Equal Representation in Two-tier Voting Systems*

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Abstract

The paper investigates how voting weights should be assigned to differently sized constituencies of an assembly. The one-person, one-vote principle is interpreted as calling for a priori equal indirect influence on decisions. The latter are elements of a one-dimensional convex policy space and may result from strategic behavior consistent with the median voter theorem. Numerous artificial constituency configurations, the EU and the US are investigated by Monte-Carlo simulations. *Penrose's square root rule*, which originally applies to preference-free dichotomous decision environments and holds only under very specific conditions, comes close to ensuring equal representation. It is thus more robust than previously suggested.

1 Introduction

The principle of "one person, one vote" is generally taken to be a cornerstone of democracy. It is not clear, however, how this principle ought to be operationalized in practice in terms of determining what are the ideal shares. This paper addresses this problem for *two-tier voting systems* that involve multiple constituencies of different population size. We concentrate on situations in which representatives of constituencies in the higher-level assembly vote as a block (as in the US Electoral College) or in which a single agent represents each constituency but is endowed with a number of votes that somehow reflect population size (as in the EU Council of Ministers). Both boil down to weighted voting.

Although it seems straightforward to allocate weights proportional to population sizes, this ignores the combinatorial properties of weighted voting, which often imply stark discrepancies between *voting weight* and actual *voting power*: In an assembly with simple majority rule and three representatives having weight 47, 43, and 10, all three possess exactly the same number of possibilities to form a winning coalition and hence the same a priori power. Moreover, direct proportionality disregards the possibly nonlinear relationship between population size and an individual's effect on the respective constituency's top-tier policy position.

The most well-known solution to this problem is the one first suggested by Penrose (1946). Starting from the ideal world in which only constituency

* This work is an abbreviated version of a paper forthcoming in *Social Choice and Welfare*. We thank M. Braham, M.J. Holler, participants of the workshop Voting Power & Procedures, Warwick, 2005, and two anonymous referees for their constructive comments.

membership[1] distinguishes voters, Penrose found that if members of any constituency are to have the same a priori chance to indirectly determine the outcome of top-tier decisions, then constituencies' voting weights need to be such that their power at the top-tier as measured by the *Penrose-Banzhaf index* (Penrose 1946; Banzhaf 1965) is proportional to the square root of the respective constituency's population size (also see Felsenthal and Machover 1998, sect. 3.4). This *square root rule* has recently become the benchmark for numerous studies of the EU Council of Ministers (see, e. g. Felsenthal and Machover 2001; 2004, Leech 2002) and it is at least a reference point for investigations concerning the US (see e. g. Gelman, Katz, and Bafumi 2004).

Applying the square root rule has, unfortunately, two weaknesses: First, Penrose's theorem critically depends on equiprobable 'yes' and 'no'-decisions by all voters (or at least a 'yes'-probability which is random and distributed independently across voters with mean exactly 0.5). If the 'yes'-probability is slightly lower or higher, or if it exhibits even minor dependence across voters – say, they are influenced by the same newspapers – then the square root rule may result in highly unequal representation (see Good and Mayer 1975 and Chamberlain and Rothschild 1981). Related empirical studies in fact have failed to confirm the predictions for average closeness of two-party elections which lie behind the square root rule (see Gelman, Katz, and Tuerlinckx 2002 and Gelman, Katz, and Bafumi 2004).

Second, rigorous justifications for using the square root rule as the benchmark have so far concerned only *preference-free binary voting*.[2] But real decisions are rarely binary, e. g., about *either* introducing a tax, building a road, accepting a candidate, introducing affirmative action, etc. *or* not. At least at intermediate levels there is a preference-driven compromise that involves *many* alternative tax levels, road attributes, suitable candidates, degrees of affirmative action, etc.

The first criticism has been addressed in the literature, at least in abstract normative terms. Namely, one can argue that constitutional design should be carried out behind a thick veil of ignorance in which no particular type of dependence or modification of equiprobability (which follows from the principle of insufficient reason) is justified. Regarding the second issue, this paper is to our knowledge the first to investigate equal representation for non-binary decisions that possibly involve strategic behavior.

We consider policy alternatives from a finite interval. Our formal model (see Section 2) imposes two key assumptions: first, the policy advocated by the top-tier representative of any given constituency coincides with the ideal point of the respective constituency's *median voter* (or the constituency's *core*).

