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Expressive Ballots for Voting Systems and Political Analysis

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Thesis

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Abstract

Voting systems that are used in practice, such as plurality with runoff or proportional systems with thresholds, often fail to capture the complexity of voter preferences. This lack of expressiveness in preference formats contributes to well-documented flaws, including the spoiler effect where similar candidates split the vote, and wasted votes for parties falling below electoral thresholds, pushing voters to vote strategically and potentially leading to the selection of undesirable outcomes.

This thesis argues that expressive preference formats (such as approval ballots, rankings, and weak orders) can significantly mitigate these issues and potentially improve the democratic process, and that they additionally provide insightful data for new types of descriptive analyses of the political landscape.

Part I concentrates on voting problems. It proposes enhancements for three widely used voting systems: improving plurality with runoff by using approval ballots in the first round; extending instant runoff voting to handle voter indifference using weak orders; and reducing the amount of wasted votes in proportional systems with thresholds by allowing ranked preferences. Part II shifts focus to using expressive preferences as a tool for understanding the political landscape. This includes developing methods to derive candidate orderings on an ideological axis using approval patterns, and techniques to identify pairs of candidates that generate significant societal conflict by analyzing ordinal preference data.

For each problem, we introduce new methods based on expressive ballot formats like approval ballots, rankings, or weak orders. We then evaluate these methods both theoretically, through axiomatic analysis, and empirically, via simulations on synthetic data as well as real-world preference datasets, including data collected during actual high-stakes political elections.

Résumé

Les systèmes de vote utilisés en pratique, tels que le scrutin uninominal majoritaire à deux tours ou les systèmes proportionnels avec seuils électoraux, ne parviennent souvent pas à saisir la complexité des préférences des électeurs. Ce manque d'expressivité des bulletins de vote est à l'origine de défauts étudiés et documentés, comme la division des voix entre candidats similaires, et la non prise en compte des votes pour les partis n'atteignant pas les seuils électoraux, incitant les électeurs au vote stratégique (dit "vote utile") et conduisant potentiellement à des résultats indésirables.

Cette thèse soutient que des formats de préférences expressifs (tels que les bulletins d'approbation et les classements, avec ou sans égalités) peuvent atténuer significativement ces problèmes et potentiellement améliorer le processus démocratique, et qu'ils fournissent en outre des données pertinentes pour mener de nouvelles formes d'analyses descriptives du paysage politique.

La première partie se concentre sur les problèmes liés aux systèmes de vote. Elle propose des améliorations pour trois systèmes de vote largement utilisés en pratique : améliorer le scrutin uninominal majoritaire à deux tours en utilisant des bulletins d'approbation au premier tour ; étendre le vote alternatif (*Instant Runoff Voting*) aux classements avec égalités pour gérer les indifférences ; et réduire le nombre de votes non pris en compte dans les systèmes proportionnels avec seuils en autorisant les classements de préférences. La deuxième partie se tourne vers l'utilisation des préférences expressives comme outil pour comprendre le paysage politique. Cela inclut le développement de méthodes pour déduire des ordonnancements de candidats sur des axes idéologiques (par exemple gauche-droite) en utilisant les motifs d'approbation, ainsi que des techniques pour identifier les paires de candidats générant le plus de conflit en analysant les préférences ordinales.

Pour chaque problème, nous introduisons de nouvelles méthodes basées sur des bulletins expressifs, comme les bulletins d'approbation, les classements ou les ordres faibles. Nous évaluons ensuite ces méthodes à la fois théoriquement, par une analyse axiomatique, et empiriquement, via des simulations sur des données synthétiques ainsi que sur des jeux de données de préférences réelles, incluant des données collectées lors d'élections politiques réelles.

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Table of Contents

Chapter 1

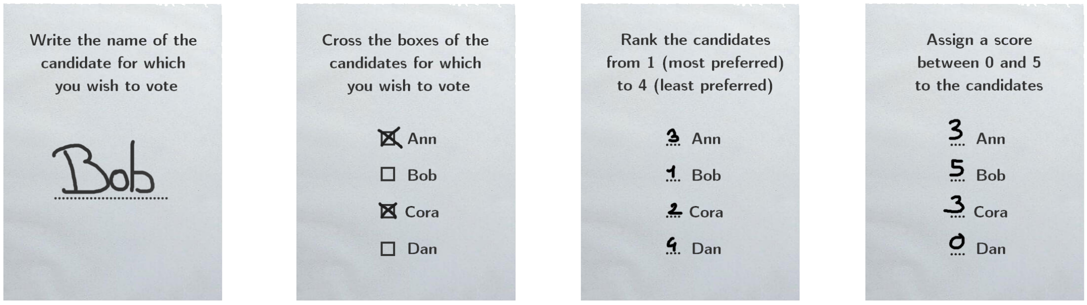
Introduction

Today is election day. French citizens are voting for their next president. They gather in polling stations for the first round of this election, after several months (even years) of campaigning, and hundreds of polls. They will have to choose between eleven candidates, and the two candidates with the most votes will go to the second round, which will take place two weeks later. The candidate with the most votes in the second round will be elected president.

Today, Sarra is voting for the first time in France. As she enters the voting station, she is faced with eleven piles of paper ballots: one for each candidate running in the election. On each ballot is written the name of one candidate. Seeing these piles of paper lined up in a random order, it occurred to her that arranging them by political position, from the most left-wing candidate to the most right-wing one, would be far more practical. However, this might be difficult, as there is no consensus on which ordering is the correct one. Considering this, she realizes that it would be useful to have a theoretical method based on voters' actual preferences that could determine a good ordering of the candidates. After taking a few paper ballots from the piles, she enters the voting booth where she has to place the ballot of her choice in a small envelope, which she will ultimately put in the voting urn.

On the booth just next to hers, Bastien has been hesitating between two candidates for five minutes now. He knows that his favorite candidate has no chance to make it to the second round, since the last polls predicted him a score of 3%. Thus, he is considering voting strategically in favor of the only candidate he finds tolerable enough and that might have a chance to get into the second round, even though he disagrees with several of this candidate's proposals. This dilemma reminds him of the last election of the members to the German parliament (Bastien is Franco-German), where he voted for his favorite party, which ended up with only 4.3% of the votes, while an electoral threshold of 5% was required for a party to be allowed to get a seat in the parliament. Bastien was frustrated, because his ballot was effectively discarded *before* the seat distribution. At the time, he wished that he could have cast a back-up vote, that would have been used in case his favorite party did not reach the threshold. On the other hand, if fewer people had voted strategically for a more popular party (and had instead voted for his favorite party), maybe it would have reached the 5% threshold.

Bastien concluded that both the German and the French voting systems were failing voters in crucial ways. Still undecided between voting sincerely and strategically in the French presidential election, Bastien thought that it would actually be less frustrating if it was possible to vote for several candidates instead of just one, and decided to do so by placing the ballots of the two candidates in the envelope. Unfortunately, this led his ballot to be invalidated.



(a) Uninominal ballot. (b) Approval ballot. (c) Ordinal ballot. (d) Cardinal ballot.

Figure 1.1: Examples of different ballot formats.

The case of Bastien demonstrates one of the numerous drawbacks of the voting systems that are currently used for most political elections. Many of these drawbacks are due to voters being often very constrained in the way they can express preferences: they have to choose one candidate (or one list), and *only one*. Why not, as Bastien wished, allow voters to vote for several candidates they like? Or, to allow them to provide a back-up vote in case their first choice does not make it? Or even to rank candidates, as it is already done in some other countries? Exploring these possibilities by allowing voters to cast more expressive ballots is the main focus of the first part of this thesis.

Once we start thinking about the ways expressive preferences can be used to improve the voting process, we notice that they can also become a great tool to *learn* the structure of the electorate, and describe the political landscape. Consider for instance Sarra's thought about ordering the candidates from the most left-wing one to the most right-wing one: if we allow voters to vote for several candidates they like, as Bastien suggested, or even to rank the candidates, then it becomes possible: if a voter votes for two candidates, then it is likely that they run on similar platforms. This is an example of a problem in which we want to *learn* the structure of the electorate, and in which expressive preferences are found to be insightful. We study this kind of problems in the second part of this thesis.

The remainder of this introduction is organized as follows. In Section 1.1, we introduce the social choice framework and the key aspects of the methodology employed in all chapters of this thesis, specifically how we evaluate and compare the different methods (or *rules*) we introduce. In Section 1.2, we give an overview of the settings and problems studied in this thesis, and summarize our main results. Finally, Section 1.3 lists all the published papers on which this thesis is based, and additionally mentions omitted works that were conducted during the PhD, but that are not included in this thesis.

1.1 Social Choice with Expressive Preferences

Formally, a social choice problem involves a set of candidates, a set of voters expressing their *preferences* over the candidates (expressed using *ballots*, and grouped in *preference profiles*), and a set of possible *outcomes*.

In a presidential election for instance, the possible outcomes are all the candidates running in the election. The way preferences are expressed varies; in many countries, including France, voters simply indicate the one candidate for whom they voted. In this case, we say that the preferences are *uninominal*: each voter chooses one candidate, and *only one*. In other countries, such as Ireland, voters rank the candidates in order of preference. In this case, we say that the preferences are *ordinal*. Other preference formats are sometimes used: in a few American cities, voters elect their mayor using *approval ballots*, in which they can approve of as many candidates as they want. Finally, in non-political contexts, such as evaluating movies, or evaluating candidates for a job, 'voters' generally give scores/grades to the 'candidates'. In this case, we say that the preferences are *valued*, or *cardinal*. Figure 1.1 provides examples of these four types of preferences.

All preference formats that go beyond uninominal open the door to many possibilities when it comes to selecting a winner. In the last three centuries, a wide variety of methods have been proposed based on inputs like ordinal preferences (rankings) and approval ballots. For rankings, fundamentally different approaches exist: one can associate positions with scores and select the candidate with the highest total (like Borda), or seek a candidate who defeats all others in pairwise majority comparisons (like a Condorcet winner). For approval ballots, the most straightforward method simply selects the most approved candidate (Approval Voting). Beyond these core ideas, numerous other methods exist, particularly for rankings. More recently, the problem of selecting a winning committee of several candidates has also gained significant interest, with methods proposed for both ranking and approval inputs.

Once the problem is well-defined, and methods have been proposed for selecting good outcomes, the question is now: which method should be used *in practice*? Unfortunately, this is generally not an easy question to answer: most of the time, each method comes with some benefits, but also with some drawbacks, and trade-offs naturally arise. For instance, the voting system that is used for the presidential election in France has the advantage of being very simple to understand and to implement due to the use of uninominal ballots, but this comes with many flaws, that could be avoided with more expressive ballots (such as rankings or approval ballots).

Thus, we need ways to *evaluate* and *compare* these methods. To this aim, we distinguish different approaches. The first to appear in the social choice literature was the *axiomatic approach* (Brandt et al., 2016), which evaluates rules based on the theoretical properties (that generally are normatively appealing) they satisfy. Thanks to the increasing availability of computers during the second half of the twentieth century, scholars started to simulate the various voting methods on computer-generated preference data, and observing their behavior. This quickly led to the definition of probabilistic models that we now use to sample such *synthetic* preference data, and this corresponds to the second approach for evaluating rules. Finally, the third approach is to simulate the different methods not on artificially generated data, but instead on datasets of *actual* preferences (we refer to it as *real* data). This approach gained in popularity in the last decades (Boehmer et al., 2024), thanks to the possibility to collect large amounts of data on the preferences of voters, generally using *online* platforms. These are the main three approaches that we will use in this thesis to evaluate the methods we propose. Let us now go over each of these approaches in more detail.

Comparing Methods using Axioms

The axiomatic approach is the traditional way to compare social choice rules. It relies on normative properties, called *axioms*, that are considered desirable for a voting method to satisfy. Consider for instance the *Condorcet criterion*, which says that if there exists a candidate who wins the majority vote against every other candidate, it should be the winner (in that case, we say that it is a Condorcet winner). This property is appealing because it means that if we select this candidate, no majority group of voters exists where all members prefer some other single candidate over the winner. Failure of this property is often observed in actual elections: a typical example is the 2007 French presidential election, in which *François Bayrou* was a Condorcet winner (according to the polls), but did not reach the second round of the election due to not being the favorite candidate of enough voters.[1] Another example is a 2022 vote in *Lille* concerning the future of one of the largest avenue of the city.[2] The four possibilities were (A) a park, (B) a park with a water surface, (C) half a park, half a canal and (D) a canal. The voting system was the simplest possible: each voter votes for one option, and the option with the most votes wins. The results were very divided: option A ("a park") won with 27% of the votes, while options C and D ("half a canal" and "a full canal") both got 26% of the votes (option B got the remaining 21%). The first option won, but it is fair to assume that if people could have *ranked* the alternatives, a different outcome would have been selected. Indeed, the supporters of option D probably have the following ranking: $D \succ C \succ B \succ A$ (the more water, the better), while the supporters of option A likely have the reverse ranking $A \succ B \succ C \succ D$ (the less water, the better). In that case, option C ("half a park, half a canal") would have been the Condorcet winner, and should have won instead of A.

This is only one of many axioms that have been studied in the social choice literature. They generally aim at formalizing a behavior that is expected from a normative point of view. For instance, in actual elections, many voters are voting *strategically* for a candidate who is not their first choice. This behavior is known and widely supported by empirical evidence. Ideally, the best voting strategy for any voter *should be* to vote sincerely, i.e., for their favorite candidate. This motivates *strategyproofness* axioms. Unfortunately, it was shown by Gibbard (1973) and Satterthwaite (1975) that for ordinal preferences, no reasonable voting method actually satisfies this axiom when there are at least three candidates. This kind of negative result is formalized as an *impossibility theorem*. On the more positive side, axiomatic analyses can also lead to *characterization theorems*, which say that a particular method is the only one from a family of methods to satisfy a given set of axioms. In the axiomatic analyses of this thesis, we prove several such impossibility and characterization results, and we consider several well-studied families of axioms, such as symmetry properties (neutrality and anonymity), efficiency and representation properties (for instance Pareto-efficiency or the Condorcet criterion), independence properties (in particular the independence of 'clones'), and strategyproofness and monotonicity properties.

Comparing Methods using Synthetic Data

In axiomatic analyses, conclusions are often quite binary: either a rule satisfies a given axiom, or it does not. This motivates the use of *experimental* analyses, which generally provide more fine-grained results. The idea is to run the different methods on datasets of preferences, and to observe their behavior. Using probabilistic models, we can for instance sample *artificial* preference profiles, on which we can simulate elections and compare the methods. Guilbaud (1952) was one of the first to study the properties of voting methods using synthetic data. He used the simplest probabilistic model, in which every ranking is equally likely (called *impartial culture*) and measured the probability that there exists a Condorcet winner in a sampled preference profile.

Other probabilistic models have been proposed in the literature, some of the most popular ones are *Mallows' model* (Mallows, 1957) and the *urn model* (Eggenberger and Pólya, 1923). An interesting feature of such probabilistic models is their modularity. Indeed, they generally rely on some parameters, which allow us to change some aspects of the model, and thus discover potential

[1] *"Sondage: au second tour, Bayrou gagnerait face à Sarkozy ou Royal"*, Libération. https://www.liberation.fr/france/2007/02/19/sondage-au-second-tour-bayrou-gagnerait-face-a-sarkozy-ou-royal_12708/

[2] *"Avenue du Peuple Belge : résultats de la consultation"*. https://www.lille.fr/Actualites/Avenue-du-Peuple-Belge-resultats-de-la-consultation

links between the outcome of a rule and the parameters of a model. This additionally allows us to test the robustness of the results we obtain on a variety of scenarios. For a long time, the social choice literature lacked a bridge connecting these synthetic probabilistic models. Moreover, it was unclear whether any of these models was actually good at simulating *actual* preferences. Recently, an approach called the *map(s) of elections* was proposed by Szufa et al. (2020) to study the structure of the space of preference profiles, and to compare profiles sampled by different probabilistic models, as well as profiles of real preferences. They showed that some models were better than others to generate preference profiles that are similar to those observed in reality. However, it still remains very hard to create artificial preferences with a realistic structure, which motivates the use of *real* data in the experimental evaluation of the different methods.

Comparing Methods using Real Data

In contrast to synthetic data, datasets of *real* preferences allow for more realistic comparisons of the different methods, and give useful insights on the behavior of the methods against actual preference patterns. In particular, applying proposed methods to political preferences is often instructive regarding the potential political consequences of the rule. The increased use of real data in social choice research motivated the creation of platforms of preference data, such as *preflib.org* (Mattei and Walsh, 2013) and *pabulib.org* (Stolicki et al., 2020).

To gather such datasets of real preferences, the traditional way is to conduct surveys on a sample of voters, generally through *field experiments*. A typical example is the *Voter Autrement* project (Laslier, 2019), which has been running since 2002 in France, and in which *in situ* experiments have been conducted on the day of each presidential election in several French cities. These experiments allow a few voters of these cities to try alternative voting methods (such as Borda or Approval Voting), thus providing expressive ballots (such as rankings or approval ballots), which we can then use in our analyses. Unfortunately, this approach is hard to scale, and the number of voters that can be surveyed is limited.

However, we can nowadays collect *larger* datasets through online platforms. For instance, we can adapt the surveys that are used in field experiments. In particular, this approach has been used in the *Voter Autrement* project since 2017 (Bouveret et al., 2018), allowing us to collect larger datasets of preferences than could be obtained from field experiments. More generally, we all frequently create new preference data when interacting with online platforms: for instance, *likes* can be seen as *approval ballots* and ratings of movies or restaurants as *valued (or cardinal) ballots*.

1.2 Overview of the Thesis

The contributions of this thesis are divided into two parts. In the first part, we study *voting* problems, in which the goal is to incorporate more expressive preferences into existing voting systems that are implemented for large-scale elections around the world, in order to tackle actual issues that arise in practice. In the second part, we study *learning* problems, in which the goal is to make use of expressive preferences to describe the political landscape, and to better understand the structure of the electorate and of the candidate set.

Even though the two parts have different goals, the problems studied, and the ways we study them have many similarities. In particular, we assume in all of them that we are given the preferences of a set of voters over a set of candidates, and that these preferences are of a particular format (approval ballots, rankings, weak orders, etc.), generally more expressive than uninominal preferences. Then, we define what kind of outcome we want to obtain using these preferences.

In the first part, outcomes can be a single winner, or a set of winners, while in the second part, outcomes aim at *learning* something about the particular structure of voters' preferences, and of the candidate set. We then define rules (or methods) that take as input the preferences, and return an outcome of the desired format. We analyze these methods using different tools, mainly axiomatic and experimental analyses, that we described previously. These five key aspects of all the problems we study (the preference format, the outcome, the methods, the axiomatic analysis, and the data analysis) will be extensively discussed in Chapter 2, where we cover the relevant state of the art for each of these aspects, giving a good starting point before delving into the more specific problems studied in the rest of the thesis. Let us now give a quick overview of these problems.

Part I: Improving Voting Systems

As we mentioned, in the first part of this thesis, we focus on voting systems that are used in practice, and on the ways to tackle their main issues using more expressive preferences.

Approval with Runoff (Chapter 3). The first voting system we improve upon is *plurality with runoff*, which works in two rounds. In the first round, every voter votes for one candidate, and the two candidates with the most votes go to the second round. The candidate who receives the most votes in the second round is declared winner. This system is used in many contexts, in particular for large-scale elections in several countries. In France, this system is almost associated with the very idea of voting: it is used for the presidential election, which is the most important one in the country, but variants of it are also used for the election of the parliament members, for departmental elections, and even the election of mayors in most cities use a form of plurality with runoff. However, this voting system has many flaws, which can be shown both theoretically and in practice. For instance, it is very sensitive to the *spoiler effect*, which occurs when several candidates are running on similar platforms and are at risk of losing the election because voters are divided between them, while one of them might have won if they were the only one to run on this platform. This causes many voters to vote *strategically* in favor of the candidate they like the most among the ones that have a chance to get into the second round according to the polls, instead of voting *sincerely* for their actual favorite candidate. Plurality with runoff also notably fails monotonicity, an axiom saying that if a candidate is elected given a set of preferences, it should still be elected if more voters support this candidate. We give concrete examples of these issues in Chapter 3.

Our proposal in Chapter 3 is to keep the two rounds of elections, but instead of asking for *uninominal* ballots in the first round, we would ask for *approval* ballots, meaning that every voter can vote for as many candidates as they want. Then, we can for instance say that the two candidates with the most votes go to the second round. However, this is not the only possibility: we can also select the two candidates such that the largest number of voters approve at least one of the two. These two rules can be different. Consider for instance a very simple election in which 51% of voters approve of Ann and Bob, and the remaining 49% only approve of Cora. In this case, the first rule would select Ann and Bob, who both received the most votes, while the second one would select Ann (or Bob) and Cora, as all voters approve of at least one of them. We show in Chapter 3 that we can actually define many other rules based on approval preferences for selecting the two finalists. We then analyze these rules using axiomatic analysis and experimental analysis, and show in particular that some of the rules we define do not suffer from most of the issues of plurality with runoff.

Instant Runoff Voting with Indifferences (Chapter 4). The second voting system we focus on is *instant runoff voting* (IRV), in which voters cast (possibly truncated) rankings of the candidates, and the candidates are eliminated one by one until only one remains. To decide which candidate to eliminate next, we look at how many voters rank each candidate first (among the non-eliminated ones), and we eliminate the one with the lowest score. The supporters of the eliminated candidate are transferred to the next candidate in their rankings that is not yet eliminated. This system (or variants of it) is used in several countries for large-scale elections, including Ireland, Australia, and New Zealand. It is also used in several states of the USA. On many normative aspects, this system is better than plurality with runoff, mainly because voters provide *rankings* of the candidates, and this expressivity gain allows for more interesting methods. However, it also has some flaws. For instance, it does not satisfy monotonicity. Moreover, some voters might still be undecided between several candidates, and may want to rank them at the same level.

Our proposal in Chapter 4 is to allow even more expressiveness in the ballots, and to ask voters to cast *weak orders* of the candidates, meaning that they can rank several candidates at the same level in the ranking they provide. With this ballot format, two generalizations of IRV have been (briefly) discussed in the literature, called Approval-IRV and Split-IRV, but Split-IRV is the only one that has actually been used in practice. In Chapter 4, we analyze these two methods axiomatically and experimentally, and argue that Approval-IRV is more appealing than Split-IRV in many contexts. In particular, we show two axiomatic characterizations of Approval-IRV among a large family of methods, making it the most natural generalization of IRV to weak orders. The first characterization says that Approval-IRV is the only method that satisfies the generalization (to weak orders) of the most appealing normative properties satisfied by IRV, and the second characterization says that it is the only method generalizing IRV that satisfies a weak monotonicity axiom.

Proportional Elections with Thresholds (Chapter 5). The third and final voting problem we address concerns parliamentary elections, focusing not on electing a single winner but on deciding which parties receive seats. In such parliamentary elections, an *apportionment method* is used: voters cast uninominal ballots for one party (or a list of candidates), and the seats are distributed among the parties, proportionally to the number of votes they received. The issue we are addressing in this thesis comes from the existence of electoral *thresholds* in these elections. Indeed, in many elections of this kind, there is a threshold (for instance, 5% of the total number of votes) such that only parties that receive more votes than required by the threshold are given seats. Other parties receive none. Because of this, all voters who voted for a party that does not reach the threshold see their ballots effectively discarded and are not represented by the final seat distribution. This causes once again many voters to vote strategically in favor of a party that is likely to reach the threshold, leaving no chance for outsider parties.

