_{i \in N} D_i}(i, \succeq_N))$

- **Example 1** *There are three agents. Allow each one to vote for her preferred alternative. Choose the median of the three voters.*
- **Example 2** *There are two agents. We fix an alternative p in $[a, b]$. Agents are asked to vote for their best alternatives, and the median of p, τ_1 and τ_2 is the outcome.*
- **Example 3** *For any number of agents, ask each one for their preferred alternative and choose the smallest.*
- *Notice that all three rules are anonymous and strategy-proof.*

- **Example 4** *There are two agents. Fix two alternatives w_1 and w_2, ($w_1 \leq w_2$). If agent 1 votes for any alternative in $[w_1, w_2]$, the outcome is 1's vote. If 1 votes for an alternative larger than w_2, the outcome is the median of w_1 and the votes of both agents.*

That rule can also be described in other ways. One way is the following. Assign values on the extended real line to the sets $\{1\}, \{2\}, \{1,2\}$. Specifically, let $a_1 = w_1, a_2 = w_2, a_{1,2} = a$ (the lowest value in the range). Now, define the rule as choosing

$$f(\succeq_1, \succeq_2) = inf_{S \in \{\{1,2\},\{1\}\{2\}\}} \left[sup_{i \in S} \left(a_s, \tau(\succeq_i) \right) \right]$$

Structure of strategy-proof social choice functions

For each coalition $S \in 2^N \setminus \emptyset$, fix an alternative a_s. Define a social choice function in a such a way that, for each preference profile $(\succeq_1, ..., \succeq_n)$,

$$f(\succeq_1, ..., \succeq_n) = inf_{S \subset N} [sup_{i \in S} (a_s, \tau(\succeq_i))]$$

The functions so defined will be called **generalized median voter schemes.**

Theorem

(Moulin, 1980a) A social choice function on profiles of single-peaked preferences over a linearly ordered set is strategy-proof if and only if it is a generalized median voter scheme.

Theorem

(Moulin, 1980a) An anonymous social choice function on profiles of single-peaked preferences over a linearly ordered set is strategy-proof if and only if there exist $n+1$ points $p_1, \ldots, p_{n+1}$ in A (called the phantom voters), such that, for all profiles,

$$f(\succeq_1, \ldots, \succeq_n) = med(p_1, \ldots, p_{n+1}; \tau(\succeq_1), \ldots, \tau(\succeq_n))$$

Definition

A left (resp. right) coalition system on the integer interval $B = [a, b]$ is a correspondence C assigning to every $\alpha \in B$ a collection of non-empty coalitions $C(\alpha)$, satisfying the following requirements:

1. if $c \in C(\alpha)$ and $c \subset c'$, then $c' \in C(\alpha)$;
2. if $\beta > \alpha$ (resp. $\beta < \alpha$) and $c \in C(\alpha)$, then $c \in C(\alpha)$, then $c \in C(\beta)$; and
3. $C(b) = 2^N \setminus \emptyset$ (resp. $C(a) = 2^N \setminus \emptyset$).

If we denote left coalition systems by L, and right coalition systems by $\Re$.

Definition

Given a left (resp. right) coalition system L (resp. $\Re$) on $B = [a, b]$, its associated generalized median voter scheme is defined so that, for all profiles $(\succeq_1, ..., \succeq_n)$

$$f(\succeq_1, ..., \succeq_n) = \beta \textit{ iff } \{i|\tau(\succeq_i) \leq \beta\} \in L(\beta)$$

and

$$\{i|\tau(\succeq_i) \leq \beta - 1\} \notin L(\beta - 1)$$

Example 5 *Let* $B = [1, 2, 3], N = 1, 2, 3.$*Let*

$$L(1) = L(2) = \left\{ S \in 2^N \backslash \emptyset : \#S \geq 2 \right\}$$

Define f to be the generalized median voter scheme associated with L. Then, for example

$$f(1, 2, 3) = 2$$
$$f(3, 2, 3) = 3$$
$$f(1, 3, 1) = 1$$

This is , in fact, the median voter rule.