[1] We take the constituency configuration to be given exogenously. See, e. g., Epstein and O'Halloran (1999) on constructing majority-minority voting districts along ethnic, religious, or social lines.

[2] For rigorous, very comprehensive treatments of the binary or *simple-game* world see Felsenthal and Machover (1998) or Taylor and Zwicker (1999). – The former (pp. 72ff) also justify the square root rule regarding voting weights by its minimal expected *majority deficit*.

Second, the decision taken at the top tier is the position of the *pivotal representative* (or the assembly's *core*), with pivotality determined by the weights assigned to constituencies and a 50% decision quota. The respective core is meant to capture the result of strategic interaction. As long as this is a reasonable approximation, the actual systems determining collective choices are undetermined and could even differ across constituencies.

In the benchmark case of voters with independent most-preferred policies, a given individual's chance to be pivotal at the bottom tier is inversely proportional to the respective constituency's population size. This makes it necessary and sufficient for equal representation of voters that the probability of any given constituency being pivotal at the top tier is proportional to its size.[3]

The population size of a constituency affects the distribution of its median. A given voter's chance to be doubly pivotal thus becomes a rather complex function of (the order statistics of) *differently distributed* independent random variables. This makes a neat analytical statement similar to Penrose's rule exceptionally hard and likely impossible, except for special limit situations. We therefore resort to Monte-Carlo simulation (see Section 3). Considering a vast number of randomly generated population configurations as well as recent data for the EU and the US, top-tier weights proportional to the square root of population turn out optimal for most practically relevant population configurations. Even for extreme artificial cases, the rule yields good results and becomes optimal if the number of constituencies gets large.

Our surprising main finding is thus that the square root rule is a much more robust norm for egalitarian design of two-tier voting systems than previous analysis suggests. In particular, it continues to apply in the presence of many finely graded policy alternatives and strategic interactions consistent with the median voter theorem. To the extent that this still produces independent median voters, the rule is even robust to the introduction of preference dependence within or across constituencies.

2 Model

Consider a large population of *voters* partitioned into m *constituencies* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ with $n_j = |\mathcal{C}_j| > 0$ members each. Voters' preferences are single-peaked with *ideal point* λ_i^j (for $i = 1, \ldots, n_j$ and $j = 1, \ldots, m$) in a bounded convex one-dimensional *policy space* normalized to $X \equiv [0, 1]$. Assume for simplicity that all n_j are odd numbers.

For any random policy issue, let $\cdot : n_j$ denote the permutation of voter numbers in constituency $\mathcal{C}_j$ such that $\lambda^j_{1:n_j} \leq \ldots \leq \lambda^j_{n_j:n_j}$ holds. In other words, $k : n_j$ denotes the k-th leftmost voter in $\mathcal{C}_j$ and $\lambda^j_{k:n_j}$ denotes the k-th leftmost ideal point (i. e., $\lambda^j_{k:n_j}$ is the *k-th order statistic* of $\lambda_1^j, \ldots, \lambda_{n_j}^j$).

[3] If voters' utility is linear in distance, the criterion also guarantees equal expected utility, i. e., a priori *power* and expected *success* are then perfectly aligned. See Laruelle, Martínez, and Valenciano (2006) for a conceptual discussion of the latter.

A policy $x \in X$ is decided on by an *electoral college* $\mathcal{E}$ consisting of one representative from each constituency. Without going into details, we assume that the representative of $\mathcal{C}_j$, denoted by j, adopts the ideal point of his constituency's *median voter*,[4] denoted by $\lambda^j \equiv \lambda^j_{(n_j+1)/2:n_j}$. Let $\lambda^{k:m}$ denote the k-th leftmost ideal point amongst all the representatives (i. e., the k-th order statistic of $\lambda^1, \ldots, \lambda^m$).

In the top-tier assembly or electoral college $\mathcal{E}$, each constituency $\mathcal{C}_j$ has *voting weight* $w_j \geq 0$. Any subset $S \subseteq \{1, \ldots, m\}$ of representatives which achieves a combined weight $\sum_{j\in S} w_j$ above $q \equiv \frac{1}{2}\sum_{j=1}^m w_j$, i. e. a *simple majority* of total weight, can implement a policy $x \in X$.