Our proposal in Chapter 5 is to allow voters to provide back-up votes, by submitting *truncated rankings* of the parties. Thus, if their favorite party does not reach the threshold, then we can look at their second one, and their third one, and so on until we find a party that reaches the threshold. However, with this ballot format, there are various ways we can select the set of parties that reach the threshold, and which may receive seats: should we simply decide that only parties that received more first-rank votes than the threshold can have a seat? Or should we use an IRV-like method that eliminates parties one by one until all remaining parties reach the threshold? We define five such methods to select the set of parties, and show that they all satisfy different normative properties. We also compare them on a dataset of real preferences that we collected for this purpose during the 2024 election of the French representatives to the European parliament.

Part II: Describing the Political Landscape

In the second part of this thesis, we discuss problems in which the goal is to use the voters' preferences to better understand the structure of the candidate and voter sets.

Ordering Candidates on an Axis (Chapter 6). We first study the problem of ordering candidates on an *axis*. This is motivated by the idea that in politics, there exists an underlying ideological left-right axis of the political landscape, on which we can approximately place the different political parties. However, this problem finds applications in many other domains that rely on binary information (which can be approximated as approval ballots).

In Chapter 6, we propose to learn such an axis of candidates using *approval* preferences of the voters, following the simple idea that candidates that are often approved together are likely to be close to each other on the axis, while candidates that are not approved together are likely to be far apart. We introduce five methods to derive orderings from approval ballots, and we analyze them axiomatically and experimentally. Our axiomatic and experimental analyses show that the different methods can be placed on a spectrum based on the complexity of the information they use from approval ballots. While methods based on less information seem more robust (satisfying for instance a monotonicity axiom), the ones based on more information behave better regarding the expected placement of particular candidates. We additionally use our methods to construct orderings of the justices of the Supreme Court of the USA, and of the candidates in French presidential elections.

Identifying Candidates Inducing Conflict (Chapter 7). Finally, we conclude this thesis with the problem of selecting pairs of candidates that induce the most conflict among the voters, based on their ordinal preferences (rankings). Like in the settings from the first part of this thesis, we want here to *select* candidates (in this case, *pairs* of candidates) based on voters' preferences. However, in this chapter, our motivation is to *learn* one aspect of the structure of the electorate, and not to *elect* winners. In particular, we want to know which pair of candidates is dividing the voters the most, motivated by the idea that identifying the sources of division is a necessary first step before aiming at reducing this division.

In Chapter 7, we first define the different aspects of conflict that can be observed with rankings by introducing quantitative *measures* of conflict. We then introduce several methods that select pairs of candidates inducing conflict, and compare them using axiomatic analysis and experimental analysis. Since this problem is very different from classical voting problems in which we select winners, we need to define completely new axioms, that are more adapted to this problem, and that we show to be incompatible with classical voting rules. Finally, we apply our methods on synthetic and real data, selected based on the amount of conflict we expect to observe between the voters.

1.3 Bibliographic Notes

Most of the work included in this thesis is derived from peer-reviewed papers published in conference proceedings (or currently under review). I contributed to all results, proofs and experiments that appear in this thesis, unless otherwise noted. Moreover, most of the text has been rewritten for this thesis, and figures have been updated.

Chapter 3 is based on the following paper:

[Delemazure et al. (2022)] *Approval With Runoff* (IJCAI 2022). **Théo Delemazure**, Jérôme Lang, Jean-François Laslier, Remzi Sanver.

The version included in this thesis heavily improves on the version published in IJCAI. In particular, it contains additional theoretical results, a deeper statistical analysis, and experimental results on more datasets.

Chapter 4 is based on the following paper:

[Delemazure and Peters (2024)] *Generalizing Instant Runoff Voting to Allow Indifferences* (EC 2024). **Théo Delemazure** and Dominik Peters.

The version included in this thesis contains the same theoretical results, but proofs have been slightly improved, and parts of the text have been rewritten. The discussion in the experimental analysis has also been extended.

Chapter 5 is based on the following paper:

[Delemazure et al. (2025b)] *Reallocating Wasted Votes in Proportional Parliamentary Elections with Thresholds* (Preprint). **Théo Delemazure**, Rupert Freeman, Jérôme Lang, Jean-François Laslier, Dominik Peters.

This paper is currently under review. The version included in this thesis contains some additional results (in particular, we consider two additional voting rules).

Chapter 6 is based on the following paper:

[Delemazure et al. (2025a)] *Comparing Ways of Obtaining Candidate Orderings from Approval Ballots* (IJCAI 2024). **Théo Delemazure**, Chris Dong, Dominik Peters, Magdalena Tydrichova.

The version included in this thesis contains a few additional results, in particular in the experimental analysis, in which more datasets are considered.

Chapter 7 is based on the following paper:

[Delemazure et al. (2025c)] *Selecting the Most Conflicting Pair of Candidates* (IJCAI 2024). **Théo Delemazure**, Łukasz Janeczko, Andrzej Kaczmarczyk, Stanisław Szufa.

The version included in this thesis contains the same theoretical results, with some changes in the proofs and discussions of the results. The experimental analysis has been extended, with a few additional results.

Other Works

During my PhD, I also worked on other topics that are not included in this thesis. In particular, I was the main contributor of the following two papers. They are not included in the thesis to obtain a shorter and more coherent document.

[Delemazure et al. (2023b)] *Aggregating Correlated Estimations with (Almost) no Training* (ECAI 2023). **Théo Delemazure**, François Durand, Fabien Mathieu.

[Delemazure et al. (2024b)] *Independence of Irrelevant Alternatives under the Lens of Pairwise Distortion* (IJCAI 2024). **Théo Delemazure**, Jérôme Lang, Grzegorz Pierczynski.

Some results of the second paper will be briefly discussed in Section 2.5.2. I also contributed to the following peer-reviewed papers that were published during my PhD, and that are related to computational social choice.

[Brill et al. (2022)] *Liquid Democracy with Ranked Delegations* (AAAI 2022). Markus Brill, **Théo Delemazure**, Anne-Marie George, Martin Lackner, Ulrike Schmidt-Kraepelin.

[Delemazure et al. (2023a)] *Strategyproofness and Proportionality in Party-Approval Multiwinner Elections* (AAAI 2023). **Théo Delemazure**, Tom Demeulemeester, Manuel Eberl, Jonas Israel, Patrick Lederer.

[Colley et al. (2023a)] *Measuring a Priori Voting Power – Taking Delegations Seriously* (IJCAI 2023). Rachael Colley, **Théo Delemazure**, Hugo Gibert.

On the more economic and political science side of social choice, I also contributed to the data analyses included in the following papers. Both papers are currently under review.

[Marsilio and Delemazure (2022)] *Are Alternative Voting Methods Ideologically Biased? A Meta-Analysis and New Insights from the 2022 Italian Election* (Working paper). Simone Marsilio and **Théo Delemazure**.

[Baujard et al. (2024)] *Do Grades Have Absolute Meaning? An Experiment on Majority Judgment* (Working paper). Antoinette Baujard, Sylvain Bouveret, Roberto Brunetti and **Théo Delemazure**.

Finally, I also contributed to the publication of several datasets. In particular, I aided in the data collection (for instance constructing websites) and the data analyses of the following two datasets.

[Delemazure and Bouveret (2024)] *Voter Autrement 2022 - The Online Experiment "Un Autre Vote"* (Dataset). **Théo Delemazure** and Sylvain Bouveret.

[Delemazure et al. (2024a)] *Voter Autrement 2024 - The Online Experiment* (Dataset). **Théo Delemazure**, Rupert Freeman, Jérôme Lang, Jean-François Laslier, Dominik Peters.

Chapter 2

The Toolbox of Computational Social Choice

2.1 Introduction

Each chapter of this thesis is centered around one voting problem. Despite their differences, these problems all share at least five key aspects:

1. **Preferences:** In all of these problems, we are given the *preferences* of *voters* over *candidates* (for instance, their rankings of the candidates).

2. **Outcomes:** The main objective is to aggregate these preferences to select *a (social) outcome* of a particular kind (for instance, select the best candidate).

The question is now: how can we aggregate the preferences of the voters to obtain the desired outcome? This is where the third aspect comes in:

3. **Rules:** We define *rules* (or methods) that aim at solving our problems. These rules are functions that take as input the preferences of the voters, and return a social outcome (or several tied outcomes).

These rules represent possible solutions to the problems. In this thesis, we define new rules, and evaluate them to understand which ones are the best to solve a given problem, depending on the context. To evaluate and compare the rules, we principally use the following two tools:

4. **Axiomatic Analysis:** We define sets of axioms/properties that we want our rules to satisfy, and check which rules satisfy which axioms.

5. **Data Analysis:** We use real-world datasets and synthetic datasets to run simulations on the rules, and we evaluate the rules based on the obtained results.

In this chapter, we cover the state of the art in (computational) social choice regarding these five aspects. As the aim is not to go over everything that has been done in computational social choice or even in voting theory, this chapter focuses on what is relevant to the problems that will be studied in this thesis. For a more general overview of computational social choice, the reader can refer to the books edited by Brandt et al. (2016) and Endriss (2017).

The remainder of this chapter is organized as follows. We first define different preference formats in Section 2.2. In Section 2.3, we go over different types of social outcomes in voting problems, and we discuss classical rules that have been defined for each of these types of outcomes. In Section 2.4, we introduce classical families of axioms that are frequently used to evaluate and compare rules in the social choice literature. Finally, in Section 2.5, we go over the different kinds of synthetic and real data used in this thesis. In particular, we introduce the *Voter Autrement* project, a collection of datasets to which I contributed, and that will be repeatedly used throughout this thesis.

2.2 Preferences

All the problems we discuss in this thesis take as input preferences of voters over candidates, sometimes called *ballots*. To make it more formal, we assume in the rest of the thesis that we have a set $V = \{1, \dots, n\}$ of n *voters* and a set $C = \{c_1, \dots, c_m\}$ of m *candidates*. Candidates are not necessarily individuals: they can be any kind of objects/concepts on which voters have preferences (political parties, movies, dates for a meeting, bills, etc.).

We say that voters have preferences over these candidates, in the sense that they like some more than others. They can express these preferences in different ways. In each of the problems we study, the preferences are restricted to some particular family of preferences $\mathcal{X}$, called the *preference domain*. In this section, we introduce the most common ballot formats used in the literature. Since every voter expresses their preferences over the candidates, we can associate some preference $X_i \in \mathcal{X}$ to every voter $i \in V$, and we can collect all these preferences in a *preference profile* $P = (X_1, \dots, X_n) \in \mathcal{X}^n$.

In the remainder of this section, we define *uninominal preferences*, *rankings* (ordinal preferences), *approval preferences*, *weak orders*, and *valued preferences* (such as utilities, costs, scores, and labels). Finally, we discuss the particular case of *structured preferences*, and more specifically single-peaked rankings and interval approval ballots.

2.2.1 Uninominal Preferences

The most common but arguably least expressive way a voter can express their preferences is by indicating the name of a single candidate, for instance the candidate they prefer among the set of possible candidates. We call these *uninominal preferences*. Formally, a uninominal preference x_i of a voter $i \in V$ is a candidate $c \in C$. For instance, if $C = \{a, b, c\}$, the preference $x_i = b$ can be interpreted by saying that voter i's favorite candidate is b. The preference profile $P = (x_1, \dots, x_n) \in C^n$ for all $i \in V$ is then a collection of uninominal preferences.

Uninominal Preferences
The domain of *uninominal preferences* is the candidate set $\mathcal{X} = C$.

This ballot format does not give a lot of information about voters' preferences: we can only compute the number of votes received by each candidate, that we call the *plurality score* and denote by $S(c) = |\{i \in V : x_i = c\}|$ for each candidate $c \in C$. Nonetheless, this ballot format is very easy to understand and to collect, which partly explains why it is widely used in practice, from the papal conclave to the election of the president of the United States. Actually, uninominal preferences are likely as old as the concept of voting, as one of its earliest known uses is the procedure of *ostracism* that was taking place in Athens twenty-five centuries ago (Pébarthe, 2020), and in which Athenians had to write the name of the person they wanted to ostracize on a piece of pottery (the "ballot"): if a person received more than 6 000 votes, they were exiled for 10 years.[1]

[1] In this case, the candidate given by a voter is not their favorite candidate, but their least favorite one.

2.2.2 Rankings

Rankings are one of the most studied preference formats in the social choice literature. The earliest written trace of a ranking-based voting method is from the fifteenth century, during which the philosopher and theologian Nicholas Cusanus described a method that will later be known as the Borda rule (McLean, 1990).[2] Nowadays, several countries use ranked ballots for political elections, such as Australia, New Zealeand, Ireland and Malta, as well as some states of the USA, though the voters are often not required to cast a full ranking.

In a ranking, a voter order the candidates from their most favorite one to their least favorite one. Note that this preference format is quite expressive, but it can be cognitively demanding for voters, especially when the number of candidates is large: imagine having to rank all the movies you have seen in your life, or all the 62 candidates of some political election, as was required in New South Wales for the 1983 election of the Australian Senate (leading to 11.1% of invalid ballots).[3]

Formally, a *ranking* $\succ$ is a linear order (irreflexive, antisymmetric, transitive and complete binary relation) over the set of candidates C. For instance, $a \succ b \succ c$ is a ranking over $\{a, b, c\}$ in which a is preferred to b which is preferred to c. Each ranking $\succ$ can be associated with a rank function σ such that for each candidate $x \in C$, $\sigma(x) = |\{y \in C : y \succ x\}| + 1$ is the rank of x in $\succ$. For instance, in the ranking $a \succ b \succ c$, $\sigma(a) = 1$ and $\sigma(c) = 3$.

Rankings
The *rankings* domain is the set of all linear orders $\mathcal{X} = \mathcal{L}(C)$.

A profile of rankings (or ordinal profile) is then a collection $P = (\succ_1, \dots, \succ_n)$ where $\succ_i$ is the ranking of voter $i \in V$. For two candidates $a, b \in C$, we define $V^{a \succ b} = \{i \in V : a \succ_i b\} \subseteq V$ as the subset of voters who rank a above b. We also define the *majority relation* $\succ^M$ such that for $a, b \in C$, we have $a \succ^M b$ if the majority of voters prefers a to b, i.e., $|V^{a \succ b}| > |V^{b \succ a}|$. Using these two notions, we can construct the *weighted majority graph*, in which there is an edge from a to b if $a \succ^M b$, and the weight of the edge is equal to $|V^{a \succ b}| - |V^{b \succ a}|$.

2.2.3 Approval Preferences

An *approval ballot* is a subset of candidates $A \subseteq C$ such that all candidates $a \in A$ are approved, and all candidates $b \notin A$ are disapproved. The earliest known major elections that used approval ballots were the papal conclaves from 1294 to 1691 (Colomer and McLean, 1998) and the election of the Doge of Venice during the late Middle Ages and the Renaissance, with a procedure mixing sortition and approval voting (Lines, 1986). Nowadays, approval ballots are used in some political elections, in particular in the United States for some local elections (in St. Louis, Missouri and Fargo, North Dakota), and in various organizations for internal elections. Moreover, approval ballots are often used for low-stakes collective decisions, such as deciding on the date for a meeting using apps like *doodle.com*, *framadate.org* or *whale.imag.fr*.

Approval ballots may be less cognitively demanding for voters than rankings, but they also seem less expressive, as voters cannot indicate which candidate they prefer between two approved (or disapproved) candidates.[4] On the other hand, approval ballots allow to express actual indifferences between candidates, which cannot be done with full rankings.

[2] McLean (1990) also discusses a voting rule proposed by Ramon Lull that follows the Condorcet principle, but which is not really based on rankings. Voting procedures were an important topic in western Europe during the Middle Ages and the Renaissance, especially for elections in the Church (Christin, 2014), though most of the discussion focused on the majority principle.

[3] See the results of this election at `theo.delemazure.fr/thesis/1983senatensw.txt`.

[4] For m candidates, there exist $m!$ possible rankings but 2^m possible approval ballots, which is significantly less.

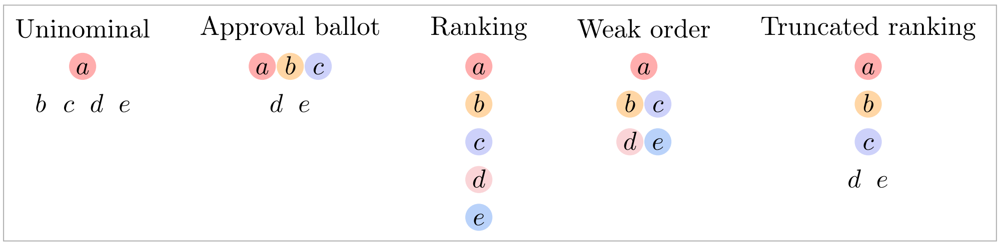


Figure 2.1: Examples of ballots for the different preference formats.

Approval Preferences
The domain of *approval preferences* is the set of all subsets of candidates $\mathcal{X} = 2^C$.

An approval profile is a collection of approval ballots $P = (A_1, \dots, A_n)$ where A_i is the approval ballot of voter $i \in V$. For a candidate $a \in C$, we say that $V^a = \{i \in V : a \in A_i\}$ is the set of voters who approve a, and $S(a) = |V^a|$ is the *approval score* of a.

2.2.4 Weak Orders

On one hand, approval preferences allow to express indifferences between candidates, but they do not allow to express the relative intensity of the preferences between approved candidates. On the other hand, rankings allow to express the relative intensity of the preferences, but they do not allow indifferences. Weak orders are a more general type of preferences that gets the best of both worlds: they allow voters to provide a ranking of the candidates, but still give the possibility to put several (tied) candidates at the same position in the ranking.

More formally, a *weak order* $\succcurlyeq$ is a complete pre-order (reflexive, transitive and complete binary relation) over the set of candidates C. For a weak order $\succcurlyeq$, we can define its asymmetric part $\succ$ such that $a \succ b$ if $a \succcurlyeq b$ and not $b \succcurlyeq a$, and its symmetric part $\sim$ such that $a \sim b$ if both $a \succcurlyeq b$ and $b \succcurlyeq a$. In this case, a and b are ranked at the same position in the ranking, and we say that they are part of the same *equivalence class*. For instance, $a \succ b \sim c \succ d$ is a weak order over $\{a, b, c, d\}$ in which a is preferred to b and c, which are tied and each preferred to d. For ease of reading, we sometimes write $a \succ \{b, c\} \succ d$ to indicate that b and c are in the same indifference class.

Weak Orders
The domain of *weak orders* is the set of all complete pre-orders over the set of candidates $\mathcal{X} = \mathcal{R}(C)$.

A preference profile of weak orders is then a collection $P = (\succcurlyeq_1, \dots, \succcurlyeq_n)$ where $\succcurlyeq_i$ is the weak order of voter $i \in V$. We can define $V^{a \succ b}$ and the majority relation $\succ^M$ as for full rankings, using the asymmetric parts of the weak orders. As for full rankings, each weak order $\succcurlyeq$ can be matched to a rank function σ such that for each candidate $x \in C$, $\sigma(x) = |\{y \in C : y \succ x\}| + 1$ is the rank of x in $\succcurlyeq$. For instance, in the weak order $a \succ b \sim c \succ d$, $\sigma(a) = 1$, $\sigma(b) = \sigma(c) = 2$ and $\sigma(d) = 4$.

There exist several interesting sub-categories of weak orders. In particular, rankings and approval ballots are sub-categories. We can also define *truncated rankings*, in which voters strictly rank a subset of the candidates, and the remaining candidates are put in the same indifference class, below the ranked candidates. For instance, $a \succ b \succ c \succ \{d, e, f\}$ is a truncated ranking (we generally write $a \succ b \succ c$ and omit the non-ranked candidates for ease of reading). This kind of weak orders is particularly useful when the number of candidates is large, and voters do not have the time or the willingness to rank all the candidates. In particular, this is the actual preference format that is used in most political elections that ask voters to rank candidates (for instance in Ireland and some local US elections). Examples of ballots for the different preference formats are shown in Figure 2.1.

Figure 2.2 summarizes the expressive power of the different types of preferences we introduced in this section. In particular, it shows which preference format can be expressed using another format. The dotted arrow between rankings and uninominal preferences indicates that it is possible to *derive* the latter from the former by ignoring a part of the ranking.

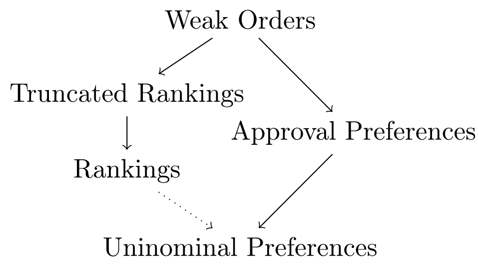


Figure 2.2: There is an arrow from one preference format to another if it is possible to express preferences from the second format using preferences from the first one.

Truncated rankings can also be seen as the combination of approval ballots and full rankings, by saying that the candidates that are ranked are the approved candidates, and the non-ranked candidates the non-approved ones. However, this does not exactly correspond to the combination between approval ballots and rankings, as we would not know the ranking of non-approved candidates. In Chapter 3, we discuss *approval-ordinal* ballots,[5] that are a true combination of approval ballots and rankings, in the sense that it combines two preference profiles: one containing approval ballots P_A, and one containing rankings $P_\succ$, giving an approval-ordinal profile $P = (P_A, P_\succ)$.

2.2.5 Utilities, Costs, Scores, and Labels

With the preference formats we mentioned so far, everything could be expressed as a binary relation, indicating for each pair of candidates whether one is preferred to the other. However, there exist completely different ways to express preferences than through pairwise comparisons of candidates. In particular, voters can express their preferences by assigning a value, taken from an ordered set, to each candidate. This is what we are doing when we are rating restaurants on *Tripadvisor*, movies on *Letterboxd* or *Blablacar* drivers. In these real-life scenarios, the set of allowed scores is often $Z = \{0, 1, 2, 3, 4, 5\}$. In some other contexts, the scores can take any positive real value, and in this case we often say that candidate c induces a *utility* $u_i(c) \in \mathbb{R}_{\geqslant 0}$ to voter i. Equivalently, candidates can be costly for the voters (for instance if the candidates are chores that need to be done), and we say that candidate c induces a *cost* $c_i(c) \in \mathbb{R}_{\leqslant 0}$ to voter i. Finally, in some other contexts, the voters can give a label to each candidate, taken from an ordered set. For instance, the *Majority Judgement* rule (Balinski and Laraki, 2011) uses the labels $\{\text{Excellent}, \text{Very Good}, \text{Good}, \text{Fair}, \text{Poor}, \text{To reject}\}$. Another typical example is the grades given to students in some countries, often taken from the set $Z = \{A, B, C, D, E, F\}$ (sometimes with additional grades such as $A+$ or $A-$). This inspired the concept of *tierlists*, which are now a part of the internet culture. To make a tierlist, voters are given a set of categories ordered from the best to the worst, and they have to say into which category each "candidate" falls (see Figure 2.3 for an example).