Example 6 *Let now* $B = [1, 2, 3, 4]$, $N = 1, 2, 3$.*Consider the right coalition system given by*

$$\Re(4) = \Re(3) = \Re(2) = \left\{ C \in 2^N \backslash \emptyset : 1 \in C \textit{ and } 2 \in C \right\}$$

In that case, both 1 and 2 are essential to determine the outcome. Let g be the generalized median voting scheme associated with $\Re$. *Here are some of the values of g:*

$$g(1, 4, 4) = 1$$
$$g(3, 3, 1) = 3$$
$$g(3, 2, 2) = 2$$

Strategy-proofness for generalized single-peaked domains I

- (Multi-dimensional social choices) Let K be a number of dimensions. Each dimension will stand for one characteristic that is relevant to the description of social alternatives. Allow for a finite set of admissible $B_k = [a_k, b_k]$ on each dimension $k \in [K]$. Now the set of alternatives can be represented as the Cartesian product $B = \prod_{k=1}^{K} B_k$. Sets like this B are called K-dimensional boxes. Representing the set of social alternatives as the set of elements in a K-dimensional box allows us to describe many interesting situations.
- (Single-peakedness) Every preference have a unique top (or ideal) and if z is between x and $\tau(_i)$, then z is preferred to x.
- (Betweenness) We endow the set B with the L_1 norm , letting, for each $\alpha \in B, ||\alpha|| = \sum_{k=1}^{K} |\alpha_k|$. Then, the minimal box containing two alternatives α and β is defined as $MB(\alpha, \beta) = \{\gamma \in B| \, \|\alpha - \beta\| = \|\alpha - \gamma\| + \|\gamma - \beta\|\}$.(Barberà, Gul, and Stacchetti (1993))

Strategy-proofness for generalized single-peaked domains II

We can interpret that z in "'between"' alternatives x and $\tau(_i)$, if $z \in MB(x, \tau(_i))$. Under this interpretation, the following is a natural extension of single-peakedness.

Definition

A preference $\succeq_i$ on B is generalized single-peaked iff for all distinct $\beta, \gamma \in B, \beta \in MB(\tau(\succeq_i), \gamma)$ implies that $\beta \succ_i \gamma$.

Strategy-proofness for generalized single-peaked domains III

(K-Dimensional) generalized median voter schemes on $B = \Pi_{k=1}^{K} B_k = \Pi_{k=1}^{K}[a_k, b_k]$ can be defined as follows:

Definition

Let L(resp. $\Re$) be a family of K left (resp. right) coalition systems, where each L_k (resp. $\Re_k$) is defined on $[a_k, b_k]$. The corresponding k-dimensional generalized median voter scheme is the one that, for all profiles of preferences on B, chooses

$$f(\succeq_1, ..., \succeq_n) = \beta \text{ } iff \text{ } \{i|\tau(\succeq_i) \leq \beta_k\} \in L_k(\beta_k)$$

and

$$\{i|\tau(_i) \leq \beta_{k-1}\} \notin L(\beta_{k-1}),$$

for all $k = 1, ..., K$

Strategy-proofness for generalized single-peaked domains IV

Example 7 *(Example of a generalized median voter scheme). Let* $B = [1, 2, 3] \times [1, 2, 3, 4], N = \{1, 2, 3\}$*. Let* L_1 *be as* L *in example 5. Let* $\Re_2$ *be as* $\Re$ *in example 6. Let* h *be the two-dimensional generalized median voter scheme associated to this coalition system. Then, for example,*

$$h((1,1),(2,4),(3,4)) = (2,1)$$
$$h((3,3),(2,3),(3,1)) = (3,3)$$
$$h((1,3),(3,2),(1,2)) = (1,2)$$

Strategy-proofness for generalized single-peaked domains V

Theorem

(Barberà, Gul, and Stacchetti (1993)). A social choice function f defined on the set of generalized single peaked preferences over a K-dimensional box, and respecting voters' sovereignty is strategy-proof iff it is a (K-dimensional) generalized median voter scheme.

A special case: voting by committees

- **Example 8** *(Barberà, Sonnennschein, and Zhou (1991)). Consider a club composed of N members, who are facing the possibility of choosing new members out of the set of K candidates. Are there any strategy-proof rules the club can use?*

Connection between the example and the n-dimensional model.

Constraints. A first approach I

- Many social decisions are subject to political or economic feasibility constraints.
- Different feasible alternatives may fulfill different requirements to degrees that are not necessarily compatible among themselves (Ex: fine arts vs a top quality kindergarden).
- Distinction between feasible and conceivable alternatives.

Constraints. A first approach II

Let Z be the set of feasible alternatives and let B be the minimal box containing Z.

Definition

A generalized median voter scheme f on B respects feasibility on $Z \subset B$ if $f(\succeq_1, ..., \succeq_n) \subset Z$ for all $(\succeq_1, ..., \succeq_n)$ such that $\tau(\succeq_i) \in Z$.