Consider the random variable P defined by

$$P \equiv \min\Big\{r \in \{1, \ldots, m\} \colon \sum_{k=1}^{r} w_{k:m} > q\Big\}.$$

Player $P\!:\!m$'s ideal point, $\lambda^{P:m}$, is the unique policy that beats any alternative $x \in X$ in a pairwise majority vote, i. e. constitutes the *core* of the voting game defined by weights and quota.[5] Without detailed equilibrium analysis of any decision procedure that may be applied in $\mathcal{E}$ (see Banks and Duggan 2000 for sophisticated non-cooperative support of policy outcomes inside or close to the core), we assume that the policy agreed by $\mathcal{E}$ is in the core, i. e. it equals the ideal point of the *pivotal representative* $P\!:\!m$.

In this setting we consider the following egalitarian norm: *Each voter in any constituency should have an equal chance to determine the policy implemented by the electoral college.* Or, more formally, there should exist a constant $c > 0$ such that

$$\forall j \in \{1, \ldots, m\} \colon \forall i \in \mathcal{C}_j \colon \Pr\big(j = P\!:\!m \wedge i = (n_j+1)/2\!:\!n_j\big) \equiv c. \tag{1}$$

We would like to answer the following question: which allocation of weights $w_1, \ldots, w_m$ satisfies this norm (at least approximately) for an arbitrary given partition of an electorate into m constituencies? In other words we search for an analogue of Penrose's (1946) rule, which calls for proportionality of a constituency's Penrose-Banzhaf index[6] and square root of population.

The probability of a voter's double pivotality in (1) depends on the distribution of all voters' ideal points. Though in practice ideal points in different

[4] We are aware of this not being appropriate in all contexts. – The possibility that two ideal points exactly coincide, in which case the median voter (in contrast to the median policy) is not well-defined, is ignored. This is innocuous for any continuous ideal point distribution.

[5] Things are more complicated if $q > \frac{1}{2}\sum_{j=1}^m w_j$ is assumed. Then, the complement of a losing coalition need no longer be winning. In this case there may not exist *any* policy $x \in X$ which beats all alternatives $x' \neq x$ despite unidimensionality of X and single-peakedness of preferences.

[6] This index equals a constituency's probability of being pivotal under equiprobable random 'yes'-or-'no' votes at the top tier. Conditions for when this is approximately the voting weight are given by Lindner and Machover (2004). In general, implementing Penrose's square root rule requires numerical solution of the *inverse problem* of finding weights which induce a desired power distribution (see e. g. Leech 2003).

constituencies may come from different distributions on X and may exhibit various dependencies, it is appealing from a normative constitutional-design point of view to presume that the ideal points of all voters in all constituencies are *independently and identically distributed* (i. i. d.).

Given that voters' ideal points in constituency $\mathcal{C}_j$ are i. i. d., each voter $i \in \mathcal{C}_j$ has the same probability to be its median. Hence,

$$\forall j \in \{1, \ldots, m\}\colon \forall i \in \mathcal{C}_j\colon \ \Pr\big(i = (n_j+1)/2 \!:\! n_j\big) = \frac{1}{n_j}.$$

Because the events $\{i = (n_j+1)/2 \!:\! n_j\}$ and $\{j = P \!:\! m\}$ are independent, one can thus write (1) as

$$\forall j \in \{1, \ldots, m\}\colon \frac{\Pr(j = P \!:\! m)}{n_j} \equiv c. \tag{2}$$

Representatives' ideal points $\lambda^1, \ldots, \lambda^m$ are independently but (except in the trivial case $n_1 = \ldots = n_m$) *not* identically distributed. If all voter ideal points come from the (arbitrary) identical distribution F with density f, then $\mathcal{C}_j$'s median position is asymptotically normally distributed (see e. g. Arnold et al. 1992) with mean $\mu^j = F^{-1}(0.5)$ and standard deviation

$$\sigma^j = \frac{1}{2\, f(F^{-1}(0.5))\sqrt{n_j}}.$$

So, the larger a constituency $\mathcal{C}_j$ is, the more concentrated is the distribution of its median voter's ideal point, λ^j, on the median of the underlying ideal point distribution (assumed to be identical for all λ_i^j). This makes the representative of a larger constituency on average more central in the electoral college and more likely to be pivotal in it for a given weight allocation.

It is important to observe that the assumption of the respective *collective preferences* having an identical a priori distribution is inconsistent with the assumption that all *individual preferences* are a priori identically distributed. We find the latter assumption considerably more fitting and will assume i. i. d. ideal points for all bottom-tier voters throughout this paper.