Considered more formally, all these examples correspond to the same family of preference formats, in which there exists a set of possible values Z (scores, labels, utilities, costs, etc.) associated with a strict total order $>$, and where each voter $i \in V$ assigns a value $z_i(c) \in Z$ to each candidate $c \in C$. We call these preferences *valued preferences*.

[5]Approval-ordinal ballots were introduced by Brams and Sanver (2009) under the name *approval-preferences ballots*, but we do not keep this name to avoid any confusion with approval preferences.

Figure 2.3: Example of a tierlist of single-winner voting rules ('S' being the best category, and 'F' the worst one).

Valued Preferences
Given an ordered set Z, the domain of *Z-valued preferences* is the set $\mathcal{X} = Z^C$.

We can then define the preference profile $P = (z_1, \dots, z_n)$ as the collection of these functions. We can induce weak orders from valued preferences. More precisely, we can define the weak order $\succcurlyeq_i$ of voter $i \in V$ such that for all candidates $a, b \in C$, we have $a \succ_i b$ when $z_i(a) > z_i(b)$ and $a \sim_i b$ when $z_i(a) = z_i(b)$. Approval preferences are a special case of valued preferences with $Z = \{0, 1\}$. Finally, note that if the elements in Z are real numbers (or more generally from a linearly ordered abelian group), we also call this preference format *cardinal preferences*.

None of the problems we study in this thesis actually take such preferences as input, however these preferences are very useful to *evaluate* rules. In particular, we sometimes use utilities (and costs) preferences, consider the induced weak orders, and see which rules return the candidates with highest utility, or lowest cost. We will discuss this use-case in more detail with the notion of distortion in Section 2.5.2.

2.2.6 Structured Preferences

Aggregation of preferences is a very hard problem in the general case. In particular, we quickly encounter paradoxes and impossibility results, some of which will be discussed in Section 2.4. Among the different ways to escape these impossibility results and paradoxes, one promising solution is to restrict the preference domain $\mathcal{X}$ to preferences having a particular structure (Arrow, 1951; Black, 1948). In addition, it later appeared that some social choice problems that are *computationally* hard in the general case (for instance computing the outcome of a voting rule) can be solved efficiently if the preferences have a particular structure (starting with the work of Walsh (2007)).

In this thesis, we focus on two kinds of structured preferences: *single-peaked rankings*, and its equivalent for approval ballots: *interval approvals*. Single-peakedness was introduced by Black (1948), and is easily the most studied type of structured preferences in the social choice literature. Interval preferences for approval ballots were introduced more recently by Elkind and Lackner (2015) among other kinds of structured preferences. We direct the reader to the survey of Elkind et al. (2017c) for a more complete overview of the recent works on structured preferences in computational social choice, which goes over many other types of structured preferences.

These two types of structured preferences have in common that they depend on a global ordering of the candidates that we call an *axis*, and denote by $\lhd$. Axes have exactly the same structure as rankings, except that the direction of an axis does not matter. For instance, $a \lhd b \lhd c \lhd d$ and $d \lhd c \lhd b \lhd a$ represent the same axis. For simplicity, we write $\lhd = abcd$ instead of $a \lhd b \lhd c \lhd d$.

The idea of single-peaked and interval preferences is that every voter has a favorite candidate on the axis, and the further away a candidate is from it *in one direction* on the axis, the less the voter likes this candidate. For rankings, this means that if two candidates a and b are on the same side of the favorite candidate, and b is further away from it than a, then the voter prefers a to b. For approval ballots, this means that if b is approved, then a is approved too (as it is closer to the voter than b), and if a is not approved, then neither is b (as it is further away from the voter than a). Note that this is only true if both candidates are *on the same side* of the voter's favorite candidate (in contrast to Euclidean preferences, see Section 2.5.1).

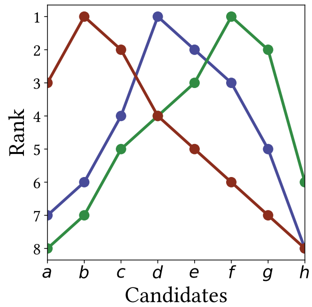


Figure 2.4: Single-peaked rankings for the axis $\lhd = abcdefgh$.

Thus, we say that a ranking is *single-peaked* on the axis $\lhd$ if when we plot the ranks of candidates along the axis, the plot has a single peak, i.e., it is always decreasing when we move away from the peak (see Figure 2.4). More formally, a ranking $\succ$ is single-peaked on the axis $\lhd$ if for any three candidates $a, b, c \in C$ such that a is the candidate ranked first, $\sigma(a) = 1$ (i.e., it is the peak), if $a \lhd b \lhd c$ or $c \lhd b \lhd a$ on the axis, then $b \succ c$. Note that every ranking is single-peaked for some axis (for instance, $\succ$ is clearly single-peaked for $\lhd = \succ$). The interest of single-peaked rankings appears when all voters' rankings in a preference profile are single-peaked on the *same* axis. In this case, we say that the preference profile is single-peaked.

We similarly define interval preferences for approval ballots, by saying that an approval ballot A is an interval for the axis $\lhd$ if the set of approved candidates A forms an interval of the axis $\lhd$ (See Figure 2.5). More formally, an approval ballot A is an interval for the axis $\lhd$ if for all $a, b \in A$ and $c \in C$, if $a \lhd c \lhd b$, then we have $c \in A$. As for single-peaked rankings, we say that a preference profile of approval ballots is *linear* if all approval ballots in the profile are intervals of the same axis.

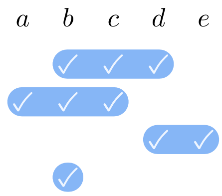


Figure 2.5: Interval approval ballots on the axis $\lhd = abcde$.

Note that there exist polynomial time algorithms to recognize if an ordinal profile is *single-peaked* for some axis (Bartholdi and Trick, 1986; Doignon and Falmagne, 1994; Escoffier et al., 2008). In the case of approval ballots, there also exist polynomial time algorithms to recognize if a preference profile is linear for some axis (Elkind et al., 2017c), as this problem boils down to check if some binary matrix satisfies the *consecutive ones property* (Booth and Lueker, 1976). We will discuss this property in more detail in Chapter 6.

An important motivation for studying these structured preferences is that in many contexts, it is sometimes assumed that there exists some 'natural' ordering of the candidates. This is especially true in politics, where it is often considered that the candidates can be ordered on some left-right ideological axis. This is also true when candidates represent different options on a numerical axis.

2.2.7 Summary

In Table 2.1, we summarize the different preference formats introduced in this section, and in Table 2.2, we indicate for each chapter of this thesis the preference format used as the input of the problem. Note that we also use preference formats that are not necessarily the input of the

Preferences	Domain	Example
Uninominal ballots	C	a
Rankings	$\mathcal{L}(C)$	$a \succ b \succ c$
Approval ballots	2^C	$\{a, b\}$
Weak orders	$\mathcal{R}(C)$	$a \succ \{b, c\} \succ d$
Truncated rankings	$\cup_{S \subseteq C} \mathcal{L}(S)$	$a \succ b$
Valued preferences	Z^C	$z(a) = 5$

Table 2.1: Summary of the preference formats introduced in this section.

Chapter	Preferences
Chapter 3: Approval with Runoff Rules	Approval-ordinal preferences
Chapter 4: Runoff Scoring Rules	Weak orders
Chapter 5: Party Selection Rules	Truncated rankings
Chapter 6: Axis Rules	Approval ballots
Chapter 7: Conflict Rules	Rankings

Table 2.2: Preference formats in the input of each chapter's problem.

problem, typically to evaluate the rules (especially in the case of valued preferences), or to compare them to similar rules that have been defined for other preference formats.

2.3 Outcomes and Rules

In the previous section, we have seen different preference formats that form the preference profiles which are the inputs of the problems studied in this thesis. Let us now discuss the different kinds of *outcomes* of these problems. A typical, and already non-trivial, example of outcome is the selection of one candidate as the winner of an election, using the voters' preferences. However, outcomes are not restricted to the selection of winner(s), and we also consider in this thesis problems that aim at *describing* the structure of the electorate.

More formally, our problems take their solutions in an *outcome* domain $\mathcal{O}$. For instance, if we want to find the winner of an election, then the possible outcomes are all the candidates: $\mathcal{O} = C$. Once we know the preference domain $\mathcal{X}$ and the domain of outcomes $\mathcal{O}$, we can define *rules* $f : \mathcal{X}^n \to \mathcal{O}$ that take as input a preference profile $P \in \mathcal{X}^n$ and return an outcome $f(P) \in \mathcal{O}$, or a set of outcomes $f(P) \subseteq \mathcal{O}$ if we allow tied outcomes in the output. In the first case, we say that the rule is *resolute*, and in the latter that it is *irresolute*. Note that any irresolute rule can be made resolute by using a tie-breaking mechanism on the possible outcomes (for instance, in many French elections, when two candidates receive the same number of votes, the oldest candidate wins). In this thesis, we will sometimes consider irresolute rules (in Chapters 3, 4, 6 and 7), and sometimes resolute rules (in Chapter 5).

In the remainder of this section, we go over the different social choice problems that are discussed in this thesis, their corresponding outcome domains, and the most common *rules* that have been defined for each preference format. More specifically, we will introduce *single-winner voting*, *multi-winner voting*, and the problem of obtaining an *ordering of the candidates* (not to be confused with obtaining a *ranking* of the candidates, which is yet another voting problem, that is not discussed in this thesis).

2.3.1 Single-Winner Voting

Single-winner voting is surely the most studied problem in (computational) social choice. On one hand, it is simple to define and widely used in our everyday life and within our political institutions, but on the other hand, it is quite complex to solve, and trade-offs naturally appear for some preferences formats. Political elections with one winner, like presidential elections, are a typical example of single-winner voting: there is a set of candidates, and only one of them is elected based on the preferences of the voters. This kind of election is not new: already during the Middle Ages, various kinds of votes were conducted in European universities, brotherhoods and Catholic institutions to decide who should hold important positions (Christin, 2014). We also encounter single-winner voting situations quite often in our social life: when deciding on a date for a meeting, when choosing a restaurant among a group of friends, or in some board games and video games to determine which player to eliminate. It is thus not surprising that this problem has been extensively studied. Before going over the different single-winner voting rules that exist for each preference format, let us first look at what can be done when there are only two candidates.

The Majority Principle

When there are only $m = 2$ candidates, all the preferences formats we defined in Section 2.2 (except valued preferences) have the same expressive power: voters can only say which of the two candidates they prefer (or potentially abstain by indicating indifference). Thus, a reasonable principle is to select the candidate that is preferred by a majority of (non-abstaining) voters. We call the corresponding rule the *majority rule*.

Majority Rule (when $m = 2$)
The *majority rule* selects the candidate(s) that are preferred by at least half of the (non-abstaining) voters.

When the number of voters n is odd, this rule is resolute. It has been shown by May (1952) that the majority rule is the only rule for $m = 2$ candidates that satisfies three very natural properties:

(1) treating all voters equally,

(2) treating all candidates equally, and

(3) if a candidate is winning and a voter who preferred the other one changes their ballot to now vote for this candidate, then it should still win.

These three properties are respectively the *anonymity*, *neutrality* and *positive responsiveness* (or *monotonicity*) properties, that we formally define in Section 2.4. Problems appear when we have more than two candidates. First, because we need to decide on which preference format to use, and second, because due to paradoxes and impossibilities, we often need to decide on some trade-off. Let us now go over the different preference formats introduced in Section 2.2, and discuss some classical single-winner voting rules for each of them.

Uninominal Ballots

With uninominal preferences, the only natural way to select a *winner* is to select the candidate(s) with the highest *plurality score* $S(c) = |\{i \in V : x_i = c\}|$. The (irresolute) rule f that selects those candidates is called *plurality*, and is widely used in practice, especially for high-stakes elections. It is sometimes referred to as *first past the post*, as the candidate that comes first in the election wins everything. This is for instance the system used in the United Kingdom for the election of the members of the House of Commons, or in the United States for the election of the members of the House of Representatives.

Plurality
The *plurality* rule selects the candidate(s) with the highest plurality score. Formally, we have $f(P) = \text{argmax}_{c \in C} |\{i \in V : x_i = c\}|$.

Despite being widely used in practice, the plurality rule has numerous drawbacks. We discuss some of them in this thesis.

Approval Ballots

If we have an approval profile $P = (A_1, \ldots, A_n)$, we can generalize the plurality rule described above by selecting the candidate(s) with the highest *approval score* $S(c) = |\{i \in V : c \in A_i\}|$. The resulting (irresolute) rule is called *approval voting* (AV). In the literature, it was formally defined by Brams and Fishburn (1978), but it was already used to elect the pope from 1294 to 1691.

Approval Voting (AV)
The *approval voting* rule selects the candidate(s) with the highest approval score. Formally, we have $f(P) = \text{argmax}_{c \in C} |\{i \in V : c \in A_i\}|$.

In most contexts, AV is the only reasonable single-winner voting rule based on approval ballots. In particular, it has been shown to be the only approval-based rule that satisfies various sets of desirable properties (Brandl and Peters, 2022). However, one might consider more complex rules, such as *split approval*. With this rule, each voter $i \in V$ distributes a total of one point, which is divided equally between the candidates they approve, thus giving $1/|A_i|$ points to each of them. The winner is the candidate with the highest split approval score. A false idea is that this rule is fairer than AV, as every voter gives the same total amount of points. However, the opposite is true, and this rule actually favors voters who approve fewer candidates. To see it, consider the simple election with one voter approving $\{a\}$ and one voter approving $\{b, c\}$. Split approval selects a, but all candidates are supported by the same amount of voters. Still, this property can be interesting in some contexts, for instance in truth-tracking (or *epistemic social choice*), in which there exists only one "correct" candidate, and we want to favor voters who approve fewer candidates, as they are more likely to be sure of their choice (Allouche et al., 2022). Note that in this context, ballots encode voters' beliefs rather than preferences (as there exists an actual best candidate).

Split Approval
The *split approval* rule selects the candidate(s) with the highest split approval score. Formally, we have $f(P) = \text{argmax}_{c \in C} \sum_{i \in V : c \in A_i} 1/|A_i|$.

Example 2.1
Consider an approval profile $P = (\{a, b\}, \{a\}, \{b, c\}, \{b, c, d\})$. The approval scores are $S(a) = S(c) = 2$, $S(b) = 3$ and $S(d) = 1$ and the split approval scores are $S(a) = 3/2$, $S(b) = 4/3$, $S(c) = 5/6$ and $S(d) = 1/3$. Thus, the approval winner is b and the split approval winner is a.

Rankings

If we have a preference profile of rankings $P = (\succ_1, \dots, \succ_n)$, the situation gets a bit trickier, as there is not *one* natural rule to select a winner, as it was the case for uninominal or approval ballots. In fact, many rules have been defined in the literature, and some of them are being used in practice. Let us go over the most common ones.

Scoring Rules In the rules we defined for uninominal and approval preferences, each voter was assigning a score to each candidate, and the winner was the candidate with the highest score. We can extend this principle for rankings with the family of *scoring rules*, by saying that each voter i assigns a score to each candidate c, depending on the position of c in the ranking $\succ_i$. Naturally, the better ranked, the higher the score. More formally, a *scoring rule* is specified by a *score vector* $\mathbf{s} = (s_1, \dots, s_m)$ with $s_1 \geqslant \dots \geqslant s_m$ and $s_1 > s_m$, such that each voter gives s_j points to the candidate they rank in position j. Then, the score of a candidate c is the sum of the scores given by every voter $S(c) = \sum_{i \in V} s_{\sigma_i(c)}$. The winners are the candidates with the highest score.

Scoring Rules
A *scoring rule* based on the score vector $\mathbf{s}$ is the rule f defined as $f(P) = \text{argmax}_{c \in C} \sum_{i \in V} s_{\sigma_i(c)}$.

One of the most common scoring rules is the *Borda* rule, that is based on the score vector $\mathbf{s}_{\text{Borda}} = (m-1, m-2, \dots, 0)$. It takes its name from *Jean-Charles de Borda*, who suggested this method at the end of the 18th century (de Borda, 1781). However, we can find traces of a similar principle in writings of Ramon Lull from the late thirteenth century (McLean, 1990). Nowadays, this method is mainly used by institutions (such as American universities) or for sports and art awards. It is also used in a few political elections, for instance in Slovenia to elect two particular members of the national Assembly.

Borda
The *Borda* rule is the rule f defined as $f(P) = \text{argmax}_{c \in C} \sum_{i \in V}(m - \sigma_i(c))$.

We can also obtain the *plurality* rule that we defined for uninominal preferences by using the vector $\mathbf{s}_{\text{Plurality}} = (1, 0, \dots, 0)$, by considering that the top-ranked candidate of each voter is a uninominal ballot. Another common rule is the *veto* rule (also called *anti-plurality*) that is based on the vector $\mathbf{s}_{\text{Veto}} = (1, \dots, 1, 0)$, and which selects the candidate that is ranked last by the fewest voters. In between these two rules, we can find k-*approval* rules for $1 \leqslant k \leqslant m-1$ that are based on the vectors $\mathbf{s}_{k\text{-app}} = (1, \dots, 1, 0, \dots, 0)$ with k ones and $m-k$ zeros. One can also define more complex vectors. For instance, Formula 1 races use the vector $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1, 0)$ that is decreasingly decreasing, and the Eurovision song contest uses the vector $(12, 10, 8, 7, 6, 5, 4, 3, 2, 1, 0, \dots, 0)$. Note that the family of scoring rules has been axiomatically characterized (Smith, 1973; Young, 1975). We will discuss this characterization in Section 2.4.

Example 2.2
Consider the following profile P:

$$6:\ a \succ b \succ c \succ d$$
$$5:\ b \succ d \succ c \succ a$$
$$4:\ d \succ c \succ a \succ b$$
$$3:\ c \succ d \succ b \succ a$$

In this profile, with plurality, each voter gives one point to their top-ranked candidate, thus the winner is clearly a. With veto, the winner is c as it is the only candidate that is never ranked last. The Borda scores are $S(a) = 22$, $S(b) = 30$, $S(c) = 28$ and $S(d) = 28$. Thus, the Borda winner is b. Finally, with 2-approval, the scores are $S(a) = 6$, $S(b) = 11$, $S(c) = 7$ and $S(d) = 12$, thus the winner is d. In this example, four different scoring rules returned four different winners.

Runoff Rules We can also use the scoring vectors in a different way: instead of directly declaring that the candidates with the highest score are the winners, we can use the scores to repeatedly eliminate the candidate with the lowest score, until only one candidate remains. Intuitively, it is like doing several rounds of voting, and at each round, the candidate with the lowest score is eliminated. This is the idea behind the family of *runoff scoring rules*. Some rules from this family have been used in practice since the early 20th century in Australia, motivating the study of these rules in social choice (Smith, 1973).

To eliminate candidates, we again use scoring vectors $\mathbf{s} = (s_1, \dots, s_m)$. The first candidate to be eliminated is the one with the lowest total score $S(c) = \sum_{i \in V} s_{\sigma_i(c)}$ (we explain later how we handle ties). However, now that this candidate is eliminated, only $m - 1$ candidates remain and we cannot use $\vec{s}$ to eliminate the next one, since it is defined for m candidates. Thus, we need to define the score vectors for all numbers of candidates between 2 and m. Therefore, a runoff scoring rule is actually specified by a sequence of score vectors $(\mathbf{s}^2, \dots, \mathbf{s}^m)$ such that the candidate eliminated when k candidates remain is the one with the lowest total score according to $\mathbf{s}^k$.

Runoff Scoring Rules
A *runoff scoring rule* based on the sequence of score vectors $(\mathbf{s}^2, \dots, \mathbf{s}^m)$ is the rule f that selects the remaining candidate when all candidates have been eliminated one by one according to the sequence of score vectors.

One might wonder what happens in case of a tie between two candidates that both have the lowest score at some point. One possibility is to assume that there exists a tie-breaking order to be used in each round, and we would thus obtain a resolute rule. Another possibility is to explore all possible ways to break ties. The result of the rule would then be the set of candidates that win with at least one tie-breaking order. This is what is called *parallel-universe tie-breaking* and results in an irresolute rule (Conitzer et al., 2009, Section 7).

The most famous runoff scoring rule is *Instant Runoff Voting* (IRV). IRV is based on the plurality score vector $\mathbf{s}^k = (1, 0, \dots, 0)$ for all k. In other words, at each step, the candidate with the lowest plurality score is eliminated, and the votes of the voters who ranked this candidate first are transferred to the next candidate in their rankings that has not been eliminated yet. This rule is used in several countries for high-stakes elections, such as Australia and Ireland.

Instant Runoff Voting (IRV)
Instant Runoff Voting is the runoff scoring rule f based on the plurality score vector $\mathbf{s}^k = (1, 0, \dots, 0)$ for all k.

Some other runoff scoring rules have been defined and studied in the social choice literature, such as *Coombs*' rule based on the veto score vector $\mathbf{s}^k = (1, \dots, 1, 0)$ (Grofman and Feld, 2004), and *Baldwin*'s rule, based on the Borda score vector $\mathbf{s}^k = (k-1, k-2, \dots, 1, 0)$ (Baldwin, 1926). There is also *Nanson*'s rule, which is similar to Baldwin's rule, but repeatedly eliminates all candidates with a Borda score below the average one (Nanson, 1882). In that sense, Nanson's rule eliminates several candidates at each step, and thus is not formally part of the family of runoff scoring rules.

Example 2.3
Consider the profile of Example 2.2. With IRV, the first candidate to be eliminated is c, as it is ranked first by the fewest voters, thus increasing the score of d to 7. The next candidate to be eliminated is b, increasing the score of d to 12, making it the winner of the election. With Coombs, a is eliminated first, then b, then there is a tie between c and d. With Baldwin, a is also eliminated first, according to the score computed in Example 2.2, then the Borda score on the profile without a are $S(b) = 22$, $S(c) = 16$ and $S(d) = 16$. Thus, there is a tie between c and d, but in any case b wins the last round, so it is the sole Baldwin winner.

We mentioned that runoff scoring rules simulate elections with several rounds, but that it removes the need to organize several rounds by directly asking for the ranking of the voters. However, we could still define rules that take place in several rounds. A famous example is *plurality with runoff* (also called the *two-round system*) that is one of the most used rule for high-stakes political elections, as it is used in 84 countries for electing the head of state. With this rule, voters cast uninominal ballots in the first round. If one candidate receives more than half of the votes, it wins. Otherwise, a second round is organized between the two candidates that received the most votes in the first round. Then, the majority rule is used in the second round. We discuss this rule and its numerous flaws in more detail in Chapter 3. In the literature, this rule is generally formalized as a one-round rule based on rankings (still, some works studied it as a two-round protocol, see for instance Sato (2016)).

Example 2.4
Consider again the profile from Example 2.2. The two candidates with the highest plurality scores are a and b, and the winner of the second round is a.