Definition

Let $Z \subset B$ and let f be a generalized median voter scheme on B, defined by the left coalition system L or, alternatively by the right coalition system $\Re$. Let $\alpha \notin Z$ and $S \subset Z$. We say that f has the intersection property for (α, S) iff for every selection $\mathrm{r}(\alpha_k)$ and $\mathrm{l}(\alpha_k)$ from the sets $\Re(\alpha_k)$ and $L(\alpha_k)$, respectively, we have

$$\cap_{\beta \in S} \left[\left(\cup_{k \in M^+(\alpha,\beta)} l(\alpha_k) \right) \cup \left(\cup_{k \in M^-(\alpha,\beta)} r(\alpha_k) \right) \right] \neq \emptyset$$

where $M^+(\alpha, \beta) = \{k \in K | \beta_k > \alpha_k\}$ and
$M^-(\alpha, \beta) = \{k \in K | \beta_k < \alpha_k\}$.
We will say that f satisfies the intersection property if it is does for every $(\alpha, S) \in (B - Z, 2^K)$.

Constraints. A first approach IV

Theorem

(Barberà, Massó, and Neme (1997)). Let f be a generalized median voter scheme on B, let $Z \subset B$, and f respect voters' sovereignty on Z. Then f preserves feasibility on Z if and only if satisfies the intersection property.

Denote by S_Z the set of all single peaked preferences with top on Z. Let f be an onto social function with domain S_Z^n and range Z.

Theorem

(Barberà, Massó, and Neme (1997)). If $f : S_Z^n \to Z$ is strategy proof, then f is a generalized median voter scheme.

Constraints. A first approach V

Theorem

(Barberà, Massó, and Neme (1997)). If $f : S_Z^n \to Z$ be an onto social choice function. Then f is strategy-proof on S_Z^n iff it is a generalized median voter scheme satisfying the intersection property.

Note.

The Gibbard-Satterhwaite Theorem is included as a corollary

Embedding alternatives in a grid

Gul's conjecture

Take any strategy-proof social choice function. There will always exist a method that identifies the alternatives in its range with some points in a grid, in such a way: (a) the rule is a generalized voter scheme, and (b) the preferences in the domain of the rule are single peaked for that embedding

- The intersection property is essential in allowing the very statement of the conjecture to have some meaning.(Barberà, Massó and Neme(1997))

The sequential inclusion condition

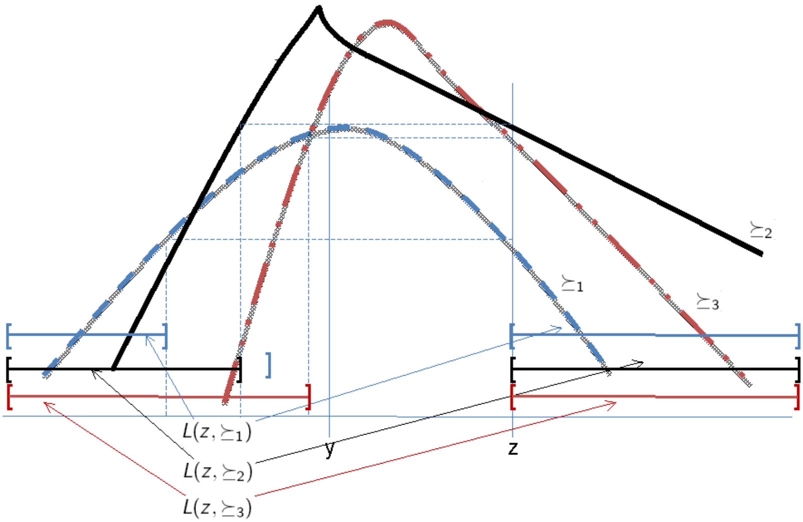
Definition

A preference profile $\succeq_N \in \times_{i \in N} D_i$ satisfies the sequential inclusion condition if for any pair $y, z \in A$ there exists an order of agents of $S = \{i : yP_i z\}$, S, say $1 < 2 < ... < s$, such that for all sequences $z_1, z_2, ..., z_{s-1}$ where $z_1 = z$ and $z_i \in L(z_{i-1}, \succeq_{i-1})$, for any $i = 2, ..., s-1$, we have that $[L(z_j, \succeq_j) \subset \bar{L}(y, \succeq_h)$ for all h, $j + 1 \leq h \leq s]$ for all $j = 1, ..., s-1$.
We say that a domain $\times_{i \in N} D_i$ satisfies the sequential inclusion condition if any preference profile in this domain satisfies it.