Probability $\Pr(j = P \!:\! m)$ in (2) depends both on the different distributions of representatives' ideal points (essentially the standard deviations σ^j determined by constituency sizes n_j) and the voting weight assignment. This makes computation of the probability of a given constituency $\mathcal{C}_j$ being pivotal a complex numerical task even for the most simple case of *uniform weights*, in which the representative of $\mathcal{C}_j$ with *median* top-tier ideal point is always pivotal, i. e. $P \equiv (m+1)/2$ for odd m. Define $N^j \equiv \{1, \ldots, j-1, j+1, \ldots, m\}$ as the index set of all constituencies except $\mathcal{C}_j$. Then, the probability of constituency $\mathcal{C}_j$ being pivotal is

$$\begin{aligned} \Pr\big(j = (m+1)/2 \!:\! m\big) &= \Pr\big(\text{exactly } \tfrac{m-1}{2} \text{ of the } \lambda^k, k \neq j, \text{ satisfy } \lambda^k < \lambda^j\big) \\ &= \int \sum_{\substack{S \subset N^j, \\ |S| = (m-1)/2}} \prod_{k \in S} F_k(x) \cdot \prod_{k \in N^j \setminus S} (1 - F_k(x)) \cdot f_j(x)\, dx, \end{aligned} \tag{3}$$

where f_j and F_j denote the density and cumulative density functions of λ^j ($j = 1, \ldots, m$). It seems feasible (but is beyond the scope of this paper) to provide an asymptotic approximation for this probability as a function of constituency sizes $n_1, \ldots, n_m$ for special cases, e. g. for $n_2 = \ldots = n_m$ (hence $F_2 = \ldots = F_m$). However, we doubt the existence of a reasonable approximation for arbitrary configurations $(n_1, \ldots, n_m)$, let alone the case of weighted voting ($P \not\equiv (m+1)/2$). A purely analytical investigation of the model is therefore unlikely to produce much insight. The following section for this reason uses Monte-Carlo simulation in order to approximate the probability of any constituency $\mathcal{C}_j$ being pivotal for given partition of an electorate or *configuration* $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ and a fixed weight vector $(w_1, \ldots, w_m)$. Based on this, we try to find weights $(w_1^*, \ldots, w_m^*)$ which approximately satisfy the two equivalent equal representation conditions (1) and (2).

3 Simulation results

The probability $\pi_j \equiv \Pr(j = P\!:\!m)$ can be viewed as the *expected value* of the random variable $H_j \equiv g_j^w(\lambda^1, \ldots, \lambda^m)$ which equals 1 if $j = P : m$ holds for given weight vector w and realized median ideal points $\lambda^1, \ldots, \lambda^m$, and 0 otherwise. The *Monte-Carlo method* (Metropolis and Ulam 1949) then exploits the fact that the empirical average of s independent draws of H_j, $\bar{h}_j^s = \frac{1}{s}\sum_{l=1}^{s} h_j^l$, converges to H_j's theoretical expectation $\mathbf{E}(H_j) = \pi_j$ by the law of large numbers. The speed of convergence in s can be assessed by the sample variance of $\bar{h}_j^s$. Using the central limit theorem, it is then possible to obtain estimates of π_j with a desired precision (e. g. a 95%-confidence interval) if one generates and analyzes a sufficiently large number of realizations.

To obtain a realization h_j^l of H_j, we first draw m random numbers $\lambda^1, \ldots, \lambda^m$ from distributions $F_1, \ldots, F_m$.[7] Throughout our analysis, we take F_j to be a *beta distribution* with parameters $\big((n_j+1)/2, (n_j+1)/2\big)$. This corresponds to the median of n_j independently $[0,1]$-*uniformly distributed* voter ideal points, i. e. all individual voter positions are assumed to be distributed uniformly.[8] Second, the realized constituency positions are sorted and the pivotal position p is determined. Constituency $\mathcal{C}_{p:m}$ is thus identified as the pivotal player of $\mathcal{E}$. It follows that $h_j^l = 1$ for $j = p\!:\!m$, and 0 for all other constituencies.

The goal is to identify a simple rule for assigning voting weights to constituencies which – if it exists – approximately satisfies equal representation conditions (1) or (2) for various numbers of constituencies m and population

[7] We use a *Java* computer program. The source code is available upon request. Directly drawing the constituency medians λ^j provides a huge computational advantage. Unfortunately, it prevents statements about the population median and, e. g., its average distance to the policy outcome.