Condorcet extensions Another popular family of rules is the family of *Condorcet extensions* that rely on the notion of Condorcet winner. We say that a candidate c is a *Condorcet winner* in an ordinal profile P if it wins the majority vote against every other candidate. More formally, c is a Condorcet winner if for all $d \in C \setminus \{c\}$, we have $c \succ^M d$, i.e. $|V^{c \succ d}| > |V^{d \succ c}|$. If it is the case, then one can argue that this candidate should be the winner of the election. This is what *Le Marquis de Condorcet* suggested at the end of the 18th century (Jean-Antoine-Nicolas de Caritat, 1785).[6] The rationale is that if we select the Condorcet winner, there is no group of more than half of the voters that agree on changing the winner for another one. Unfortunately, it is not always possible to find a Condorcet winner, as we can have a *Condorcet cycle*, that is a cycle of candidates such that each candidate wins the majority vote against the next one, and the candidates in the cycle win against all candidates outside the cycle. Consider for instance the profile $P = \{a \succ_1 b \succ_1 c, b \succ_2 c \succ_2 a, c \succ_3 a \succ_3 b\}$. In this profile, a wins the majority vote against b, b wins the majority vote against c, and c wins the majority vote against a. Thus, there is no Condorcet winner. We similarly have no Condorcet winner in the profile from Example 2.2. However, if we assume that the preferences of the voters are single-peaked (see Section 2.2.6), then there cannot be a Condorcet cycle and there necessarily exists a Condorcet winner (Black, 1948).

To generalize the Condorcet principle to the case where there is no Condorcet winner, various rules have been defined. These rules are called *Condorcet extensions*, as they always select the Condorcet winner when there is one. Many Condorcet extensions are based on the weighted majority graph described in Section 2.2.2. In other words, these rules solely rely on the pairwise

[6] Note that this principle has already been discussed prior to Condorcet, the earliest traces of it being again from the thirteenth century in the work of Ramon Lull (McLean, 1990).

majority margins $M_{a \succ b} = |\{i \in V : a \succ_i b\}| - |\{i \in V : b \succ_i a\}|$. Fishburn (1977) categorized these rules as the *C2 rules*. Moreover, rules that only use the majority relation $\succ^M$ (i.e., they do not use the strength of the majority relation and only care if $M_{a \succ b} > 0$ or not) are called *C1 rules* in Fishburn's classification (Fishburn, 1977). A typical example of a C1 rule that is also a Condorcet extension is the *Copeland* rule that selects the candidate(s) that win the most majority votes against other candidates (Copeland, 1951). More formally, it selects the candidate(s) with the highest Copeland score $S(c) = |\{d \in C : M_{c \succ d} > 0\}| = |\{d \in C : c \succ^M d\}|$.[7]

Borda is an example of a C2 rule that is not a Condorcet extension. Baldwin's rule and Nanson's rule are C2 rules that also are Condorcet extensions. Many other C2 Condorcet extensions have been defined in the social choice literature, such as *Ranked Pairs* (Tideman, 1987), *Minimax* (Young, 1977), *Black* (Black et al., 1958), and *Schulze* (Schulze, 2011).

Another Condorcet extension is Kemeny's rule (Kemeny, 1959). This rule was originally not defined as a single-winner voting rule, but as a rule returning a *ranking* of the candidates. However, it is possible to turn it into a single-winner voting rule by saying that the winner is the candidate ranked first in the output ranking. If there exists a Condorcet winner, then it will be ranked first in the ranking returned by Kemeny's rule.

Intuitively, Kemeny's rule returns the ranking of candidates that is the "closest" on average to all the rankings in the profile. The distance between two rankings of the candidates $\succ$ and $\succ'$ is defined as the number of swaps of adjacent candidates that need to be done to transform $\succ$ into $\succ'$. More formally, we have:

Kendall-tau Distance
The *Kendall tau distance*, or *swap distance* between two rankings $\succ$ and $\succ'$ is equal to

$$\mathrm{KT}(\succ, \succ') = |\{(a, b) \in C^2 : a \succ b \text{ and } b \succ' a\}|$$

For instance, the Kendall-tau distance between $a \succ b \succ c \succ d$ and $b \succ d \succ a \succ c$ is 3. This distance will be used several times in this thesis, especially in Chapters 6 and 7. The Kemeny rule is then defined as the rule f that selects the ranking(s) $f(P) = \mathrm{argmin}_{\succ^* \in \mathcal{L}(C)} \sum_{\succ \in P} \mathrm{KT}(\succ, \succ^*)$, minimizing the average Kendall-tau distance to the rankings in the profile.

Other Rules There exist other ranking-based single-winner voting rules, that we will not discuss in this thesis. In particular, we can allow randomness in the selection of the winner, leading to randomized voting rules that return a lottery over the candidates. See the recent survey of Brandt (2017) for more details.

Weak Orders

If the preferences given by the voters are weak orders, we can adapt most of the rules we defined for rankings. For C1 and C2 rules, there exists a natural generalization to weak orders, as the pairwise majority margins can be computed in the same way as for rankings $M_{a \succ b} = |\{i \in V : a \succ_i b\}| - |\{i \in V : b \succ_i a\}|$, and we ignore the voters for which $a \sim_i b$. We focus now on generalizations of scoring rules and runoff scoring rules to weak orders.

Truncated Rankings Most of the work covering weak orders is focused on truncated rankings (also called top-truncated ballots), as they are the most common type of weak orders in practice.

[7]This version of Copeland is actually Copeland^0, as pairwise ties don't give any points, while they give half a point in some other versions.

Actually, most if not all major elections using ranking-based rules such as IRV and Borda are actually using truncated rankings, and voters are generally not required to rank all candidates.

In particular, Borda is one of the most studied rule for truncated rankings, with generalizations being discussed by Emerson (2013) and later by Baumeister et al. (2012), among other scoring rules and Copeland's rule. Terzopoulou and Endriss (2021) characterized different generalizations of Borda to truncated rankings. Mainly four generalizations have been discussed in the case of Borda, but they can easily be adapted to any other scoring rule. If we consider a truncated ranking in which k candidates are ranked and $m-k$ are not, these generalizations are:

1. *Optimistic*: every ranked candidate $c \in C$ receives $s_{\sigma_i(c)}$ points, and every unranked candidate receives s_{k+1} points.

2. *Pessimistic*: every ranked candidate $c \in C$ receives $s_{\sigma_i(c)}$ points, and every unranked candidate receives s_m points (generally 0).

3. *Averaged*: every ranked candidate $c \in C$ receives $s_{\sigma_i(c)}$ points, and every unranked candidate receives $(\sum_{j=k+1}^{m} s_j)/(m-k)$ points (see Dummett (1997)).

4. *Round-down*: every ranked candidate $c \in C$ receives $s_{\sigma_i(c)+m-k-1}$ points, and every unranked candidate receives s_m points (generally 0). In other words, we only keep the last $k+1$ values of the score vector (see Narodytska and Walsh (2014)).

For instance, assume that we have the weak order $a \succ b \succ \{c, d\}$, and the Borda score vector $\mathbf{s}_{\text{Borda}} = (3, 2, 1, 0)$. The optimistic generalization gives $S(a) = 3$, $S(b) = 2$, and $S(c) = S(d) = 1$. The pessimistic generalization gives $S(a) = 3$, $S(b) = 2$, and $S(c) = S(d) = 0$. The averaged generalization gives $S(a) = 3$, $S(b) = 2$, and $S(c) = S(d) = 0.5$. Finally, the round-down generalization gives $S(a) = 2$, $S(b) = 1$, and $S(c) = S(d) = 0$.

Note that for plurality, the first three generalizations are equivalent (which makes sense, as plurality can be defined for uninominal ballots, which are truncated rankings with only one candidate ranked). For veto, the round-down and pessimistic generalizations are equivalent, and for Borda, the round-down and optimistic generalizations are equivalent, corresponding to what is called *modified Borda* in the literature (Emerson, 2007).

We can similarly define generalizations of runoff scoring rules to truncated rankings, by generalizing the score vectors in one of the four ways mentioned above. Similarly to plurality, there is only one sensible generalization of IRV to truncated rankings, in which all non-ranked candidates receive 0 points. Kilgour et al. (2020) study the prevalence and the impact of the use of truncated rankings with the IRV rule using synthetic and real data.

Weak Orders If it is possible to easily extend the first three generalizations of scoring rules and runoff scoring rules proposed for top-truncated ballots to weak orders, there seems to be little work on the topic in the social choice literature. The optimistic generalization can be extended by giving to each candidate the maximum number of points that they can obtain in their indifference class, the pessimistic generalization by giving them the minimum, and the averaged generalization by giving them the average. For instance, in the ranking $a \succ \{b, c\} \succ d$, the Borda score of a is always 3, the one of d always 0, but the score of b and c depends on the generalization: it is 2 for the optimistic one, 1 for the pessimistic one, and 1.5 for the averaged one. The round-down generalization cannot be extended to weak orders.

In a series of articles, Meek (1994, Section 6), Warren (1996), and Hill (2001) developed a way in which IRV (and in fact, every ranking-based voting rule) can be generalized to weak orders.

Their idea was to replace every weak order by several (weighted) ranking votes, corresponding to all possible ways in which the indifferences can be broken. For example, a voter with a weak order $a \succ \{b, c\} \succ d$, would be replaced by a voter of weight $1/2$ with ranking $a \succ b \succ c \succ d$ and another voter of weight $1/2$ with $a \succ c \succ b \succ d$. Similarly, a voter who reports indifference between all candidates would be replaced by $m!$ votes each with weight $1/m!$. After this replacement operation, the original (scoring or runoff) rule is applied on the obtained profile. This actually corresponds to the averaged generalization. Consider the case of IRV: at each step, the candidates in the top indifference class are receiving $1/k$ points each, where k is the size of the indifference class. We study the particular case of the generalizations of IRV to weak orders in more detail in Chapter 4.

Partial Orders, Possible Winners and Necessary Winners Finally, the different rules based on rankings have also been studied for partial orders, which is a preference format more general than weak orders. For instance, $\{a \succ b, c \succ d\}$ is a possible partial order in which a and c are not comparable, but they are not part of an equivalence class either. Among other works, Pini et al. (2009) generalized some classical social choice axioms and impossibility results to the case of partial orders, and Terzopoulou and Endriss (2019) proposed a way to average such preferences by assigning different weights to different ballots depending on their size (i.e., the number of pairs of candidates that can be compared).

Another interesting approach is the *possible* and *necessary winners* approach, in which we have a profile of partial orders, and a rule based on full rankings, and the goal is to find candidates that are winning in every possible completion of the partial orders into full rankings, giving necessary winners, or in at least one completion, giving possible winners (Konczak and Lang, 2005). Many subsequent works provide solutions to these problems and show their computational complexity for various ranking-based rules, especially scoring rules (Xia and Conitzer, 2011; Betzler and Dorn, 2010; Baumeister and Rothe, 2012). Some works also considered the particular case of weak orders (Kenig, 2019; Chakraborty et al., 2021). Informally, these works assume that voters' preferences can actually be expressed as rankings, but we are given *incomplete* preferences. There can be several possible reasons we only have partial information, for instance because of communication costs. In this thesis however, we have another interpretation of weak orders, and assume that the voters are actually indifferent between the candidates they put in the same indifference class.

Valued Preferences

Let us now mention the different principles that guide the possible ways to aggregate valued (or cardinal) preferences, without delving into the details.

We have an ordered set of possible values Z. A first possibility is to aim at maximizing the *median* value given to a candidate over all voters. In other words, each candidate is associated to its median value $z^*(c)$ such that half of the voters give a better value to c, $z_i(c) \geqslant z^*(c)$ and the other half a lower value $z_i(c) \leqslant z^*(c)$. The candidates with the highest median are then the winners. This is the idea behind the *majority judgement* rule proposed by Balinski and Laraki (2011), which uses a nominal scale of five to seven values (e.g., $Z = \{$Excellent, Very Good, Good, Acceptable, Poor, Reject$\}$). We can also select the candidate(s) that maximize the *lowest* value over all voters $z^*(c) = \min_{i \in V} z_i(c)$, corresponding to the *egalitarian* principle (Rawls, 2017).

We can define more rules when the set of possible values is numerical, e.g. $Z = \mathbb{R}$. In that case, we can associate each candidate to the *average* value they receive from the voters $z^*(c) = (1/n) \sum_{i \in V} z_i(c)$, and select the candidates with the highest average value (Gaertner and Xu, 2012). This is the idea behind the *utilitarian* principle (Mill, 2016). When Z is a finite set of numerical values (e.g., $Z = \{-1, 0, 1, 2\}$), we often call *range voting* or *evaluative voting* (Hillinger, 2005) the idea to vote using such scale and select the candidate with the highest average (or equivalently, total) value. Finally, we can also select the candidates that maximize the product of the values they receive $z^*(c) = \prod_{i \in V} z_i(c)$, which is the idea behind the *Nash* rule (Nash et al., 1950; Kaneko and Nakamura, 1979).

2.3.2 Multi-Winner Voting

In multi-winner voting, the goal is to select a subset of $k > 1$ candidates $W \subseteq C$ (also called a *committee*) based on some criterion. We encounter multi-winner voting situations quite often: in politics, to elect an assembly of representatives; in companies, to shortlist the best applicants; in recommendation systems, to select which items to recommend to the user; in facility location, to select where to build infrastructures; or in participatory budgeting, to select projects to fund.

Formally, the outcome domain of multi-winner voting is the set of all subsets of candidates $\mathcal{O} = 2^C$. However, in many cases, there is a size constraint on the number of candidates in the committee. Consider for instance that we have to choose an assembly of representatives with a fixed number of seats. In that case, we say that we want to select a committee of size k and the outcome domain is then $\mathcal{O} = \{W \subseteq C : |W| = k\}$. In the following, and unless stated otherwise, we consider that the number of candidates we want to select is fixed to $k > 1$. Thus, multi-winner voting rules take as input a profile P and a desired committee size k, and we write $f(P, k)$ for the result of the rule. For a more detailed overview of the multi-winner voting problem, we recommend the survey by Faliszewski et al. (2017).

Different Goals

Before going over the different rules that have been defined for multi-winner voting, we must distinguish different goals that can be pursued when selecting a committee of candidates. In single-winner voting, the goal is generally clear: we want to select the *most preferred* candidate. However, in multi-winner voting, we might want to achieve different goals, depending on the context. Elkind et al. (2017b) identify three main objectives that are often pursued in multi-winner elections:

1. *Excellence:* select the candidates that are *individually* the best according to voters' preferences. Consider for instance a company that wants to shortlist applicants for a job.

2. *Proportionality:* select candidates such that each group of voters is represented in the committee in proportion to its size. Typical examples include parliamentary elections, and participatory budgeting.

3. *Diversity:* select candidates such that each voter can be associated with one candidate in the committee that is high in their preferences. In that sense, voters only care about the candidate they like the most in the committee, and not about the other ones. Consider for instance a facility location problem in which infrastructures need to be built in a city: it is better to build the infrastructures uniformly around the city such that everyone is close to one infrastructure than to build all of them in the city center, even if it is the best location for the majority of the population.

Of course, other goals can be pursued in multi-winner voting. For instance, the goal in Chapter 7 is to select a committee that is inducing the most *conflict* among the voters. Another example is the study by Dong et al. (2024), in which they aim at selecting a committee of candidates that are the most *interlacing*, in the sense that the selected candidates have overlapping sets of supporters. In the rest of this section on multi-winner voting though, we focus on these three objectives, and discuss the rules that have been defined for different ballot formats.

Uninominal Preferences

With uninominal ballots, the only possibility is to select the k candidates with the highest plurality score, which is the rule called *single non-transferable vote* (SNTV).

Approval Ballots

Things get more interesting with approval preferences. Multi-winner voting with approval ballots is often referred to as *approval-based committee* voting (ABC voting for short), and rules as ABC rules. In this section, we introduce the rules that are discussed in Chapter 3. We redirect to the survey on ABC rules by Lackner and Skowron (2023) for a more detailed overview of the literature.

Generalization of Single-Winner Rules The most natural rule is *multi-winner approval voting*, which selects the k candidates with the highest approval scores. As this rule selects the k candidates that perform best according to the single-winner approval rule, it is an interesting rule for the excellence objective (Lackner and Skowron, 2018b). However, this rule is not good if we want to achieve proportionality or diversity. Consider for instance a profile with candidates $C = \{a, b, c, d\}$ in which 51% of voters approve $\{a, b\}$ and 49% approve $\{c, d\}$. With a committee size $k = 2$, the multi-winner approval rule selects $\{a, b\}$, and almost half of the voters are not represented at all by the committee.

We can also generalize *split approval* to the multi-winner setting, by selecting the k candidates with the highest split approval scores $S(c) = \sum_{i:c \in A_i} 1/|A_i|$. In the multi-winner setting, this rule is sometimes called *satisfaction approval voting* (Brams and Kilgour, 2015). It is not known to be a particularly interesting rule for any of the three objectives.

Example 2.5
Consider the following approval profile P with $k = 2$:

4: $\{a, b, c\}$ 6: $\{a, b\}$ 4: $\{c, d\}$ 2: $\{b\}$ 1: $\{d\}$

The approval scores of the candidates are $S(a) = 10$, $S(b) = 12$, $S(c) = 8$ and $S(d) = 5$. Thus, multi-winner approval voting returns the committee $\{a, b\}$. The split approval scores are $S(a) = 4/3 + 6/2 = 26/6$, $S(b) = 4/3 + 6/2 + 2 = 38/6$, $S(c) = 4/3 + 4/2 = 20/6$ and $S(d) = 4/2 + 1 = 3 = 18/6$. Thus, multi-winner split approval also returns the committee $\{a, b\}$.

Thiele Rules An interesting family of ABC rules is the family of *Thiele* rules (Thiele, 1895), named after the Danish astronomer Thorvald Nicolai Thiele. These rules follow a similar idea than scoring rules for single-winner voting with rankings, in the sense that every voter assigns a score to each committee, and the committee with the highest total score is selected. To determine the score given by a voter to a committee, a *Thiele* rule is based on a score vector $\mathbf{s} = (s_1, \dots, s_k)$ such that $s_j \geqslant 0$ for all j. Then, voter $i \in V$ gives $S_i(W) = \sum_{j=1}^{l_i} s_j$ points to the committee W where $l_i = |W \cap A_i|$ is the number of candidates from the committee that are approved by i. The total $\mathbf{s}$-score of a committee W is then $S(W) = \sum_{i \in V} S_i(W)$, and the rule selects the committee(s) with the highest score.

Thiele Rules

A *Thiele* rule based on the score vector $\vec{s}$ is the rule f defined as:

$$f(P,k) = \underset{W \subseteq C, |W|=k}{\text{argmax}} \sum_{i \in V} \sum_{j=1}^{|W \cap A_i|} s_j.$$

Note that the multi-winner approval rule is a Thiele rule and is based on the vector $\mathbf{s} = (1,\dots,1)$, as all voters give one point to the committee for each candidate they approve. Some other particular Thiele rules have been extensively studied in the literature:

- *Chamberlin-Courant Approval Voting* (CCAV), based on the score vector $\mathbf{s} = (1,0,\dots,0)$, and which was independently proposed by Chamberlin and Courant (1983). With this rule, a voter gives a point to a committee if they approve one candidate in it, and no point otherwise. This rule follows the *diversity* objective (Lackner and Skowron, 2018b).
- *Proportional Approval Voting* (PAV), introduced by Thiele (1895), is based on the harmonic score vector $\mathbf{s} = (1, 1/2, 1/3, \dots, 1/k)$. This enables PAV to balance between the demands of large and small groups, and to satisfy strong *proportionality* properties (Aziz et al., 2017a).

Other less studied families of Thiele rules include *threshold methods* (Fishburn and Pekec, 2004), in which a voter gives a point to a committee if and only if they approve at least a certain number of candidates in the committee, and *p-geometric rules* (Skowron et al., 2016), that are based on the score vector $(p^{k-1}, p^{k-2}, \dots, 1)$ with $p \geqslant 1$, rules with high values of p being closer to CCAV, and rules with low values of p to AV. Do et al. (2022) study generalizations of Thiele rules to an online setting, in which candidates are proposed one by one. They show that it is still possible to achieve some interesting proportionality and diversity properties in this setting.

Example 2.6

Consider the approval profile P from Example 2.5, still with $k = 2$. For a committee W, we denote n_1 the number of voters who approve exactly one candidate from the committee and n_2 the number of voters who approve both candidates. The CCAV score is then equal to the total number of voters who approve at least one candidate $n_1 + n_2$. To compute the PAV score, observe that voters who approve one candidate from the committee give 1 point, and voters who approve two candidates give 1.5 points. The following table shows values of n_1 and n_2 for each committee, as well as their PAV and CCAV scores:

	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$
n_1	2	10	15	12	17	5
n_2	10	4	0	4	0	4
CCAV $(n_1 + n_2)$	12	14	15	16	**17**	9
PAV $(n_1 + 1.5n_2)$	17	16	15	**18**	17	11

Thus, CCAV selects the committee $\{b,d\}$ and PAV selects the committee $\{b,c\}$.

One major drawback of Thiele rules such as CCAV and PAV is that they are NP-hard to compute (see Procaccia et al. (2008) for CCAV, Skowron et al. (2016) for PAV, and Godziszewski et al. (2021) for a more general result). A notable exception is the multi-winner approval voting rule, which is clearly computable in polynomial time, as one just needs to compute the approval score of each candidate. Peters and Lackner (2020) showed that the hardness of Thiele rules does not hold anymore when the approval profile is linear (i.e., all approval ballots are intervals of the same axis $\lhd$, see Section 2.2.6).

Sequential Thiele Rules To escape this computational hardness, a possible solution is to use greedy approximations of these rules instead of computing the exact committee that maximizes the score. This is the idea behind *sequential Thiele rules*, that were also defined in the original paper by Thiele (1895). With sequential rules, the idea is to add the candidates to the committee one by one instead of computing the best committee all at once. More precisely, for a sequential Thiele rule based on the score vector $\mathbf{s} = (s_1, \dots, s_k)$, we start with an empty committee $W = \emptyset$, and at each step $l \leqslant k$, we compute for each candidate $c \in C \setminus W$ the $\mathbf{s}$-score of the committee $W \cup \{c\}$, and we select the candidate that maximizes this score. We then add this candidate to the committee. We repeat the process until the committee is full, i.e., $|W| = k$.

We can thus define sequential-PAV and sequential-CCAV, the sequential versions of PAV and CCAV. Note that the sequential version of the multi-winner approval rule is equivalent to the non-sequential version: both select the k candidates with the highest approval score. Lu and Boutilier (2011) and Skowron et al. (2016) showed that this greedy approach is actually a good approximation of the exact rule, as the committee selected by the sequential Thiele rule based on the vector $\mathbf{s}$ is guaranteed to have a score greater than $1-1/e \approx 0.63$ times the score of the optimal committee for the original Thiele rule.

Example 2.7
Consider again the approval profile P from Example 2.5, with $k = 2$. For all sequential Thiele rules, it is clear that the first candidate to be added to the committee is the approval winner b. Now, using the table from Example 2.6, we can directly conclude that sequential CCAV selects the committee $\{b, d\}$ and sequential PAV selects the committee $\{b, c\}$.