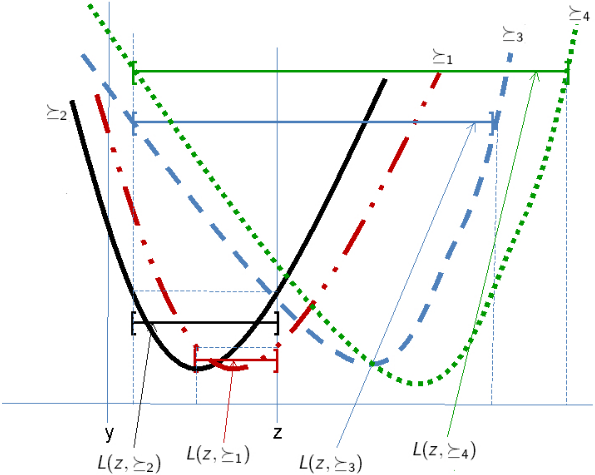
Theorem

Let $\times_{i \in N} D_i$ be a domain satisfying the sequential inclusion condition. Then, any strategy-proof social choice function is group strategy-proof.

Example:single-peaked preferences



Example: single-dipped preferences



Example: separable preferences and its subdomains - quota rules

- Consider the case $k = 2, n = 2$, $q = 1$. The following preferences are separable.

P_1	P_2
$(1, 0)$	$(0, 1)$
$(0, 0)$	$(0, 0)$
$(0, 0)$	$(0, 0)$
$(0, 1)$	$(1, 0)$

Indirect sequential inclusion I

Definition

For preferences $\succeq_i, \succeq_i' \in D_i$ and alternative $x \in A$, $\succeq_i'$ is a strict monotonic transformation of $\succeq_i$ at x if either $\succeq_i = \succeq_i'$ or else $\succeq_i'$ is such that for all $y \in A \setminus \{x\}$ such that $x_i \succeq_i y$, $xP_i'y$.

Lemma

Let f be a strategy-proof social choice function. For any $\succeq_N \in \times_{i \in N} D_i$, any $i \in N$, if $\succeq_i' \in D_i$ is a strict monotonic transformation of $\succeq_i$ at $f(\succeq_N)$ we have that $f(\succeq_N) = f(\succeq_i', \succeq_{-i})$.

Indirect sequential inclusion II

Definition

A preference profile $\succeq_N \in \times_{i \in N} D_i$ satisfies the indirect sequential inclusion condition if for each pair $y, z \in A$ there exists $\succeq'_N \in \times_{i \in N} D_i$ where $\succeq'_{N \setminus S} = \succeq_{N \setminus S}$ and $S = \{i \in N : yP_i z\}$, such that

1. for any $j \in S$, $\succeq'_j$ is a strict monotonic transformation of $\succeq_j$ at z.
2. for any $i \in S$ such that $yP_i z$, $yP'_i z$.
3. $\succeq'_N$ satisfies sequential inclusion for y,z.

We say that a domain $\times_{i \in N} D_i$ satisfies indirect sequential inclusion if this condition holds for $\succeq_N \in \times_{i \in N} D_i$.

Theorem

Let $\times_{i \in N} D_i$ be a domain satisfying indirect sequential inclusion. Then, any strategy-proof social choice function is group strategy-proof.

K-size group manipulation

Definition

A social choice function f is k-group strategy-proof on $\times_{i \in N} D_i$ if for any $\succeq_N \in \times_{i \in N} D_i$, there is no coalition $C \subseteq N$ with $\#C \leq k$ that manipulates f on $\times_{i \in N} D_i$ at $\succeq_N$.

Definition

A preference profile $\succeq_N \in \times_{i \in N} D_i$ satisfies the k-size sequential inclusion condition if for any pair $y, z \in A$ and any $K \subset S = \{i \in N : yP_i z\}$, where K has cardinality $k \leq s$, there exists an order of agents in K, say $1 < 2 < ... < k$, such that for all sequences $z_1, z_2, ..., z_{k-1}$ where $z_1 = z$ and $z_i \in L(z_{i-1}, \succeq_{i-1})$, for any $i = 2, ..., k-1$, we have that $[L(z_j, \succeq_j) \subset \bar{L}(y, \succeq_h)$ for all h, $j + 1 \leq h \leq k]$ for all $j = 1, ..., k-1$.
We say that a domain $\times_{i \in N} D_i$ satisfies the k-size sequential inclusion condition if any preference profile in this domain satisfies it.

Corollary

Let $\times_{i \in N} D_i$ be a domain satisfying the k-size sequential inclusion condition. Then, any strategy-proof social choice function is k-group strategy-proof.