[8] The mentioned asymptotic results for order statistics imply that only F's median position and density at the median matter when constituency sizes are large. So below findings are *not* specific to the assumption of uniform distributions at the bottom tier.

configurations $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$. A natural focus is the investigation of *power laws*

$$w_j = {n_j}^{\alpha} \tag{4}$$

with $\alpha \in [0,1]$. For big m this approximately includes Penrose's square root rule as the special case $\alpha = 0.5$ (see Lindner and Machover 2004 and **?**).[9]

For any given m and population configuration $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ under consideration, we fix α and then approximate π_j by its empirical average $\hat{\pi}_j$ in a run of 10 million iterations. This is repeated for different values of α, ranging from 0 to 1 with a step size of 0.1 or 0.01, in order to find the exponent α which comes 'closest' to implying equal representation for the given configuration.

Our criterion for evaluating distance between the (estimated) probability vector $\hat{\pi} \equiv (\hat{\pi}_1, \ldots, \hat{\pi}_m)$ realized by weights w and the ideal egalitarian vector $\pi^* \equiv (\sum_{k=1}^{m} n_k)^{-1} \cdot (n_1, \ldots, n_m)$ considers cumulative quadratic deviations between the realized and the ideal chances of an *individual.* Any voter in any constituency $\mathcal{C}_j$ would ideally determine the outcome with the same probability $1/\sum_{k=1}^{m} n_k$, but vector $\hat{\pi}$ actually gives him or her the probability $\hat{\pi}_j/n_j$ of doing so. Treating all n_j voters in any constituency $\mathcal{C}_j$ equally then amounts to looking at

$$\sum_{j=1}^{m} n_j \cdot \left(\frac{1}{\sum_{k=1}^{m} n_k} - \frac{\hat{\pi}_j}{n_j} \right)^2 . \tag{5}$$

We refer to measure (5) as *cumulative individual quadratic deviation* below.

3.1 Randomly generated configurations

Table 1 reports the optimal values of α that were obtained for four sets of configurations $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$.[10] For $m \in \{10, 15, 20, 25, 30, 40, 50\}$, constituency sizes $n_1, \ldots, n_m$ were independently drawn from a uniform distribution over $[0.5 \cdot 10^6,\ 99.5 \cdot 10^6]$. Numbers in column (I) are the optimal $\alpha \in \{0, 0.1, \ldots, 0.9, 1\} \subset [0,1]$, where probabilities $\hat{\pi}_j$ were estimated by a simulation with 10 mio. iterations. Cumulative individual quadratic deviations for optimal α's are shown in brackets. Column (II) reports the respective values obtained for an independent second set of constituency configurations; columns (III) and (IV) do likewise but based on the finer grid $\{0, 0.01, 0.02, \ldots, 0.99, 1\}$ that contains α.[11]

[9] For comparison purposes, we also considered the exact version of Penrose's rule for a selected number of population configurations. Although there are exceptions to this, Penrose's rule tends to perform worse than (4) with the respective optimal exponent α. This extends to $\alpha = 0.5$ when this is close to being optimal. In other cases, e.g., when in fact uniform weights produce the most equal representation, Penrose's square root rule performs better at least than its approximation by $w_j = \sqrt{n_j}$. We leave a more systematic investigation of alternatives to (4) – like "w_j s.t. β_j is proportional to ${n_j}^{\alpha}$" with β_j referring to j's Penrose-Banzhaf index, as suggested by an anonymous referee – for future research.

[10] The configuration draws are independent across different values of m. Thus, the table actually reports optimal values obtained for 28 *independent* configurations.

[11] Hence columns (III) and (IV) each report on 101·7 simulation runs (with 10 mio. iterations each).

# const	(I)	(II)	(III)	(IV)
10	**0.5**	**0.6**	**0.39**	**0.00**
	(1.22×10^{-11})	(1.04×10^{-11})	(2.20×10^{-12})	(2.39×10^{-11})
20	**0.5**	**0.5**	**0.49**	**0.49**
	(4.80×10^{-14})	(8.59×10^{-14})	(5.66×10^{-15})	(6.91×10^{-15})
30	**0.5**	**0.5**	**0.49**	**0.49**
	(1.11×10^{-15})	(5.12×10^{-15})	(7.36×10^{-15})	(2.38×10^{-15})
50	**0.5**	**0.5**	**0.50**	**0.50**
	(3.06×10^{-15})	(4.70×10^{-15})	(3.10×10^{-15})	(3.30×10^{-15})