Thiele (1895) also proposed greedy variants of the rules in which we start with a full committee $W = C$ and we repeatedly remove the candidate that contributes the least to the $\mathbf{s}$-score of the committee until we reach the desired size. This is the idea behind *reverse sequential Thiele rules*, and is similar to the principle behind *sequential runoff rules* for ranking-based single-winner voting.

Other Proportional Rules We conclude this overview of ABC voting rules by mentioning a few other rules that aim for proportionality and that are used in Chapter 3.

Sequential Phragmén is one of the many rules proposed by the Swedish mathematician Lars Edvard Phragmén (Phragmén, 1894). As for sequential Thiele rules, we construct the committee by adding the candidates one by one until it is full. The intuition behind the rule is the following (Lackner and Skowron, 2023): each voter $i \in V$ starts with a 'budget' of 0, and their budget continuously increases over time. Thus, after a time $t \in \mathbb{R}_{>0}$, the budget of every voter is t. Now, assume that the 'cost' of every candidate is 1, and as soon as there is a candidate c that is approved by voters who have a total budget of 1 together, then this candidate can be 'bought', and is added to the committee $W = W \cup \{c\}$. The budget of all the voters who approved this candidate (but only of these voters) is reset to 0. We repeat this process until the committee is full. Peters and Skowron (2020) proposed a variant of this rule, called the *Method of Equal Shares* (MES) that satisfies even stronger proportionality properties. The main difference resides in the fact that the voters are given a fixed budget $y(i) = k/n$ from the start, while it is continuously increasing in Phragmén's rule. Note that this rule does not always return a complete committee, and might require a second phase to complete the committee. Finally, Phragmén also discussed another sequential rule, which is based on the idea that each candidate in the committee should represent a quota Q of voters (for instance $Q = n/k$). This rule was independently proposed by another Swedish mathematician Gustaf Eneström (Eneström, 1896), and is thus called the *Eneström-Phragmén* rule.

We note that in the profile from Example 2.5 and $k = 2$, it can be computed that all these rules select the committee $\{b, c\}$, like PAV. This hints that these rules might be good for the *proportionality* objective. Indeed, these rules have been shown to satisfy strong proportionality properties (Peters and Skowron, 2020; Skowron, 2021). We refer to Lackner and Skowron (2023) for a formal definition and a deeper analysis of these rules.

Rankings

Let us now move to ranking-based multi-winner voting rules, with still the three objectives of excellence, proportionality and diversity in mind. Again, we will not discuss all existing rules, but only the ones that are relevant to this thesis.

Excellence: Extending Single-Winner Rules A safe principle to construct a multi-winner rule aiming for excellence is to start with a single-winner rule that assigns a score to each candidate, and select the k candidates with the highest scores. This works for all scoring rules, but also for Copeland and Minimax. Consider for instance the k-Borda rule:

k-Borda Rule
The *k-Borda* rule is the rule that selects the k candidates with the highest Borda scores.

This rule has been axiomatically characterized by Debord (1992). Another popular rule is *Bloc*, which returns the k candidates with highest k-approval scores, using the same value of k as for the desired committee size. Finally, as already mentioned for the case of uninominal ballots, the *single non-transferable vote* (SNTV) selects the k candidates with the highest plurality scores. Note that in the case of SNTV, it is not clear that it is a good rule for the excellence objective. These "best-k" rules based on single-winner scoring rules are part of the more general family of *committee scoring rules*, introduced by Elkind et al. (2017b).

Example 2.8
Consider the following preference profile P with $k = 3$:

6: $a \succ d \succ e \succ b \succ c$ 5: $b \succ e \succ d \succ a \succ c$

4: $c \succ e \succ d \succ b \succ a$ 2: $e \succ b \succ a \succ d \succ c$

In this profile, the Borda scores are $S(a) = 33$, $S(b) = 36$, $S(c) = 16$, $S(d) = 38$ and $S(e) = 47$. Thus, k-Borda returns the committee $\{e, d, b\}$. Bloc returns $\{a, e, d\}$, as these are the three candidates that are ranked in the top 3 the most. Finally, SNTV returns the committee $\{a, b, c\}$.

With the excellence objective in mind, Barberà and Coelho (2008) proposed another rule, called *sequential plurality*, which sequentially constructs the committee by adding the candidate with the highest plurality score at each step, and removing it from the profile. By eliminating the candidate, this might lead to a very different committee than the one we obtain with SNTV, even though both rules are based on plurality scores. Consider for instance the profile $P = \{3 : a \succ b \succ c, 2 : c \succ a \succ b\}$ with $k = 2$. In this profile, SNTV selects $\{a, c\}$, since the plurality score of b is 0. However, sequential plurality first selects a and eliminates it from the profile, and then selects b, giving the committee $\{a, b\}$.

Diversity: Chamberlin-Courant Intuitively, a committee is diverse if many voters put one of the candidates from the committee among the first candidates of their ranking. In a sense, the SNTV rule is achieving some sort of diversity, as it maximizes the number of voters who rank one of the candidates in the committee first. Still, a better rule for diversity is the *Chamberlin-Courant* rule (Chamberlin and Courant, 1983), which takes into account the whole rankings of the voters and not only the first-ranked candidates. It is based on the Borda score vector, but instead of giving $m - \sigma_i(c)$ points to each candidate c in the committee W like k-Borda, it only takes into account the candidate from the committee that is ranked the highest by the voter. In other words, each voter $i \in V$ gives $\max_{c \in W} m - \sigma_i(c)$ points to the committee W.

Chamberlin-Courant
The *Chamberlin-Courant* rule selects the committee(s) W (of size k) that maximize the score

$$S(W) = \sum_{i \in V} \max_{c \in W} (m - \sigma_i(c)).$$

The Chamberlin-Courant rule is arguably one of the best rules based on rankings to achieve the diversity objective (Elkind et al., 2017b). However, as for CCAV in the case of approval ballots, (Procaccia et al., 2008) showed that Chamberlin-Courant is NP-hard to compute in general, and polynomial time computable if the preference profile is single-peaked. Similarly to sequential variants of Thiele rules in the approval case, we can define a greedy variant of Chamberlin-Courant that is polynomial time computable. Skowron et al. (2015) showed that this greedy variant achieve the same approximation bound of $1 - 1/e \approx 0.63$ as sequential Thiele rules in the approval case.

Example 2.9
Consider again the profile from Example 2.8 with $k = 3$, and let us run the greedy Chamberlin-Courant rule. According to Example 2.8, the candidate with the highest Borda score is e with $S(e) = 47$. Thus, it is the first candidate added to the committee. Now, we can compute the Chamberlin-Courant scores we obtain when we add each candidate in the committee. We obtain 59 for $\{e, a\}$, 52 for $\{e, b\}$, 51 for $\{e, c\}$ and 53 for $\{e, d\}$. Thus, a is the second candidate added to the committee. The Chamberlin-Courant scores when we add the remaining candidates are now 64 for $\{e, a, b\}$, 63 for $\{e, a, c\}$ and 59 for $\{e, a, d\}$. Thus, the committee returned by greedy Chamberlin-Courant is $\{e, a, b\}$. It is possible to verify that this committee is also the one selected by Chamberlin-Courant.

Proportionality: STV (and Monroe) Finally, we consider the *proportional representation* objective. While finding the best rules for proportional representation is an active area of research (Faliszewski et al., 2019b; Elkind et al., 2017b,a), the *single transferable vote* (STV) rule has been shown to satisfy desirable proportionality properties (Tideman, 1995; Tideman and Richardson, 2000). Intuitively, STV can be seen as a generalization of IRV to the multi-winner setting, as it proceeds by repeatedly eliminating or adding candidates to the committee. The rule is based on a quota Q, for instance the Droop quota $Q = \left\lfloor \frac{n}{k+1} \right\rfloor + 1$, and is defined as follows.

Single Transferable Vote
The *Single Transferable Vote* (STV) rule constructs the committee W as follows:

1. Start with the empty committee $W = \emptyset$ and the full set of candidates $C' = C$. Set the weight of each voter to $w(i) = 1$.

2. Repeat until $|W| = k$:
   (a) If there is a candidate $c \in C'$ ranked first among the remaining candidates C' by a set of voters S with total weight at least $\sum_{i \in S} w(i) \geqslant Q$, then add this candidate to the committee W, and remove it from the set of remaining candidates: $C' = C' \setminus \{c\}$. Remove a total weight of Q from voters in the set S.
   (b) If no candidate reaches the quota, we eliminate the candidate $c \in C'$ with lowest total weight $\sum_{i \in S} w(i)$, where S is the set of voters ranking c first among candidates in C'. Thus, we set $C' = C' \setminus \{c\}$.

Note that the way we remove weight from voters in step 2.a does not affect the proportionality properties satisfied by the rule. However, it might affect the outcome of the rule. A common way to do it is to determine the minimal value α such that $\sum_{i \in S} \min(w(i), \alpha) = Q$, and then set $w(i) = w(i) - \min(\alpha, 0)$ for all voters $i \in S$.

Example 2.10
Let us consider again the profile from Example 2.8 with $k = 3$. We have $n = 17$, so the quota is $Q = \left\lfloor \frac{17}{3+1} \right\rfloor + 1 = 5$. Since a is ranked first by 6 voters, it is added to the committee and removed from the profile. We also remove 5 of the 6 voters who ranked a first. We obtain:

$$1\colon\ d \succ e \succ b \succ c \qquad 5\colon\ b \succ e \succ d \succ c$$
$$4\colon\ c \succ e \succ d \succ b \qquad 2\colon\ e \succ b \succ d \succ c$$

Now, b is reaching the quota, as it is ranked first by 5 voters. Thus, it is added to the committee and removed from the profile. We also remove the 5 voters who ranked b first. We obtain:

$$1\colon\ d \succ e \succ c \qquad 4\colon\ c \succ e \succ d \qquad 2\colon\ e \succ d \succ c$$

No candidate reaches the quota, so the candidate with lowest plurality score d is eliminated. This gives one more supporter to e, however it is not enough for any candidate to reach the quota, thus e is eliminated too, and c is now reaching the quota. Thus, STV returns $\{a, b, c\}$.

One of the earliest mention of the STV system is from the 19th century and the treatise by Hare (1861). STV has been used in many large-scale political elections for over a century, in particular in Australia, Ireland and Malta.

Finally, another well-studied rule that aims at returning a committee satisfying some form of proportional representation is the Monroe rule (Monroe, 1995). Informally, it forces every candidate in the committee $c \in W$ to represent exactly $Q = n/k$ voters (more precisely between $\lfloor n/k \rfloor$ and $\lceil n/k \rceil$ voters), and each candidate c in the committee receives a score $m - \sigma_i(c)$ from each voter i they represent. In that sense, it is quite similar to Chamberlin-Courant, except that the voters are not necessarily represented by the candidate they prefer the most in the committee. Like Chamberlin-Courant, Monroe's rule is NP-hard to compute in general (Procaccia et al., 2008), but in contrast to Chamberlin-Courant it has not been shown to be easier to compute with single-peaked preferences (Betzler et al., 2013). For this rule, Skowron et al. (2015) also proposed a greedy variant called *greedy Monroe*, that interestingly satisfies good proportionality properties, that are not satisfied by the original rule.

Weak Orders

There is much less work on multi-winner voting with weak orders, especially if we do not restrict preferences to truncated rankings. If preferences are truncated rankings, most of what we saw in Section 2.3.1 for the single-winner case can be adapted to the multi-winner case. In particular, the definition of STV still works if rankings are top-truncated. If we now consider weak orders and not only truncated rankings, there exist different generalizations of STV, that are analogous of the generalizations of IRV to weak orders that we mentioned in Section 2.3.1 (and that we will extensively discuss in Chapter 4). These generalizations of STV to weak orders are discussed on the original version of the paper on which Chapter 4 is based (Delemazure and Peters, 2024).

Aziz and Lee (2020) introduced another rule based on weak orders that satisfies interesting proportionality properties, called the *expanding approval rule*. On a high-level, it is similar to STV, as it starts by considering the top-1 preference of each voter (i.e., their favorite candidate) and adds to the committee all candidates that are reaching some quota Q in number of supporters, but instead of eliminating candidates when none is reaching the quota, the rule instead considers the top-2 preferences of the voters, and so on until it finds a candidate that reaches the quota.

Valued Preferences

We do not discuss multi-winner voting rules based on valued preferences in this thesis. However, we note that all the rules discussed in Section 2.3.1 for the single-winner case can be adapted to the multi-winner case by taking the top-k candidates, thus going in the direction of the individual excellence objective.

Committees of Variable Size

We assumed so far that the size of the desired committee k was fixed. However, in some cases, it might be more interesting to have a committee that is not constrained to a fixed size if this committee is qualitatively better. For instance, in the shortlisting context, we might want to select all candidates that are good enough to be part of the committee, without restricting its size. Such rules have been studied in the literature, and are sometimes referred as *dichotomy functions* (Duddy et al., 2014).

For the individual excellence objective, the rules based on scores can be adapted to the unconstrained size case by deciding to select in the committee all candidates that have a score higher than some threshold τ (Lackner and Maly, 2021). For instance, with approval ballots, we can include in the committee all candidates approved by more than half of the voters. This is actually what the *net approval* rule, proposed by Kilgour (2010), does. In other words, with this rule each voter gives $+1$ to each candidate in the committee they approve, and -1 to each candidate in the committee they disapprove, which is equivalent to saying we maximize the value of $\sum_{i \in V} |W \cap A_i| - |W \setminus A_i|$ over all committee W. Faliszewski et al. (2020) proposed a family of rules generalizing this principle by considering that approvals and disapprovals can have different weights (for instance, a voter could give $+1$ to each approved candidate, and $-1/2$ to each disapproved candidate).

Another family of rules are *next-r rules* for $r \geqslant 2$ (Kilgour, 2016; Brams and Kilgour, 2012), which select the smallest non-empty committee W such that every candidate in W has an approval score higher than the sum of the scores of any set of r candidates that are not in the committee. Kilgour (2016) also proposed *first majority*, which returns the smallest committee such that the sum of the approval scores of the candidates in the committee is greater than the sum of the approval scores of the candidates outside the committee.

Lackner and Maly (2021) also proposed approval-based shortlisting rules. For instance, the *largest gap* rule returns the committee that maximizes the minimal gap between the approval score of a candidate in the committee and a candidate outside it. They also proposed the *size priority* rule which comes with a preference order over possible sizes of the committee $>$, and returns the committee of the most preferred size that maximizes the approval score, and which does not require ties to be broken. Faliszewski et al. (2020) discussed another approval-based rule, this time aiming for diversity, that returns all committees of minimal size such that every voter (with a non-empty approval ballot) approves at least one candidate in the committee. They call this rule the *minimal representing committee* rule.

Finally, another possible rule is to select all candidates that have an approval score higher than the average approval score (Duddy et al., 2016). This is also the idea behind the ranking-based rule *Borda mean*, proposed by Brandl and Peters (2019) and that returns a committee W containing all candidates c that have a Borda score higher than the average Borda score. Note that this idea can be applied to any scoring rule.

In Chapter 5 of this thesis, we discuss the problem of selecting a subset of candidates from rankings such that each candidate in the committee represents at least a certain number of voters (in the sense that these voters rank them higher than any other candidate from the committee).

Apportionment

We conclude this section on multi-winner voting by briefly discussing the closely related problem of apportionment (Balinski and Young, 2010; Pukelsheim, 2014). A typical example of apportionment is the allocation of seats in a parliamentary election. The main difference with the multi-winner voting problem is that candidates (in general called *parties*) can receive more than one seat. More formally, there is a total number of seats k to distribute to parties.

Most parliamentary elections in real life use uninominal ballots. However, contrary to the other settings, there exists a variety of interesting rules to distribute the seats in a proportional fashion among the parties with only uninominal preferences. The two most famous families of rules are the *largest remainder* rules and the *highest average* rules. The most popular rule in practice is the *D'Hondt* method, which is a highest average rule (Balinski and Young, 1978).

Brill et al. (2018) proposed adaptations of approval-based multi-winner rules to the apportionment problem, which we further studied in Delemazure et al. (2023a). Airiau et al. (2023, Section 7) considered apportionment based on complete rankings of the competing parties. The problem studied in Chapter 5 of this thesis is also closely related to this problem, as it discusses the problem of selecting the subset of parties that will be allowed to have seats in a parliament from a preference profile of rankings.

2.3.3 Candidate Ordering

Let us now move to another kind of problem, in which the goal is not to select winners, but instead to *learn* the structure of the electorate and of the candidate set, based on the preferences. More specifically, we consider the problem of obtaining an ordering (or *axis*) of the candidates $\lhd$. The intuition is that candidates supported by similar sets of voters should be close to each other on the axis, while candidates who have very different sets of supporters should be far apart. For instance, if the axis is $a \lhd b \lhd c \lhd d$, candidates b and c probably have overlapping sets of supporters, while it is less likely for a and d. Thus, this problem is tied to the notions of structured preferences that we discussed in Section 2.2.6, in particular single-peakedness and interval approval preferences, which both rely on the idea that there exists an underlying ordering of the candidates. In a sense, we are looking for an axis for which the preference profile is *almost* single-peaked (for rankings) or linear (for approval ballots).

This problem has numerous applications, from political science (where it is often assumed that political parties can be approximately ordered on some left-right ideological axis), to relative dating in archeology and geology, or scheduling (which all rely on temporal axes). We discuss these various applications in more detail in Chapter 6.

Formally, the outcome domain of this problem is the set of all linear orders over candidates $\mathcal{O} = \mathcal{L}(C)$. However, this problem should not be confused with the one of obtaining a *ranking* of the candidates: a ranking represents the preferences of the society, while the ordering *describes* their structure. In particular, recall that the direction of the ordering does not matter, for instance the orderings $\lhd = abcd$ and $\overleftarrow{\lhd} = dcba$ are equivalent. We call the rules that return such orderings *axis rules*.

Clearly, if the preferences are uninominal, it is not possible to identify any relationship between the candidates. The approval case is the topic of Chapter 6, and has not been discussed previously (to our knowledge) in the computational social choice literature. Finally, a few papers have studied the problem of obtaining an ordering using rankings. In the remainder of this section, we briefly go over their main contributions.

Rankings

The main approach to uncover a good ordering of the candidates is to find the ones for which the profile of rankings is *nearly single-peaked*, in the sense that the preference profile is almost single-peaked for this axis. The most common definition of near single-peakedness is called *partial single-peakedness* by Niemi (1969) and *k-voter deletion single-peakedness* in more recent works (Elkind and Lackner, 2014; Faliszewski et al., 2014). Intuitively, a preference profile of rankings is k-voter deletion single-peaked for an axis $\lhd$ if it is possible to delete k voters from the profile such that the remaining profile is single-peaked for $\lhd$. This inspired the following axis rule:

Voter Deletion
The *voter deletion* rule selects the axes $\lhd$ that minimize the number of voters to remove from the profile to make it single-peaked for $\lhd$.

Erdélyi et al. (2017) proposed a similar notion of near single-peakedness that relies on deleting candidates instead of voters, which is called *k-candidate deletion single-peakedness*. They also introduced *k-global swaps single-peakedness* that says we can perform k swaps of adjacent candidates in voters' rankings and obtain a single-peaked profile. Then, the *global swaps* rule selects the axes that minimize the value of k for which the profile is k-global swaps single-peaked for them.

Faliszewski et al. (2014) introduced another interesting notion of near single-peakedness that they call *PerceptionFlip*$_k$ *single-peakedness*. Intuitively, a profile is PerceptionFlip$_k$ single-peaked for an axis $\lhd$ if for each ranking in the profile, it is possible to make it single-peaked for $\lhd$ by performing at most k swaps of adjacent candidates *on the axis*. Erdélyi et al. (2017) proved that this is equivalent to *k-local swaps single-peakedness*, in which swaps are not done on the axis, but on the rankings, but unlike for the global swaps single-peakedness, here k is the maximum number of swaps that can be done *per ranking*, and not globally.

Finally, Escoffier et al. (2021) introduced a notion of near single-peakedness based on the idea of *forbidden triples*. Informally, a triple of candidates (a, b, c) is a forbidden triple for an axis $\lhd$ and a ranking $\succ$ if they are breaking the single-peakedness of $\succ$ for $\lhd$. More formally, this means that a is ranked first in $\succ$ and $a \lhd c \lhd b$ or $b \lhd c \lhd a$, but $a \succ b \succ c$. Then, a profile is *k-forbidden triples*

Chapter	Outcome
Chapter 3: Approval with Runoff Rules	Single-winner
Chapter 4: Runoff Scoring Rules	Single-winner
Chapter 5: Party Selection Rules	Multi-winner (unconstrained size)
Chapter 6: Axis Rules	Orderings
Chapter 7: Conflict Rules	Multi-winner ($k = 2$)

Table 2.3: Outcome types of each chapter's problem.

single-peaked if there exists at most k forbidden triples in the profile for some axis. This inspired the following rule:

Forbidden Triples
The *Forbidden Triples* rule selects the axes $\lhd$ that minimize the number of forbidden triples in the profile for $\lhd$.

Tydrichová (2023) axiomatically analyzed the different axis rules and showed that Forbidden Triples satisfies nice axiomatic properties. Some of the rules we discuss in Chapter 6 for the approval case are inspired by the rules we just discussed.

Distance-rationalizable Rules

Before concluding this section, note that these axis rules, as well as some of the single-winner and multi-winner rules, can be seen as what is called *distance-rationalizable rules* in the social choice literature (Elkind et al., 2015). On a high level, a rule is *distance-rationalizable* if there exists a distance function between ballots $d(\cdot,\cdot)$, and a class of preference profiles $\mathcal{C} \subseteq X^n$ in which there is always a consensual outcome, such that for any preference profile $P = (X_1, \dots, X_n)$, the rule will necessarily return the consensus outcome of the preference profile from the (consensus) class $P' = (X'_1, \dots, X'_n) \in \mathcal{C}$ that is the closest to P according to the distance d. More formally, we have $f(P) = f(P')$ where $P' = \text{argmin}_{P' \in \mathcal{C}} d(P, P') = \text{argmin}_{P' \in \mathcal{C}} \sum_{i \in V} d(X_i, X'_i)$.

For instance, in single-winner voting with rankings, it is clear by its definition that Kemeny's rule is distance-rationalizable for the Kendall-tau distance and the consensus class of *strongly unanimous* profiles, in which all voters submit the exact same ranking. Meskanen and Nurmi (2008) showed that many other well-known single-winner voting rules are distance-rationalizable. For instance, Borda is distance-rationalizable for the Kendall-tau distance and the class of *unanimous* profiles, in which all voters rank the same candidate first, and plurality is distance-rationalizable for the same class of unanimous profiles, but using the *discrete distance*, which is equal to 1 if the two rankings are different, and 0 otherwise. In other words, this distance counts the number of voters that need to be removed to make the profile part of the consensus class.

Finally, for the rules returning an ordering of the candidates, that we discussed in this section, it is clear by their definitions that they are distance-rationalizable if we consider the consensus class of single-peaked profiles. In particular, voter deletion is distance-rationalizable using the discrete distance, and global swaps is distance-rationalizable using the Kendall-tau distance.

2.3.4 Summary

Table 2.3 summarizes the outcome types of the problems discussed in this thesis. Note that in Chapter 3, the outcome is a single candidate, but the rules we introduce are based on ABC rules.