Table 1: Optimal value of α for uniformly distributed constituency sizes (cumulative individual squared deviations from ideal probabilities in parentheses)

While results for $m = 10$ are still inconclusive, $\alpha \approx 0.5$ emerges as the very robust ideal exponent for larger number of constituencies. The reported cumulative individual quadratic deviations are so small that even if the power laws assumed in (4) do not contain the theoretically best rule for equal representation in our median-voter context (because possibly constituencies' sizes are not the right reference point, but rather something like their Penrose-Banzhaf or Shapley-Shubik index), they allow a sufficiently good approximation for most practical purposes.

Results in Table 1 are strongly suggesting that (an approximation of) Penrose's square root rule holds also in the context of median voter-based policy decisions in $[0, 1]$. But optimality of $\alpha \approx 0.5$ could be an artifact of considering uniformly distributed constituency sizes $n_1, \ldots, n_m$, which perhaps unrealistically makes small constituencies as likely as large ones. We therefore conduct similar investigations using other distributional assumptions.

Constituency sizes seem usually a matter of history, geography, or deliberate design. In the latter case, one might expect them to be clustered around some 'ideal' intermediate level. This makes a (truncated) normal distribution around some value $\bar{n}$ a focal assumption for constituency configurations. Table 2 indicates that, in this case, $\alpha = 0.5$ is no longer the general clear winner from the considered set of parameters $\{0, 0.1, \ldots, 0.9, 1\}$. This is neither very surprising nor – from a square-root-rule point of view – very disturbing: Moderately many and more or less equally sized constituencies give rather little scope for discrimination between constituencies. Assigning slightly larger constituencies substantially more weight risks overshooting the mark, but assigning them only slightly more weight may not translate into an increased number of pivot positions at all. So, first, the optimal α can be expected to be rather sensitive to the precise constituency configuration at hand, especially when a small number of constituencies creates relatively few distinct opportunities to achieve a majority. And, second, in the wide range where extra weight to an above-the-average constituency translates into no or few extra winning coalitions, the objective function is very flat. This is nicely illustrated by Figure 1. Its minimization via Monte Carlo techniques is then particularly sensitive to remaining estimation

# const	(I)	(II)	(III)	(IV)
10	**0.0**	**0.0**	**0.0**	**0.0**
	(1.22×10^{-9})	(1.65×10^{-9})	(9.21×10^{-9})	(1.83×10^{-9})
30	**0.1**	**0.2**	**0.4**	**0.5**
	(1.07×10^{-10})	(1.07×10^{-10})	(6.94×10^{-11})	(6.76×10^{-11})
50	**0.4**	**0.2**	**0.3**	**0.3**
	(1.60×10^{-11})	(7.39×10^{-12})	(3.56×10^{-11})	(4.72×10^{-11})
100	**0.5**	**0.5**	**0.5**	**0.5**
	(1.01×10^{-13})	(2.30×10^{-12})	(1.99×10^{-13})	(3.44×10^{-13})

Table 2: Optimal value of α for normally distributed constituency sizes ($\mu =$ 1 mio., $\sigma =$ 200,000; truncated below 0)

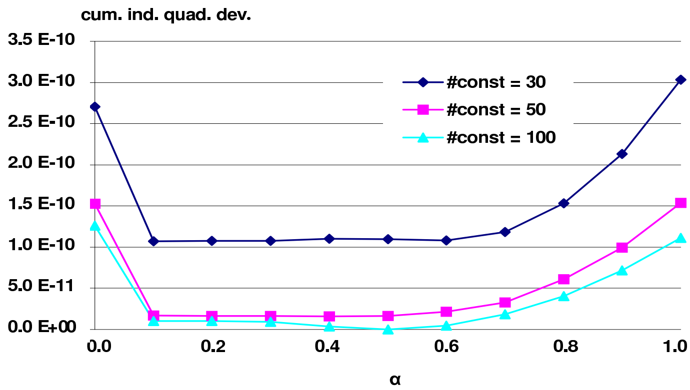

Figure 1: Cumulative individual quadratic deviation in normal-distribution runs (I) for different numbers of constituencies

errors. But note that the importance of these issues decreases as m gets large. This indicates that the applicability of the square root rule rests on enough flexibility regarding the formation of distinct winning coalitions.

When historical or geographical boundaries determine a population partition, a yet more natural distributional benchmark for n_j is a power law such as *Zipf's law* (or *zeta distribution*). In summary, simulations results with constituency sizes drawn from Pareto distributions correspond nicely to those for the uniform distribution as long as the distribution is only moderately skewed. $w_j = \sqrt{n_j}$ performs best and gets close to ensuring equal representation provided that the number of constituencies is sufficiently large. The former is no longer the case for a heavily skewed distribution of constituency sizes, i. e. when there are mostly small constituencies and only one or perhaps two large constituencies (reminiscent of atomic players in an otherwise oceanic game). Giving all constituencies equal weight does reasonably well. As in the normal-distribution case, this problem gets less severe, the greater is the total number of constituencies: For $m = 100$ or larger, $\alpha = 0.5$ turns out to be clearly optimal even for high skewness.

The above analysis of many different population configurations reveals three things. First, as Table 1 and Figures 1 show, $\alpha = 0.5$ results in representation close to being as equal as possible for the given partition of the electorate. Second, for a moderately large number m of constituencies $\alpha \approx 0.5$ is optimal in the considered class of power laws unless all constituency sizes are very similar (e. g., n_j normally distributed with small variance) or rather similar with one or two outliers (corresponding to a heavily skewed distribution). Third, even in these extreme cases the optimal α converges to 0.5 as m gets large. We now turn to two prominent real-world two-tier voting systems.

3.2 EU Council of Ministers

Together with Commission and Parliament, the Council of Ministers is one of the European Union's chief legislative bodies. It is widely held to be the most influential amongst the three and most voting power analysis concentrates on it.[12] It consists of a national government representative from each of the EU member states, endowed with voting weight that is (weakly) increasing in share of total population.[13]

Figure 2 illustrates the probabilities that representatives from differently sized member states are pivotal in the Council assuming a 50% decision quota and assigning voting weight based on populations size via $w_j = {n_j}^{\alpha}$.[14] In line with above findings for randomly generated two-tier voting systems, $\alpha = 0.5$ performs best amongst all coefficients in $\{0, 0.1, \ldots, 1\}$. The figure shows how close the implied probability of country j being pivotal comes to the respective ideal value, which would implement a priori perfectly equal representation. Only the most populous country, Germany, would be visibly misrepresented (here: over-represented).

Note that this analysis not only puts historical voting patterns and preference similarities between some members behind a veil of ignorance but also, as do the mentioned applied studies, it disregards differences between the bottom-tier voting procedures which determine national governments. For example, the UK uses plurality rule or a "first-past-the-post" system, whilst Germany uses a roughly proportional system.[15] This difference might have a systematic effect on the respective accuracy of our median voter assumption at the constituency level. To the extent that it does not, our findings are robust.

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[12] See Felsenthal and Machover (2004), and Leech (2002) for examples. Napel and Widgrén (2006) argue formally that the Commission's and Parliament's positions are nearly irrelevant in the EU25's most common *codecision procedure*.

[13] The current voting rule (based on the Treaty of Nice) is actually quite complex. In addition to standard weighted voting it involves the requirement that the majority weight supporting a policy represents a simple majority of member states and 62% of population.

[14] These and the following numbers are Monte-Carlo estimates obtained from six runs with 10 million iterations each. In case of qualified majority voting, the pivot is identified by assuming a status quo $q = 0 \in X$.

[15] Germany's system is actually complex: some members of parliament are directly elected in a first-past-the-post manner, others get seats in proportion to their party's vote.

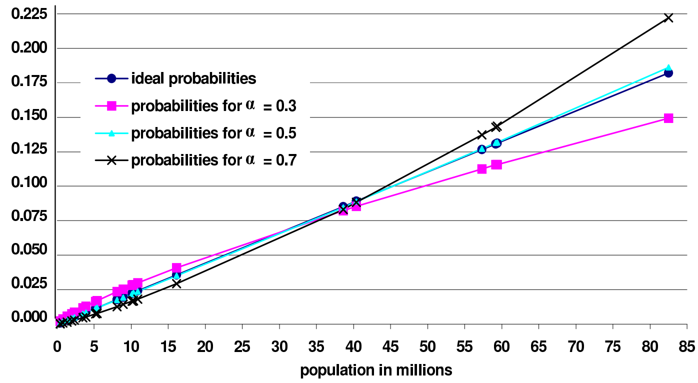

Figure 2: EU25 with weights $w_j = {n_j}^{\alpha}$ compared to ideal probabilities

Investigation of a quota variation even for a very idealized Council illustrates that the decision threshold is not only affecting the balance of 'external costs' and 'decision-making costs' (Buchanan and Tullock 1962) or challenging the so-called 'efficiency' of a decision-making body (operationalized as the probability that a random proposal is passed in the classical 0-1 setting by Felsenthal and Machover 2001 and Baldwin et al. 2001 amongst others). The quota also has important implications for equality of representation and hence the legitimacy of decisions.