2.4 Axioms

We saw that for a given ballot format and a desired outcome type, we can generally define many rules returning an outcome of the desired type based on the preferences of the voters. Once several rules have been defined, the next step is to evaluate them, in order to better understand in which context it is relevant to use each rule. The two main tools we use in this thesis to evaluate and compare rules are the *axiomatic analysis*, in which we define desirable properties and check which rules satisfy them, and the *experimental analysis*, in which we simulate preference profiles to see how the rules behave in practice. These profiles can be artificially generated using some random model (giving *synthetic data*), or based on real preferences (giving *real data*). In this section, we focus on the axiomatic analysis, and we cover the experimental analysis in Section 2.5.

The idea of the axiomatic analysis is to define a set of normative *axioms* $\mathcal{A}_1, \dots, \mathcal{A}_k$, that generally sound desirable. There is no in-between: either a rule f satisfies the axiom, or it does not satisfy it. However, one axiom can be strictly stronger than another one (i.e., if a rule satisfies the first, stronger axiom, then it must also satisfy the second, weaker one). The axiomatic analysis relies on mathematical proofs to show whether or not rules satisfy axioms.

Besides proving which rules satisfy which axioms, the axiomatic analysis can lead to other meaningful results. In particular, we often find incompatibilities between axioms, by saying that no rule of a certain family can satisfy all of the axioms that are in some set $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$. Some of these theorems are foundational results of social choice theory, such as Arrow's and Gibbard's impossibility theorems. Impossibility theorems are quite negative, as they state that the set of rules that satisfy a certain set of axioms is *the empty set*. We can also look for more positive results, that state that the set of rules that satisfy a certain set of axioms is *a singleton*, meaning that there exists exactly one rule f that satisfies all of the axioms. We call this an axiomatic *characterization* of the rule f. A famous example of such result is May's characterization of the *majority rule* in the case we only have $m = 2$ candidates, that we discussed in Section 2.3.1. We can also characterize families of rules instead of single rules. A typical example is the family of scoring rules for ranking-based single-winner voting (Smith, 1973; Young, 1975).

In this section, we introduce several *categories* (or families) of axioms. Of course, each voting problem has its own set of axioms, as many desirable properties depend on the particular context of the problem. Thus, as for our overviews of preferences, outcomes and rules, we will not be exhaustive in this section, and we will focus on the most studied families of axioms, that appear in several chapters of this thesis. The different families of axioms we will introduce are the following:

- *Symmetry (Section 2.4.1):* What should happen if we permute the candidates or the voters in the profile?
- *Efficiency and representation (Section 2.4.2):* What should happen in some specific profiles for which there are clearly good outcomes or clearly bad outcomes?
- *Reinforcement (Section 2.4.3):* What should happen if we combine two profiles for which the outcome is the same?
- *Independence axioms (Section 2.4.4):* What should happen if we remove one or several candidates from the profile?
- *Strategyproofness and monotonicity (Section 2.4.5):* What should happen if one voter changes their preferences?

We can separate these axioms into two families based on Fishburn (1973) categorization: *intra-profile* properties, that state what should be the outcome for some specific profiles, and *inter-profile* properties, that state how the outcomes of different profiles are related. They are sometimes also referred to as respectively *functional* and *relational* properties. Efficiency and representation axioms say what should happen given *one* preference profile, so they are intra-profile (or functional) properties, while the other axioms are inter-profiles (or relational) properties.

Zwicker (2016) proposes yet another categorization of axioms (in the single-winner voting context) into three groups: the ones of low strength that *must* be satisfied by any reasonable rule, the ones of middle strength that *might or might not* be satisfied by rules, and finally the stronger ones, that are *never* satisfied by reasonable rules, generally due to impossibility theorems.

2.4.1 Symmetry

Neutrality and *anonymity* are some of the least controversial social choice axioms (at least, for irresolute rules). Neutrality states that the rule should treat all candidates the same way, and Anonymity states that the rule should treat all voters the same way. More formally, we have:

Anonymity
A rule f is *anonymous* if for any permutation π of the voters and any preference profile $P = (X_1, \dots, X_n)$, we have $f(\pi(P)) = f(P)$.

In this definition, we have $\pi(P) = (X_{\pi(1)}, \dots, X_{\pi(n)})$. We define neutrality similarly:

Neutrality
A rule f is *neutral* if for any permutation π of the candidates and any preference profile P, we have $f(\pi(P)) = \pi(f(P))$.

Here, $\pi(P) = (\pi(X_1), \dots, \pi(X_n))$ indicates that we permute the names of the candidates in the preferences $X_1, \dots, X_n$ of the voters according to the permutation π. For instance, with the permutation $\pi = (a \to b, b \to c, c \to a)$ we have $\pi(a \succ b \succ c) = b \succ c \succ a$ if preferences are rankings, and $\pi(\{a, b\}) = \{b, c\}$ if preferences are approval ballot.

For irresolute rules, these axioms are generally required. Actually, all the irresolute rules introduced in Section 2.3, as well as the ones that are introduced in the different chapters of this thesis satisfy anonymity and neutrality. Things get more difficult for resolute rules, as they need to break ties in case there are several winners. Consider for instance a single-winner election with two candidates a and b, and a profile P containing two voters with preferences respectively $a \succ_1 b$ and $b \succ_2 a$. Any resolute rule needs to return exactly one candidate, but the profile is perfectly symmetric, so the choice needs to be arbitrary, or according to a tie-breaking order. Let us assume without loss of generality that $f(P) = a$. Then using the permutation $\pi = (a \to b, b \to a)$ of the candidates, we switch the names of a and b, but because the profile is symmetric, we fall back on the same profile with voter 1 and 2 swapped. If we now consider a permutation π' of the voters, that swaps voter 1 and 2, then we exactly have $\pi'(\pi(P)) = P$. However, by neutrality we have $f(\pi(P)) = \pi(f(P)) = \pi(a) = b$, and by anonymity we have $f(\pi'(\pi(P))) = f(\pi(P)) = b$, which is a contradiction with $f(P) = a$. A generalization of this proof gives Moulin's impossibility theorem (Moulin, 1983) that states that no reasonable rule (i.e., that satisfies Pareto) can be resolute, neutral and anonymous.

The issue is that if several candidates are tied as winners, we somehow need a way to break the ties. The two main ways to break ties are to either favor some voter(s) (breaking anonymity), or favor some candidate(s) (breaking neutrality). In practice, breaking anonymity would means breaking the democratic principle *"One man, one vote"*. Thus, the latter is often done by using a tie-breaking order over the candidates. It is also fair to assume that in most cases, if the number of voters is large enough, the probability of ties is very low for most rules (Xia, 2021; Janeczko and Faliszewski, 2023). For this reason, neutrality is often considered a too strong axiom when we are dealing with resolute rules, and the weaker axiom of *non-imposition* is sometimes used instead. It states that for every possible outcome in the outcome domain $\mathcal{O}$, there exists at least one preference profile for which this outcome is selected.

2.4.2 Efficiency and Representation

We now consider axioms that impose a constraint on the outcome for a subset of profiles. In single-winner voting with rankings, the *Condorcet criterion* is an example of such an axiom: it states that if there exists a Condorcet winner, then this candidate should be the winner. This gives the *Condorcet extension* family of rules that we discussed in Section 2.3.1. We can also define Condorcet losers, analogously to Condorcet winners: in an ordinal profile, the Condorcet loser is the candidate c that loses the majority vote against any other candidate, i.e., for all $d \in C$ such that $d \neq c$, we have $d \succ_M c$. Then, the *Condorcet loser criterion* states that if there exists a Condorcet loser, then this candidate should *not* be the winner. While the Condorcet criterion *imposes* the outcome, the Condorcet loser criterion *excludes* one possible outcome. These two examples are instances of respectively *positive efficiency* axioms (if there is a clearly correct outcome, then select it) and *negative efficiency* axioms (if there is a clearly incorrect outcome, then do not select it). Finally, we will also discuss *representation* axioms, that are a kind of positive efficiency axioms.

Negative Efficiency

We start with negative efficiency axioms. For the single-winner voting setting with rankings, we already mentioned the *Condorcet loser criterion.* A less controversial axiom is the Pareto-efficiency axiom, or *Pareto principle* (Pareto, 1919). Informally, it states that if all voters weakly prefer some candidate a to another candidate b, and at least one voter strictly prefers a to b, then b should not be the winner. Formally, we say that a candidate a *Pareto-dominates* another candidate b in a profile P of weak orders if (1) for all $i \in V$, $a \succcurlyeq_i b$ and (2) there exists $i \in V$ such that $a \succ_i b$.

If we restrict the preferences to rankings instead of weak orders, then a Pareto dominates b if and only if $a \succ_i b$ for all voters, and if we restrict the preferences to approval ballots, then a Pareto-dominates b if all voters who approve b also approve a and at least one voter approves a and not b. The definition also works for valued preferences, since any evaluation function z_i from candidates to an ordered set Z induces a weak order $\succcurlyeq_i$.

Pareto-Efficiency
A rule f is *Pareto-efficient* if it never returns a Pareto-dominated candidate.

Note that this axiom can apply to resolute and irresolute rules. In particular, one can easily check that all of the single-winner rules discussed in Section 2.3.1 satisfy this axiom. A weaker version of this axiom, designed for irresolute rules, states that if for a preference profile P, f returns a Pareto-dominated candidate b, then all the candidates that Pareto-dominate b should also be returned by the rule for this profile (tied with b).

There are several ways to generalize Pareto-efficiency to multi-winner voting. For approval ballots, Lackner and Skowron (2019) proposed an efficiency axiom which relies on a domination relation between committees. They say that a committee W dominates a committee W' if all voters approve weakly more candidates in W than in W', and at least one voter approves strictly more. As with Pareto-efficiency, we can define a weak version of this axiom for irresolute rules. Lackner and Skowron (2020) later showed that surprisingly few ABC rules satisfy even the weak version of the axiom: among the rules we introduced in Section 2.3.2, only Thiele rules and SAV satisfy the weak version of the axiom, and only AV, PAV and SAV satisfy the strong version. For the ordinal setting, Lainé et al. (2025) consider several notions that first transform voters' rankings over candidates into weak orders over committees (for instance using a lexicographic order), and then apply the classical Pareto principle to these weak orders.

Positive Efficiency

In a sense, positive efficiency axioms are stronger than negative efficiency ones, as they *impose* the outcome. We already mentioned the Condorcet criterion, which states the following:

Condorcet Criterion
A rule f satisfies the *Condorcet criterion* if for every profile P that admits a Condorcet winner, it returns the Condorcet winner.

Fishburn (1981) generalized this principle to multi-winner voting by saying that a committee W is a Condorcet committee if for any other possible committee W', a majority of voters prefers W to W'. In the approval case, we use the same preference notion between committees as the one we used for Pareto-efficiency: a voter prefers the committee in which they approve the most candidates. Another way to generalize the Condorcet criterion to the multi-winner setting is to say that W is a Condorcet committee if every candidate $c \in W$ is preferred by a majority of voters to every candidate $d \notin W$ (Gehrlein, 1985; Ratliff, 2003).

Related to the idea of *distance-rationalizable rules* (Section 2.3.3), we can define a more general family of axioms that says that if the preference profile is part of a consensus class, then the outcome must be the 'consensual' one. The Condorcet criterion is an axiom of this family, with the consensus class of profiles with a Condorcet winner. Similarly, we can define the *unanimity* axiom for single-winner voting, that states that if all voters put the same candidate first in their ranking (in case of rankings), or if all voters approve this candidate (in case of approval preferences), then it should be the winner. This can be generalized to multi-winner voting with committee size k by saying that if all voters put the same k candidates first in their rankings (not necessarily in the same order) or approve all of them, then they should be selected (Elkind et al., 2017b).

For the problem of finding the best candidate ordering using voters' rankings (that we discussed in Section 2.3.3), an axiom of this sort would be the following:

Consistency with Single-Peakedness
An axis rule f is *consistent with single-peakedness* if for every profile P that is single-peaked for an axis $\lhd$, the rule returns all axes $\lhd$ for which P is single-peaked.

Representation

Finally, let us briefly discuss some *representation* axioms. On a high level, these axioms states that if a group of voters is large enough and cohesive in their preferences, they should be represented somehow in the outcome. Axioms of this family have been particularly studied in the context of multi-winner voting, and more specifically for the *proportional representation* objective. In the case preferences are rankings, Dummett (1984) proposed the following axiom.

Proportionality for Solid Coalitions
A rule f satisfies *proportionality for solid coalitions* if for every profile P and committee size k, if there exists a value ℓ such that $1 \leqslant \ell \leqslant k$ and a group of $\ell \cdot n/k$ voters who all rank the same ℓ candidates first, then these ℓ candidates should be in the committee(s) returned by f.

This is based on the principle that each candidate in the committee represents a quota $Q = n/k$ of voters, and if there exists a group of voters of size ℓ times the quota who all support the same ℓ candidates, then these candidates should all be selected, as representatives of this group. None of the rules we discussed in Section 2.3.2 satisfy proportionality for solid coalitions in general, however Elkind et al. (2017b) showed that STV, as well as SNTV and greedy Monroe, satisfy a weaker version of the axiom with ℓ fixed to $\ell = 1$. Woodall (1994) proposed a variant of this axiom called the *Droop proportionality criterion*, which uses the Droop quota $Q = \lfloor n/(k+1) \rfloor + 1$ instead of the Hare quota $Q = n/k$, and which is satisfied by a variant of STV. Bardal et al. (2025) showed that solid coalitions rarely occur in real-world elections, and propose other ways to measure the proportionality of ranking-based multi-winner voting rules. Finally, this proportionality principle has also been extended to the approval voting case through a series of proportionality axioms (Aziz et al., 2017a; Sánchez-Fernández et al., 2017; Aziz et al., 2018; Peters et al., 2021).

Let us now consider the single-winner case. In a sense, the *unanimity* axiom can be seen as a representation axiom: a large group of voters (here, all voters) is cohesive in their preferences (they all put the same candidate first, or all approve it), so this group should be represented in the outcome, by this candidate. It is actually equivalent to proportionality for solid coalitions with $k = 1$. A weaker axiom in the case of rankings is the *majority criterion*, which corresponds to the Droop proportionality criterion with $k = 1$. Formally, we have:

Majority Criterion
A rule f satisfies the *majority criterion* if for every profile P and every candidate $c \in C$, if a majority of voters rank c first, then c should be the winner.

This follows the idea that the majority rule is the only good rule when $m = 2$ (May's theorem). It is at the same time a strengthening of the unanimity axiom and a weakening of the Condorcet criterion, since a candidate ranked first by a majority of voters is necessarily a Condorcet winner. It is actually referred to as the weak Condorcet criterion by Lepelley (1992), who showed that plurality is the only scoring rule satisfying this axiom in the ranking setting (see also Sanver (2002)). One (other) way to generalize the majority criterion to the multi-winner case is to say that if a majority of voters rank all the candidates in a committee W among the first k candidates of their rankings, then this committee should be selected (Debord, 1993). Brandl and Peters (2022) generalized the majority criterion to approval preferences with the *respect for unanimous majorities* axiom, which they used to characterize the single-winner approval voting rule.

Respect for Unanimous Majorities
A rule f satisfies *respect for unanimous majorities* if for every approval profile P, if a majority of voters cast the same approval ballot A^*, then the candidate returned by f should be in A^*.

In Chapter 4, we define generalizations of the majority criterion to the case of weak orders, and use it to characterize one of the generalization of IRV to weak orders.

2.4.3 Reinforcement

For two profiles $P^1 = (X^1_1, \dots, X^1_{n_1})$ and $P^2 = (X^2_1, \dots, X^2_{n_2})$, we denote by $P = P^1 + P^2$ the profile that is the combination of the two profiles, i.e., $P = (X^1_1, \dots, X^1_{n_1}, X^2_1, \dots, X^2_{n_2})$. *Reinforcement* (Smith, 1973; Young, 1974), sometimes referred as *consistency*, states that if a rule returns the same outcome in two different profiles P_1 and P_2 with the same set of candidates C, but distinct sets of voters V_1 and V_2 (with $V_1 \cap V_2 = \emptyset$), it should *also* return this outcome in the combination of the two profiles $P = P^1 + P^2$.

Reinforcement
A rule f satisfies *reinforcement* if for all profiles P^1 and P^2 with the same set of candidates but distinct sets of voters, if $f(P^1) \cap f(P^2) \neq \emptyset$, then $f(P^1 + P^2) = f(P^1) \cap f(P^2)$.

The rationale behind this axiom is quite natural: if a candidate wins in districts A and B, it should still win if we merge the two districts. Surprisingly, many rules, such as plurality with runoff and IRV, fail this axiom. However, any rule that is based on a score function $S(X, O)$ that assigns a score to each possible outcome $O \in \mathcal{O}$ based on a ballot $X \in \mathcal{X}$, and that returns the outcome(s) O^* maximizing the sum of scores $\sum_{i \in V} S(X_i, O^*)$ clearly satisfies reinforcement. This is for instance the case of the scoring rules in single-winner voting with rankings, and Smith (1973) and Young (1975) actually showed that scoring rules are the only rules that satisfy neutrality, anonymity and reinforcement in this setting. For the approval-based multi-winner setting, Lackner and Skowron (2018b) discussed this axiom and proved that a family of rules they called *ABC scoring rules*, and that includes Thiele rules and SAV, can be characterized by reinforcement and a few other axioms. A weakening of reinforcement is *homogeneity*, which states that if we duplicate a profile ℓ times, the outcome should be the same as in the original profile (Smith, 1973). More formally, if $P' = \ell P = P + \dots + P$, then $f(P') = f(P)$.

2.4.4 Independence Axioms

We now move to independence axioms. Consider for instance the single-winner voting setting. Independence axioms say that if we remove some candidates from the profile, the outcome should not change (except of course if we remove the winner). In the following, we write $P_{C'}$ the restriction of the profile P to the subset of candidates $C' \subseteq C$, i.e., the profile in which we removed all candidates in $C \setminus C'$ from all preferences in the profile. For simplicity, if $C \setminus C' = \{c\}$ (i.e., we remove only one candidate), we write P_{-c}. Independence axioms link the outcome of the rule for a profile P to the outcome of the rule for a reduced profile $P_{C'}$.

Independence of Irrelevant Alternatives

Kenneth Arrow's *independence of irrelevant alternatives* (IIA) axiom was originally introduced for ranking-based rules returning a ranking of the candidates (Arrow, 1950). It informally states that the society's preference between two candidates a and b should only depend on the individual preferences of voters between a and b, as opposed to taking into account how they rank the other candidates. In other words, for a rule f that takes as input an ordinal profile and outputs an aggregated ranking of the candidates, the preferences between a and b in the output ranking should be the same in all pairs of profiles P, P' such that $P_{\{a,b\}} = P'_{\{a,b\}}$. Consider for instance the following profiles:

P :	51: $a \succ c \succ d \succ e \succ b$	49: $b \succ a \succ c \succ d \succ e$
P' :	51: $a \succ b \succ c \succ d \succ e$	49: $b \succ c \succ d \succ e \succ a$

These two profiles are such that $P_{\{a,b\}} = P'_{\{a,b\}}$, so IIA imposes that the relative order of a and b should be the same in the output rankings $f(P)$ and $f(P')$.

Then, Arrow's theorem states that the only rules that take as input any possible ordinal preference profile, and that satisfy IIA and Pareto-efficiency, are dictatorships, i.e., the outcome is always the ranking of some voter (the dictator). More formally, there exists a voter $i \in V$ such that for all profiles P, we have $f(P) = \succ_i$ the ranking of voter i in P. Arrow's theorem has had, until today, a tremendous importance in social choice. It is often seen as negative, since there are many compelling arguments in favor of IIA (see for instance Maskin (2020)). Still, IIA has been criticized for not taking into account preference intensities, and this may result in a loss of social welfare. To understand this argument, consider the two profiles P and P' displayed above. If it is clear that a should be better ranked than b in the outcome of the first profile P, can we really say the same for the second profile P', in which almost half of the voters rank a last, while all voters rank b among their top 2? This issue has been previously discussed in the literature (Coakley, 2016; Pearce, 2021; Osborne, 1976; Hillinger, 2005; Lehtinen, 2011), and we quantitatively analyzed in Delemazure et al. (2024b), in which we used the notion of distortion to show that enforcing this axiom may cause a loss of social welfare in many cases. This issue motivated Maskin (2020) to propose a weak variant of IIA that can be satisfied by some rules, such as Borda.

If we adapt the definition of IIA to single-winner voting, we obtain the following axiom.

Independence of Losers

A rule f is *independent of losers* if for all profiles P with $c \notin f(P)$, we have $f(P) = f(P_{-c})$.

In other words, removing losers from the profile should not change the result of the rule. Intuitively, it seems appealing for a voting rule to satisfy this axiom: a losing candidate should not be able to change the result of an election by withdrawing from the election. Situations in which a losing candidate is preventing another candidate from winning because it is "stealing" votes from them are very frequent in real-world elections, and are known as instances of the *spoiler effect*: by running in the election, a third-party candidate is spoiling the election for another candidate, generally one who is ideologically close to them, and is more likely to win.

Unfortunately, following Arrow's impossibility theorem, independence of losers is not satisfiable by any ranking-based rule. However, Black (1948) showed that Arrow's impossibility result does not hold anymore if we consider single-peaked preferences, since Condorcet extensions would select the Condorcet winner, which does not change if we remove losers from the profile. For the approval-based setting, Brandl and Peters (2022) showed that approval voting satisfies the axiom, and even characterized the rule with this axiom and a few other uncontroversial axioms. If we adapt the axiom to the multi-winner setting by saying that removing a candidate that is not in the committee (or in any selected committee in the case of irresolute rules) should not change the outcome of the rule, then we can check that Thiele and sequential Thiele rules satisfy it (including the multi-winner approval rule), as the score of each committee only depends on the candidates inside the committee. Finally, for valued preferences, most rules satisfy this axiom if we assume that voters do not change their preferences depending on the set of candidates C on which they can express preferences (a debatable assumption according to Sen (1993), as the set of candidates might bring context and new information to the voters).

Finally, for the problem of finding an ordering of candidates using rankings, Tydrichová (2023) proposed the following *heredity* axiom, which is inspired by IIA and informally says that the axes returned by the rule on the profile restricted to a subset of candidates should correspond to the restriction of the axes returned by the rule on the original profile with all candidates.

Independence of Clones

Independence of clones was introduced by Tideman (1987) for ranking-based single-winner voting rules, and it relies on the notion of clones and clone sets. Informally, we say that a set of candidates are clones if all voters rank them next to each other. They could be for instance candidates from the same political party, and running on the same platform. Independence of clones says that the result of the election should not change if we remove one of the clones from the election. Conversely, this implies that the result should not change if we *add* a clone of a candidate in the election. This is a particular case of the *spoiler effect*, which we discussed to motivate independence of losers. Most voting rules that are used for actual elections fail independence of clones. Indeed, we observe that voters from the same side of the political spectrum are often split between several candidates, which might cause their side to lose an election that they might have won otherwise. We will study the particular cases of plurality with runoff (using the example of the French presidential election) in Chapter 3, and of parliamentary elections with thresholds in Chapter 5.