3.3 US Electoral College

US citizens elect their president via an Electoral College. The 50 states and Washington DC each send representatives to it. Their number is weakly increasing in the represented share of total population. Although most Electors are not legally bound to vote in any particular way, all state representatives cast their vote for the presidential candidate who secured a plurality of the respective state's popular vote with only minor exceptions. The US Electoral College is therefore commonly treated as a weighted voting system.

Decisions in the Electoral College have in the recent past been essentially binary. The pivotal player amongst the states' median voters might, however, feature prominently in a more sophisticated model of how the two main contestants are selected. In any case, consideration of strategic policy choices in a convex space provide a useful benchmark for the preference-free dichotomous model considered by Penrose (1946) and, specifically addressing the Electoral College, Banzhaf (1968).[16] Figure 3 illustrates the result of determining (hypothetical) weights for state representatives based on current US state population data. Corroborating the findings of Penrose and Banzhaf, the square root rule

[16] Early weighted voting analysis of US presidential elections also includes Brams (1978, ch. 3).

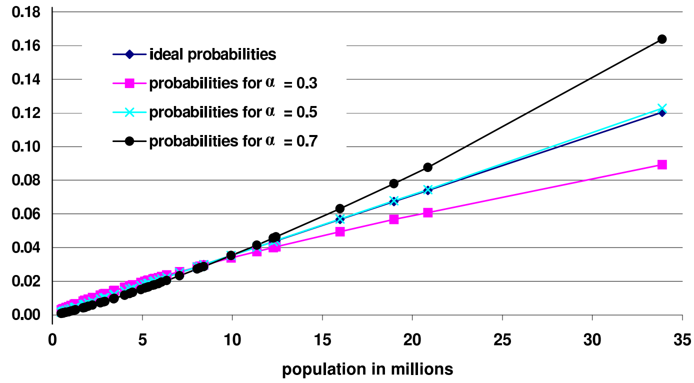

Figure 3: US Electoral College with weights $w_j = {n_j}^{\alpha}$ compared to ideal probabilities

corresponding to $\alpha = 0.5$ is again extremely successful in ensuring equal representation.

4 Concluding remarks

As highlighted, e. g., by Good and Mayer (1975) and Chamberlain and Rothschild (1981), even slight changes regarding decision making at the individual or collective level can produce very different recommendations for operationalizing the one-person, one-vote principle, interpreted here as identical (and positive) indirect expected influence on final outcomes by all voters. Apart from our 'veil of ignorance' perspective with a priori identical but independent voters, the setting considered in this paper is very remote from the preference-free binary model considered by Penrose (1946), Banzhaf (1965, 1968) and others. It is thus surprising that *voting weight proportional to square root of population*, which corresponds to Penrose's original suggestion for most practical purposes,[17] emerges as optimal for both prominent real-world examples as well as many artificial population configurations.

This result matters not only from an abstract point of view. It shows that numerous applied studies have indeed used a robust benchmark. This is also highlighted by recent work of Beisbart and Bovens (2005), which discovers optimality of the square root rule in a very different binary, utility-based egalitarian model. And at least for large constituency populations consisting of many small blocks, Barberà and Jackson (2005) produce similar conclusions in an entirely utilitarian framework. In summary, the square root rule is a simple and trustworthy norm, not an artifact of a particular objective function or setting.

[17] In fact, Penrose (1946) seems to have deliberately blurred the distinction between *voting weight* and *voting power* in his discussion of equal representation in a world assembly. Penrose was aware, however, that approximate proportionality of weight and power generally holds only for sufficiently many constituencies.

This insight can hopefully increase its effect on constitutional design in the real world.[18]

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[18] The square root rule already played a significant role in the public discussion of a possible EU Constitution. See, for example, the open letter by **?**) to the EU members' governments with repercussions in various national news outlets.

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