More formally, given a profile P of weak orders, we say that a set of candidates $T \subseteq C$ is a *clone set* if for all candidates $x \notin T$ and all voters $i \in V$, we have either

$$c \succ_i x \text{ for all } c \in T, \qquad \text{or} \qquad c \sim_i x \text{ for all } c \in T, \qquad \text{or} \qquad x \succ_i c \text{ for all } c \in T.$$

Note that this definition of clones corresponds to the definition for weak orders given by Schulze (2011) and by Holliday and Pacuit (2023, Section 5.3.2). When preferences are rankings, this matches the definition of Tideman (1987), and when preferences are approval ballots, this matches the definition of Brandl and Peters (2022). However, it is stronger than the way Tideman (1987, p. 186) proposed to define independence of clones for weak orders (in his definition, it is not allowed for a clone to be ranked equally to a non-clone).

We can then define independence of clones, which states that adding or removing clones should not alter (significantly) the result of the election. In particular, non-clones either stay winning or stay losing. The only thing that might change is that if a clone was winning, then another clone can win instead. Formally, we have:

Independence of Clones
f satisfies *independence of clones* if for all profiles P with a clone set $T \subseteq C$, if $\hat{P}$ is the profile obtained by removing from P all candidates of T except one candidate $\hat{c} \in T$, from C, we have:

1. for every $x \notin T$, $x \in f(P)$ if and only if $x \in f(\hat{P})$, and
2. $\hat{c} \in f(\hat{P})$ if and only if there exists $c \in T$ such that $c \in f(P)$.

This definition is for irresolute rules. For resolute rules, strange things might happen, as the rules will likely use a tie-breaking order, so the clones need to be next to each other in the tie-breaking order too for the axiom to hold. We sometimes define independence of clones for pairs of candidates only (i.e., for $|T| = 2$), giving a weaker axiom. However, most rules that satisfy the weak version also satisfy the stronger version.

Of course, it seems highly unlikely to have perfect clones in a real-life election, even more when there are thousands of voters.[8] However, this axiom gives a good intuition of the behavior of a rule in the case similar candidates run in an election: can the fact that they are similar make them lose the election? For instance, in the 2002 French presidential election that used plurality with runoff, eight left-wing candidates were running, dividing the votes of the left-wing electorate, and allowing a far-right candidate to reach the second round, illustrating that plurality with runoff fails independence of clones. Tideman (1987) showed that IRV satisfies independence of clones and introduced the ranked pairs rule, that satisfies this axiom. Most of the other rules do not satisfy independence of clone, and only a few other Condorcet extensions, such as Schulze's rule, and some tournament solutions, are known to satisfy it (Schulze, 2011; Holliday and Pacuit, 2023). In the case of approval ballots, Brandl and Peters (2022) characterized the single-winner approval voting rule in yet another way using this axiom, reinforcement and a weak efficiency axiom. Elkind et al. (2011) studied a problem related to the cloning axiom, with a deeper analysis of how prone to manipulation by cloning were different voting rules, and on which specific profiles introducing clones is the most harmful.

For the multi-winner problem, Woodall (1994) proposed several variants of independence of clones, the most relevant ones being:

1. Removing clones from a clone set should not increase the number of candidates from the clone set that are in the committee (or, the other way around, a candidate should not be losing because of a clone).

2. Replacing a candidate by a clone set T should not make another candidate $x \notin T$ part of the committee if it was not before.

He claimed that for approval preferences, the approval rule satisfies both variants, and for ordinal preferences, STV satisfies the first one, but not the second one. Baumeister et al. (2024) recently provided a proof that STV indeed satisfies the first variant when the clone set is of size 2. Finally, Neveling and Rothe (2020) generalized the work of Elkind et al. (2011) on manipulation by cloning to the multi-winner setting.

Finally, a strengthening of independence of clones in the single-winner voting setting is *composition consistency*, introduced by Laffond et al. (1996). On a high-level, composition consistency says that if in a profile P we replace every set of clones T by a single candidate c_T (such that c_T represents the clone set T), giving a new profile P', if a single-winner voting rule returns the candidate $f(P') = c_T$ on this new profile, then the winning candidate in P should be the winning candidate in the restriction of P to candidates from the clone set T, i.e., $f(P) = f(P_T)$. One way to see it is to think that each clone set is a political party representing many candidates, and the winning candidate should be the same if all candidates from all parties run at the same time and if we first run primary elections in each party, and then run an election with all winners of primary elections. The relationship between independence of clones and composition consistency has been explored by many papers in the literature, see for instance the work of Berker et al. (2025).

[8] Still, there are several examples of opposition candidates in Russia that found themselves to run against spoiler candidates with the *exact* same name and surname recruited by the government party to confuse the voters. The spoilers sometimes even changed their appearance to look like the original candidate from the opposition. See https://www.theguardian.com/world/2021/sep/06/three-near-identical-boris-vishnevskys-on-st-petersburg-election-ballot

2.4.5 Strategyproofness and Monotonicity

We finally consider strategyproofness and monotonicity axioms, which played a key role in the development of social choice theory. Axioms from these two categories link the outcome of a rule for a profile P to the outcome of the rule for a profile P' that is obtained by altering the preferences of one or several voters.

Strategyproofness

Informally, a rule is strategyproof if the best strategy for a voter is to report their true preferences, or in other words, they cannot be better off by misreporting their preferences. For single-winner voting with weak orders, we can define strategyproofness as follows:

Strategyproofness
A rule f is *strategyproof* if for all profiles $P = (\succcurlyeq_1, \dots, \succcurlyeq_n)$ and all voter $i \in V$, for all profiles $P' = (\succcurlyeq'_1, \dots, \succcurlyeq'_n)$ such that for all $j \neq i$, $\succcurlyeq_j = \succcurlyeq'_j$, we have $f(P') \not\succ_i f(P)$.

Recall that approval ballots and rankings are special cases of weak orders, so the definition also works in these settings. This definition applies to resolute rules, but many generalizations to irresolute rules have been proposed (see Barberà (2011) for a survey). One of them is *Kelly-strategyproofness* (Kelly, 1977), which says that a manipulation is successful if all candidates in $f(P')$ are weakly preferred to all candidates in $f(P)$ by the manipulating voter i.

Unfortunately, Gibbard (1973) and Satterthwaite (1975) independently showed that in the case of rankings and with $m \geqslant 3$, the only resolute, non-imposed and strategyproof rules are dictatorships. This result might sound dramatic, but it can be tempered by the fact that the manipulating voter is very unlikely to be pivotal would have to know the exact preferences of the other voters to accurately compute the correct manipulation (which can be a hard computational problem, see Conitzer and Walsh (2016)). The case of approval ballots is more hopeful, as Brandl and Peters (2022) showed that the approval voting rule is strategyproof, and characterized it (again) with this axiom, neutrality and reinforcement (see also Fishburn (1979)).

Generalizations to multi-winner voting with resolute rules are inspired by the ones for single-winner irresolute rules (Gärdenfors, 1979). In particular, Kelly-strategyproofness is still relevant in this context: a manipulation is successful if all candidates in the committee $W' = f(P')$ are weakly preferred to all candidates in the committee $W = f(P)$. In the approval setting, Peters (2021) proposed the *superset-strategyproofness*, which states that a manipulation by a voter i is successful if $W \cap A_i \subsetneq W' \cap A_i$, where $W = f(P)$ is the committee obtained before the manipulation, and $W' = f(P')$ is the committee obtained after. A weaker variant, called *Hylland's free riding condition* (Hylland, 1992), states that a manipulating voter should not be able to obtain a superset of approved candidates by removing candidates from their approval ballot, i.e., by casting $A'_i \subsetneq A_i$. Peters (2021) showed that even this weak strategyproofness notion is incompatible with a weak proportional representation notion.[9] On the positive side, we showed in Delemazure et al. (2023a) that an even weaker notion of strategyproofness that we called *strategyproofness for unrepresented voters*, and in which the possible manipulators are the voters who originally approve no candidates from the committee, is compatible with proportionality, and that CCAV satisfies both properties. Finally, Lackner and Skowron (2018a) considered strategyproofness notions that apply to *irresolute* approval-based multi-winner rules (i.e., how voters compare sets of committees).

[9] However, multi-winner approval voting satisfies this notion of strategyproofness, and even stronger notions, such as *cardinality-strategyproofness*, which states that no manipulating voter should approve more candidates in W' than in W.

Since it is quite unlikely for a voter to be pivotal in an election with a large set of voters (Peleg, 1979; Slinko, 2002), some works have been studying *coalitional strategyproofness* (Saari, 1990; Durand, 2015), which states that coalitions of voters (instead of single voters) should not be able to collectively misreport their preferences and all obtain a preferred outcome.

A related axiom to *strategyproofness* is the one of *participation* (Brams and Fishburn, 1978), which states that a voter should not prefer the outcome of the election when they are abstaining to the outcome when they are participating (and casting a sincere ballot). This failure of participation is also known as the *no-show paradox*. This axiom is generally satisfied by rules that are based on scoring functions S that associates a score $S(X, O)$ to each possible outcome $O \in \mathcal{O}$ based on the ballot $X \in \mathcal{X}$, for similar reasons as why they satisfy reinforcement. In particular, in the single-winner setting with rankings, scoring rules satisfy participation. On the contrary, no Condorcet extension satisfies participation when $m \geqslant 4$ (Moulin, 1991).

Monotonicity

The general idea behind monotonicity is the following: if, given a profile P, the selected outcome is $O = f(P)$, and we alter the preferences of the voter in order to make O an *objectively better* outcome, then the rule should still select O. For instance, if a single-winner voting rule selects some candidate $c \in C$ as the winner, and we increase the position of c in a voter's ranking without changing the relative order of other candidates, then c should remain the winner (Fishburn, 1982). More formally, for a candidate $c \in C$ and a weak order $\succcurlyeq$, we say that $\succcurlyeq'$ is a c-improvement of $\succcurlyeq$ if (1) $\sigma'(c) < \sigma(c)$ where σ and σ' are the rank functions respectively associated with $\succcurlyeq$ and $\succcurlyeq'$, and (2) for all $a, b \neq c$, $a \succcurlyeq b$ if and only if $a \succcurlyeq' b$. Thus, the only difference is that c is ranked higher in $\succcurlyeq'$ than $\succcurlyeq$. This definition naturally applies to rankings (for which it was originally defined) and to approval ballots, in which c can only go from non-approved to approved. A preference profile P' is a c-improvement of P if there is a voter $i \in V$ such that $\succcurlyeq'_i$ is a c-improvement of $\succcurlyeq_i$, and for all other voters $j \neq i$ we have $\succcurlyeq_j = \succcurlyeq'_j$. We define monotonicity as the following.

Monotonicity
A rule f is *monotonic* if for any profile P and any candidate $c \in C$, if $c \in f(P)$ and P' is a c-improvement of P, then $c \in f(P')$.

Monotonicity can be seen as a weak version of strategyproofness. Indeed, if a rule fails monotonicity for a profile P and a c-improvement P', this means that a voter i can change the status of c from losing in P' to winning in P by decreasing its rank. Many other monotonicity properties have been proposed, generally lying in strength between strategyproofness and standard monotonicity (see Sanver and Zwicker (2012) for a survey). It is clear that scoring rules, as well as plurality and approval, satisfy monotonicity. However, Smith (1973) showed (Theorem 2) that all runoff scoring rules, including IRV, fail monotonicity.

Elkind et al. (2017b) adapted this monotonicity property to the multi-winner setting, as *candidate-monotonicity*, which says that if a candidate c is part of the winning committee for a profile P, it should also be part of the committee for any c-improvement P', and they showed that with ordinal preferences, committee scoring rules (like k-Borda) satisfy this axiom. The same axiom has also been considered in the approval case (Lackner and Skowron, 2018a; Janson, 2016), for which variants have also been introduced (Sánchez-Fernández and Fisteus, 2019).

In the multi-winner setting, another (quite different) monotonicity axiom says that if for a profile P a candidate is part of the committee $W = f(P, k)$ when the desired size of the committee is k, it should also be part of the committee $W' = f(P, k+1)$ when the desired size is $k+1$: adding

	Ch. 3	Ch. 4	Ch. 5	Ch. 6	Ch. 7
Symmetry	✓	✓	✓	✓	✓
Efficiency	✓	✓	✓	✓	✓
Representation		✓	✓		
Reinforcement	✓		✓	✓	
Independence of losers			✓	✓	
Independence of clones	✓	✓	✓	✓	
Strategyproofness	✓		✓		
Monotonicity	✓	✓	✓	✓	✓

Table 2.4: Summary of the axiom families discussed in this section, and the chapters of this thesis in which they will be used.

more seats in the committee should not eliminate a candidate that was part of the committee. This axiom is known as *committee monotonicity* (Elkind et al., 2017b) or *enlargement consistency* (Barberà and Coelho, 2008). It is clear that every sequential rule satisfies committee monotonicity, as it either constructs the committee by successively adding candidates to the committee (and thus will only do *one more* 'addition' step) or by successively eliminating candidates from the committee (and thus will do *one less* 'elimination' step). Similarly, all committee scoring rules (like k-Borda) satisfy this axiom, as the score of each candidate is computed independently. Committee monotonicity is particularly desirable for the excellence objective, but it often clashes with the other two objectives. Finally, this axiom is also related to *house monotonicity* (Balinski and Young, 2010) in the apportionment setting, in which candidates are parties and can get several seats in the committee. This axiom says that increasing the total number of seats should not decrease the number of seats received by any party.

2.4.6 Summary

Table 2.4 summarizes the axiom families that we discussed in this section, and the chapters in which they will be studied. As each chapter corresponds to a particular problem with a specific context, we will additionally consider context-specific axioms.

2.5 Experiments

In the previous section, we saw how we can evaluate rules by defining axiomatic properties and checking which rules satisfy which properties. It is often desirable to complement this approach with an *experimental analysis*, in which we run the rules on actual data and observe their behavior.

Ideally, we want to run the rules on real-world preference profiles, as it allows us to better interpret the outcomes returned by the rules and the differences between them. We refer to this kind of datasets as *real data*. In this thesis, we are particularly interested in datasets of political preferences, as many of our problems are motivated by political elections. Unfortunately, it is not always possible to have access to real data, especially if we are using non-standard ballot formats: remember that most elections, especially political ones, use uninominal ballots, which are not particularly interesting to us. Thus, we also use *artificial* preference profiles, that we create by sampling ballots according to some probabilistic model. We refer to this kind of datasets as *synthetic data*. Synthetic data has another interesting feature in comparison to real data: it is possible to control the parameters of the model, such as the number of voters n and of candidates

m. This modularity allows us to see how the rules' behavior depends on these parameters, and check the robustness of the observed results.

Experiments on such data have always been an important tool in social choice, but their use has been increasing in the last years, as shown by Boehmer et al. (2024) in their recent survey on experiments in computational social choice. In this thesis, we will run experiments with the following objectives in mind:

1. *Rule behavior:* We will run the different rules on the datasets to see how they behave, and how their behaviors relate to the axiomatic analysis. For instance, if a preference profile contains very similar candidates, we can approximate them as *clones*, and check if the rules satisfy independence of clones on this particular profile.

2. *Similarities between rules:* we will also compare the rules to each other, to see how similarly they behave. For instance, we can check on which percentage of the sampled profiles the rules agree on the outcome, and when they disagree, on how much they disagree.

3. *Outcome quality:* we want (if possible) to measure the quality of the outcomes returned by our rules, and see which rules return in general the "best" outcomes. However, the existence of a quality measure is not guaranteed: for instance, in real-life data of political elections, we cannot really say that one candidate is *objectively* the best, and should win the election. One approach to overcome this is to use the *distortion*, which relies on the idea that rankings or approval ballots are derived from utility or cost functions (see Section 2.5.2). Another approach is the one of *epistemic voting* in which there exists a ground truth and voters' ballots are noisy estimates of this ground truth. We use this approach in Chapter 6.

4. *Dataset analysis:* Finally, the analysis of the dataset itself is sometimes already insightful and might give a better understanding of the problem considered, of the structure of the dataset, and of the challenges it poses. For instance, we analyze in Chapter 5 how voters are strategizing their votes in parliamentary elections, and in Chapter 7 we compare the polarization of voters in different datasets.

In the literature, some other objectives that we omit here are sometimes considered. In particular, if we are interested in the computational complexity of the rules, we can run experiments to see how long the proposed algorithms take to compute or approximate some voting rule.

Importantly, a good experimental analysis should ideally be based on several distinct datasets to ensure that the results are not specific to the particular dataset, or probabilistic model, that was used to generate the data. It should also be easily reproducible, so that other scholars can check the results and build upon them. This includes providing (as much as possible) the protocol and the parameters used, and ideally giving access to the dataset if it is not already available, and to the code used to run the experiments.[10]

The remainder of this section is organized as follows. We first introduce common probabilistic models in Section 2.5.1, namely *impartial culture* (IC), *Mallows' model*, and the *Euclidean model*. In Section 2.5.2, we introduce the notion of *distortion*. Next, we discuss some of the real-world datasets that are used in this thesis in Section 2.5.3, and put a special focus on the *Voter Autrement* series of datasets in Section 2.5.4.

[10] All the experiments of this thesis have been done in Python and heavily rely on the *numpy* and *pandas* libraries. Most of the code of the experiments that are in this thesis can be found on my GitHub `github.com/TheoDlmz`.

2.5.1 Probabilistic Models

Many models have been introduced in the literature for generating preferences profiles, especially for rankings and approval ballots. We will go over some of them in this section.

Impartial Culture

Let us start with the most natural and simple model one can think of: the *impartial culture* model. In this model, we sample the preferences of each voter independently at random, and the probability of each possible preference $X \in \mathcal{X}$ is the same, i.e. $\mathbb{P}(X_i = X) = 1/|\mathcal{X}|$ (assuming that $\mathcal{X}$ is finite). Note that this implies a complete symmetry between the candidates. In particular, as the number of voters increases, all candidates in the sampled profile tend to have almost the same Borda score, plurality score, approval score, etc.

This model was first introduced for ordinal preferences by Guilbaud (1952) to study the probability of a Condorcet paradox, and is the most common model to generate rankings in the social choice literature (Boehmer et al., 2024). For rankings, the set of possible preferences is the set of all permutations of the candidates, and the probability of each ranking is $1/m!$. For approval ballots, impartial culture says that every approval ballot A_i has the same probability $1/2^m$ to be selected. This is equivalent to saying that for each voter i, each candidate c is approved with probability $p = 1/2$ independently of the other candidates. Generalizing on this idea, Bredereck et al. (2019) introduced p-impartial culture, in which each candidate is approved with probability p, independently of the other candidates. Since in real-world elections voters generally approve only a few candidates, it is more realistic to use low values of p. Finally, in the case preferences are utilities (or costs), the principle behind impartial culture corresponds to the *uniform distribution* of utilities. In this case, the utility of each candidate is sampled independently at random from some interval, typically $I = [0, 1]$.

The impartial culture is the most popular model to generate synthetic preference profiles, and is often used as a benchmark to compare the different rules. However, it is highly unrealistic and it is recommended to use other, more interesting models to complement the analysis.

Mallows Model

The second most popular model to sample rankings in computational social choice is, according to Boehmer et al. (2024), *Mallows' model* (Mallows, 1957). The idea behind Mallows' model is that there exists an average ranking, or *central ranking*, $\succ^*$ of the society, and that voters are more or less deviating from this central ranking depending on the dispersion parameter $\phi \in [0, 1]$. The central ranking can also be seen as a "ground truth" ranking that the voters are trying to estimate, with some noise. More formally, let $\succ^*$ be the central ranking, and $\phi \in [0, 1]$. Then, the probability for a voter to submit a ranking $\succ \in \mathcal{L}(C)$ is given by

$$\mathbb{P}(\succ | \succ^*, \phi) = \frac{1}{K} \phi^{\mathrm{KT}(\succ, \succ^*)}$$

where $\mathrm{KT}(\succ, \succ^*)$ is the Kendall-Tau distance between $\succ$ and $\succ^*$ (defined in Section 2.3.1), and $K = \sum_{\succ \in \mathcal{L}(C)} \phi^{\mathrm{KT}(\succ, \succ^*)}$ is a normalizing constant, such that the sum of all probabilities is equal to 1. Thus, if $\phi < 1$, the more similar $\succ$ is to the central ranking, the more likely it is to be sampled. On the extremes, if $\phi = 0$, the only ranking with non-zero weight is the central ranking $\succ^*$, so every voter will submit this ranking, and if $\phi = 1$, all rankings are equally likely, and we fall back on impartial culture. Lu and Boutilier (2014) proposed a way to sample rankings according

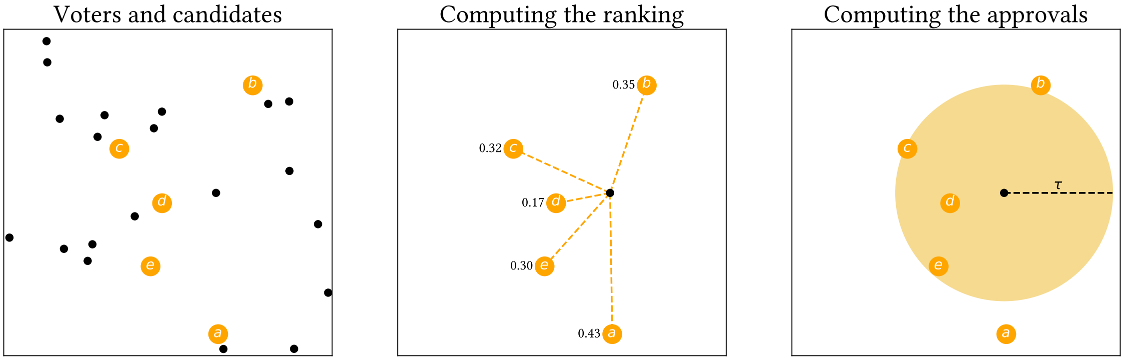


Figure 2.6: Example of a 2-dimensional Euclidean model with 20 voters (black dots) and 5 candidates $C = \{a, b, c, d, e\}$ (orange dots). The center plot shows how to compute the ranking of one voter (here, we obtain $d \succ e \succ c \succ b \succ a$). The right plot shows the approval preferences of the same voter with a threshold τ (here $\tau = 0.33$ and $A_i = \{d, e, c\}$).

to the probabilities given by the Mallows model, by means of the *random insertion model*.

Mallows models are particularly interesting, as they seem to provide a good coverage of the space of real-life preference profiles (Boehmer et al., 2022a), and the dispersion parameter ϕ allows us to cover a wide variety of preference profiles. However, Boehmer et al. (2021) recently showed that the parameter ϕ in the definition of Mallows' model was not adapted to high numbers of candidates ($m > 25$) and proposed a re-parameterization of Mallows' model with a parameter *norm-*ϕ. In this thesis, we will still use ϕ, since we only have $m = 10$ candidates in all experiments involving Mallows' models.

Mixtures of Mallows are a variant in which the voters are divided into several subsets, and the preferences of voters in each subset are drawn according to a Mallows model with a different central ranking. Generally, the different Mallows models share the same dispersion parameter ϕ, but it could be different. More formally, for a k-Mallows, we have for all $j \in [1, k]$, the probability for the voters to be in the jth subset $p_j \in [0, 1]$ (with $\sum_j p_j = 1$, generally $p_j = 1/k$), the central ranking $\succ_j^*$, and the dispersion parameter ϕ_j. Then, for each voter we first select a subset $j \in [1, k]$ according to the probabilities $p_1, \dots, p_k$, and then sample the ranking of the voter according to the Mallows model with central ranking $\succ_j^*$ and dispersion parameter ϕ_j.

For approval preferences, some models have been proposed that adapt the idea of sampling preferences that deviate from a central vote $A^* \subseteq C$, for instance the (p, ϕ)-resampling (Szufa et al., 2022) where p is the desired approval rate and ϕ a dispersion parameter.

Euclidean Preferences

The *Euclidean model* (Enelow and Hinich, 1984, 1990) is another popular model in computational social choice, for both ordinal and approval preferences. The idea behind the Euclidean model is that voters and candidates are associated with positions in the d-dimensional Euclidean space $[0, 1]^d$, and the preferences of voters are determined by their distance to the candidates: the closer the voter is to a candidate, the more they prefer this candidate. This model is particularly relevant for some problems, such as facility location problems. The earliest works using a similar model in game theory are Hotelling (1929) and Downs (1957), who analyzed the positions that a candidate should take on a line (so, with $d = 1$) in order to maximize their chances of being elected. In particular, they observed that all candidates end up having the same position. In this thesis, we

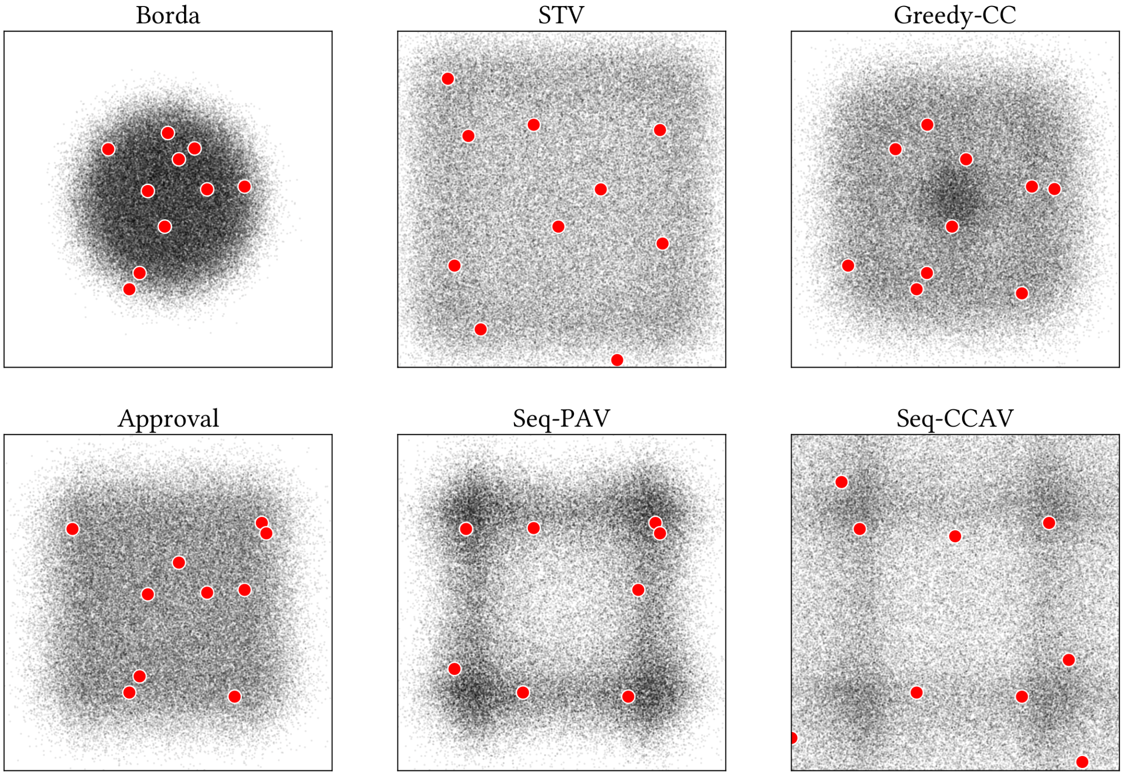


Figure 2.7: Distributions of the positions of the candidates in the committees for selected rules based on rankings (top row) and approval ballots (bottom row). The winners from a random instance (the same for all rules) are highlighted in red. Reproduced from Elkind et al. (2017a); Godziszewski et al. (2021)

will only consider the 1-dimensional and 2-dimensional setting.

More formally, let us denote $\mathsf{pos}(i) \in [0,1]^d$ the position of a voter $i \in V$ and $\mathsf{pos}(c) \in [0,1]^d$ the position of a candidate $c \in C$. Then, the ranking of the voter i is given by the increasing order of their distances to the candidates, i.e., $a \succ_i b$ if and only if $\|\mathsf{pos}(i) - \mathsf{pos}(a)\| < \|\mathsf{pos}(i) - \mathsf{pos}(b)\|$. For approval preferences, we assume that there exists a threshold τ such that a candidate a is approved if and only if $\|\mathsf{pos}(i) - \mathsf{pos}(a)\| < \tau$ (this approach is used for instance by Bredereck et al. (2019) and Faliszewski et al. (2023b)). We give a visual example on Figure 2.6. Note that τ could possibly be different for each voter, but in this thesis we always use the same τ for all voters. An interesting aspect of this model is that from the same distribution of positions, we can derive rankings, approval preferences and even weak orders, and thus compare rules that take as input different preference formats. Finally, observe that the 1-dimensional Euclidean model always gives single-peaked rankings, and interval approval preferences.

The way we described the Euclidean model so far does not say how we should choose the positions of voters and candidates, only how we can derive preferences from these positions. To do so, we use probability distributions over positions in the space $[0,1]^d$. In this thesis, we will use uniform distributions over the whole space $[0,1]^d$, and Gaussian distributions, which are particularly interesting as they put a high concentration of voters and candidates around the center.

Another interesting feature of the Euclidean model is that it allows us to *visualize* the behavior of the rules, by seeing for instance in which area of the Euclidean space are the candidates elected by the different rules. For instance, in the case of rankings, Elkind et al. (2017a) uses the 2-

dimensional Euclidean model to visualize the positions of the candidates selected by different ranking-based multi-winner voting rules, clearly observing differences between rules depending on whether they aim at excellence, proportionality, or diversity. Godziszewski et al. (2021) reproduced this experiment for approval-based multi-winner voting rules.

We reproduced these experiments in Figure 2.7 for some selected rules. We sampled 10 000 profiles of $n = 100$ voters and $m = 50$ candidates and we computed the winning committees of size $k = 10$ for different rules (in the paper by Elkind et al. (2017a), they use $n = m = 200$ and $k = 20$). The positions of voters and candidates are selected uniformly at random in the 2-dimensional Euclidean space $[0, 1]^2$. To compute the approval ballots of the voters, we used the threshold $\tau = 1/4$, leading to an average of ~ 15 candidates approved by voter. We compared three ranking-based rules: k-Borda, greedy Chamberlin-Courant, and STV; and three approval-based rules: approval voting, sequential-CCAV and sequential-PAV. Note that in both cases, we consider one rule aiming for the excellence objective (k-Borda and approval voting), one aiming for diversity (greedy Chamberlin-Courant and sequential-CCAV) and one aiming for proportionality (STV and sequential-PAV). Figure 2.7 shows the distributions of the positions of candidates that are in the committees selected by each rule (black dots), as well as the winners for one random instance (red dots). We can see that the rules aiming for excellence select candidates that are close to the center, while the rules aiming for diversity and proportionality select candidates that are more spread out. However, since we used a small radius for approval ballots ($\tau = 1/4$), the positions of the selected candidates with the approval rule still cover a large area of the space.

Other Models and the Maps of Elections

The three models we discussed so far are the main ones that are used in this thesis, but there exist many other models, for both rankings and approval ballots.

Among the other popular models to sample rankings, there are the Urn model (Eggenberger and Pólya, 1923; Berg, 1985), and models designed to only return single-peaked profiles (Conitzer, 2007; Walsh, 2015). The similarities between preference profiles sampled from different models have been studied by Faliszewski et al. (2019a), by defining distances between preference profiles. These distances were then used to construct *"maps of elections"* (Szufa et al., 2020; Boehmer et al., 2021, 2022b): profiles that are close according to some distance appear close to each other on the map. Figure 2.8 shows an example of such map, in which the distance used is the isomorphic swap distance. In particular, it shows profiles sampled with impartial culture on the bottom left corner, near the uniform profile UN, in which every ranking appears exactly once. Mallows models are represented by blue dots, which cover the space between the uniform profile UN (for $\phi = 1$) and the identity profile ID (for $\phi = 0$), in which all voters submit the same ranking. Finally, Euclidean preferences correspond to some of the green dots (*1-Dim* and *Multi-Dim*) in the center of the map. They are a bit closer to the antagonistic profile AN, in which half of the voters submit the same ranking, and the other half the reverse ranking. Such map can also be used to evaluate the behavior of the rules on a large variety of profiles, and to see where on the map (i.e., for which models) are the interesting

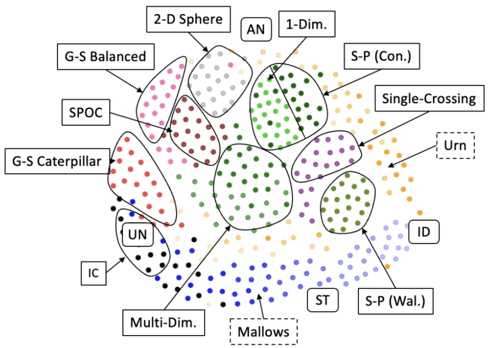


Figure 2.8: Map of elections with the isomorphic swap distance. Picture from Boehmer et al. (2022b).

cases. We will for instance use it in Chapter 4 to test the robustness of our results.

For approval preferences, some other models than the ones we discussed have been proposed by Szufa et al. (2022), who also introduced distances and maps of approval profiles.

2.5.2 Distortion

We mentioned that one of the goals of the experimental analysis is to evaluate the quality of the outcomes returned by the rules. One possibility to do so is to use the concept of voting rules' *distortion*, introduced by Procaccia and Rosenschein (2006). We now describe this tool, and we recommend the survey by Anshelevich et al. (2021) for a broader view on the topic.

Normalized Distortion

Let us first describe the principle of distortion for ranking-based single-winner voting rules. The idea is to assume that each voter $i \in V$ has a true *underlying* utility function u_i such that $u_i(c)$ is the utility of candidate c for voter i (i.e., valued preferences). We denote $U = (u_1, \dots, u_n)$ the profile of utilities and $S_U(c) = \sum_i u_i(c)$ the social welfare of a candidate c in this profile. Then, we can derive preferences from these utilities: the utility function u_i can be associated with a ranking $\succ_i$ such that if $u_i(c) > u_i(c')$, then $c \succ_i c'$. We say that a preference profile $P = (\succ_1, \dots, \succ_n)$ is *consistent* with the utility profile U if for all voters $i \in V$, the ranking $\succ_i$ can be derived from the utility function u_i, and we denote it by $P \approx U$. Then, the *distortion* of a voting rule *for profiles P and U* is the ratio between the social welfare of the optimal outcome (i.e., the outcome that maximizes the social welfare) and the social welfare of the outcome of the rule. Formally, the distortion of a (resolute) rule f for profiles P and U such that $P \approx U$ is defined as:

$$\mathsf{dist}(f, P, U) = \frac{\max_{c \in C} S_U(c)}{S_U(f(P))}.$$

In particular, the distortion is always greater than or equal to 1, and is best when equal to 1.

Then, the traditional way to study distortion is to look at the *worst-case* over all profiles. Thus, in the literature, the distortion of a rule f is generally defined as the worst-case distortion over all possible preference and utility profiles P and U such that $P \approx U$, i.e.,

$$\mathsf{dist}(f) = \sup_{P \approx U} \mathsf{dist}(f, P, U).$$

In the original distortion framework (Procaccia and Rosenschein, 2006), it is assumed that utilities are normalized for each voter, i.e., $\sum_c u_i(c) = 1$ for all i, otherwise the distortion of all rules can get infinitely large (Aziz, 2020). One of the main results in this model is that plurality has distortion $O(m^2)$ (Caragiannis and Procaccia, 2011), and that it is the optimal distortion (Caragiannis et al., 2017). On the other hand, Borda has unbounded distortion (Procaccia and Rosenschein, 2006). The multi-winner setting was also studied by Caragiannis et al. (2017).

Metric Distortion

Anshelevich et al. (2018) later initiated the study of distortion in the metric setting. The general definition of metric distortion can be based on any pseudometric space, but for simplicity we will consider here that the space is the Euclidean space $[0, 1]^d$. In this case, we consider the Euclidean model defined in Section 2.5.1, in which every voter i is associated to a position $\mathsf{pos}(i) \in [0, 1]^d$ and every candidate c is associated to a position $\mathsf{pos}(c) \in [0, 1]^d$. We obtain a distribution of positions

$E \in ([0,1]^d)^{V \cup C}$. Then, the preferences of the voter i are determined by their distances to the candidates. As in the other setting, we say that the preference profile P we obtain is *consistent* with the distribution of positions E, and we denote it by $P \approx E$.

In this model, the positions are not inducing utilities, but costs: the closer a candidate is to the voter, the less costly it is for the voter if this candidate is selected. Thus, we say that the cost of a candidate c for voter i is $\mathsf{cost}_i(c) = \|\mathsf{pos}(i) - \mathsf{pos}(c)\|$. The social cost of a candidate c in the profile P is given by $\mathsf{cost}_E(c) = \sum_i \mathsf{cost}_i(c)$. The distortion of a rule f for profiles P and E is then:

$$\mathsf{dist}(f, E, P) = \frac{\mathsf{cost}_E(f(P))}{\min_{c \in C} \mathsf{cost}_E(c)}.$$

which is again always greater than or equal to 1. The worst-case distortion of a rule f is then:

$$\mathsf{dist}(f) = \sup_{P \approx E} \mathsf{dist}(f, E, P).$$

The metric distortion has been studied for numerous rules: Anshelevich et al. (2018) showed that Copeland has a constant distortion of 5. A rule with optimal distortion of 3 called *pluralityVeto* was proposed by Kizilkaya and Kempe (2022) (see also Kizilkaya and Kempe (2023) for a more practical rule). Many distortion variants have been considered in this model, we refer to the survey by Anshelevich et al. (2021) for more details.

To see why no rule can achieve a better worst-case distortion than 3, consider an election with two candidates a and b, which have position $\mathsf{pos}(a) = 0$ and $\mathsf{pos}(b) = 1$, and $n = 2k + 1$ voters such that k voters have position $\mathsf{pos}(i) = 0$ (thus voting $a \succ b$) and $k + 1$ voters have position $\mathsf{pos}(i) = 1/2 + \varepsilon$ with $\varepsilon = 1/(k + 1)$ (thus voting $b \succ a$). Then, if the rule follows the majority principle, it will select b as the winner, while the optimal outcome is a. The distortion is:

$$\mathsf{dist}(f) = \frac{\mathsf{cost}_E(b)}{\mathsf{cost}_E(a)} = \frac{k + (k+1)(1/2 - \varepsilon)}{(k+1)(1/2 + \varepsilon)} = \frac{2k + (k+1)(1 - 2\varepsilon)}{(k+1)(1 + 2\varepsilon)} = \frac{3k - 1}{k + 3}$$

which goes to 3 when k goes to infinity.

Average Distortion

Most of the work on distortion focused on the theoretical worst-case bounds of distortion, and on the different ways we can optimize it. However, worst-case bounds generally rely on quite unrealistic examples, such as the one we just discussed to show that no rule can achieve metric distortion lower than 3. Thus, worst-case bounds are not necessarily representative of the behavior of the rules on actual preference profiles.

This motivates the idea of *average distortion*, in which we consider a distribution over utility functions (in the case of classical distortion), or over positions in the metric space (in the case of metric distortion), and we compute the expected distortion of the rule over this distribution. For instance, for a distribution over utility functions $\mathcal{D}$, we define the average distortion of f as:

$$\mathsf{dist}_{\mathsf{avg}}(f) = \mathbb{E}_{U \sim \mathcal{D}, P \approx U}[\mathsf{dist}(f, U, P)].$$

For simplicity, we assume that there are no ties in utility profiles U, and that there is only one preference profile P verifying $P \approx U$ (this assumption does not change the results). We can similarly define average distortion in the metric setting.

In the case of utilities, this model was first studied by Boutilier et al. (2012), who showed that if the distribution over utility functions is neutral (i.e., all candidates are treated the same), then the rule that minimizes the average distortion is a scoring rule. In particular, the rule that minimizes the average distortion for the uniform distribution of utilities is Borda. Gonczarowski et al. (2023) later proposed a scoring rule called *binomial voting* that performs well for all distributions. Another related work is that of Cheng et al. (2017), who studied average distortion in the metric setting, in the particular case where the candidates' positions are drawn uniformly at random from the distribution of voters' positions, and obtained much better distortions than the worst-case bounds.

Finally, in Delemazure et al. (2024b), we explored a more experimental approach to the average distortion. To compute the average distortion of the different rules for a given number of voters and candidates, we used the *Monte Carlo approach*: We considered several distributions of utilities and positions in the metric space, and sampled thousands of random profiles with these distributions, then computed the average distortion of voting rules over these profiles. For high sample sizes, Monte Carlo methods are known to converge to the true value of the expectation.

For instance, Figure 2.9 shows the average distortion of different voting rules with the uniform distribution of utilities in $[0, 1]$, based on the experiments from Delemazure et al. (2024b) that we reproduced here. We used profiles of $n = 30$ voters (which is intentionally low in order to get high enough average distortion) and m candidates, with m going from 2 to 15. We computed the average distortion over 10 000 instances for each value of m. We compare the average distortion of the following rules: plurality, veto, half-approval (i.e., k-approval with $k = \lceil m/2 \rceil$), Borda, STV, and Copeland. We also considered the pluralityVeto rule proposed by Kizilkaya and Kempe (2022), which has optimal worst-case distortion in the metric setting.

This experiment confirms the theoretical findings from Boutilier et al. (2012) that Borda is optimal for the uniform distribution, but more surprisingly, it shows that the more candidates there are, the lower the distortion (for Borda and Copeland), and thus the better the social welfare. This indicates that in some contexts, "irrelevant alternatives" are useful for increasing the welfare, and it hints that *independence of irrelevant alternatives* (Section 2.4.4), is not always as appealing as it seems. On the other hand, the distortion of veto and plurality quickly get worse as the number of candidates increases, which is not surprising since these two rules only use uninominal information. PluralityVeto is not very good either, and we show in Delemazure et al. (2024b) that this is true even in the metric setting, highlighting that the worst-case distortion is not a good indicator of the average performance of a rule.

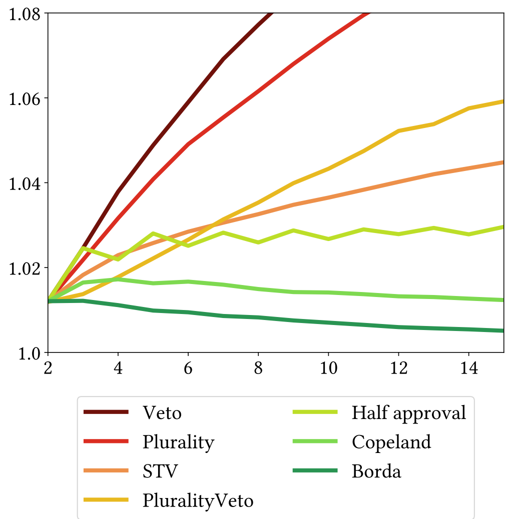


Figure 2.9: Average distortion of voting rules with the uniform distribution of utilities and different values of m (over 10 000 instances).

2.5.3 Real-world Data

In addition to the synthetic datasets generated using the models described in Section 2.5.1, we will also test rules on real-world data. Some of these datasets correspond to political elections, but this is not the case for all of them, and we will often use data related to various different topics. To gather real-world datasets, there are several possible approaches.

A first approach is to use an existing dataset from the literature or from a specialized bank of

Dataset	Source	Chapter	Preference type
San Sebastian poster competition		Chapter 3	Approvals
Camp songs		Chapter 3	Approvals
2002 Irish election data		Chapter 4	Truncated rankings
Sushi data	Kamishima (2003)	Chapter 7	Full rankings
Skate data	Smith (2000)	Chapter 7	Full rankings

Table 2.5: Preflib datasets that are used in this thesis.

dataset. For voting problems, the *Preflib* library of datasets (Mattei and Walsh, 2013) is a very useful resource. It contains a large number of datasets of preferences over various topics. Most of the profiles in these datasets are rankings, but the ballots often contain ties. Table 2.5 summarizes the different Preflib datasets that are used in this thesis. Another popular library in social choice is *Pabulib*, that contains datasets of participatory budgeting votes (Stolicki et al., 2020).

A second approach is to collect datasets from the web, either by converting open-source datasets to preference profiles, like we will do in Chapter 6 with the votes of the justices of the Supreme Court of The United States, or by *scraping* data and converting it into a preference profile. This is for instance the approach of Boehmer and Schaar (2023), who collected profiles of rankings from various sources (like music charts and results of sports competitions). In this thesis, we used this approach to build additional datasets of approval preferences used in Chapters 3 and 6. These datasets are based on *tierlists* that were created on the website *Tiermaker.com*. As a reminder, a tierlist is a list of items that are ranked by the user in different categories (generally $Z = \{S, A, B, C, D, F\}$, with S being the best one and F the worst one). The *Tiermaker* website allows users to create tierlists of any possible group of items (movies of one franchise, albums of one singer, characters of a movie, school subjects, colors, etc.), and share them to other people so that they can give their own tierlist on the same set of items. We collected tierlists submitted by the users on some popular topics (of course, everything is anonymized), and converted the set of items ranked in the best category into an approval ballot for each tierlist.

The last, and most time-consuming approach, is to collect the data ourselves. To do this, one needs to either conduct an online survey, an *in situ* experiment on the field, or a lab experiment in a controlled environment and with paid subjects. We explore the different ways to collect such data in more detail in the next section, through the example of the *Voter Autrement* project.

2.5.4 Collecting Data: the *Voter Autrement* Project

We now present a series of datasets, that are used in all chapters of this thesis, and that were collected during experiments conducted in parallel to the (first round of) French presidential elections since 2002. Most of these experiments are part of the *Voter Autrement* project.[11]

These experiments were conducted at the same time or around the time of a presidential election. In each of these experiments, participants were given the possibility to vote with alternative voting methods, such as approval voting, range voting, Borda or IRV. Thus, they could provide ballots that were more expressive than the uninominal ones that they are allowed to cast in the actual elections. These experiments had several goals, which varied between years, but in general they aimed at (1) analyzing how participants react to alternative voting methods, if they understand them and if they like them, (2) computing the results of the elections with these alternative

[11] For more information (in French): https://www.gate.cnrs.fr/vote/

methods, and (3) collecting reliable data for evaluating rules (what we do in this thesis). Let us now go over the different ways these data were collected.

The Different Kinds of Experiments

The first kind of experiments for collecting these data, which was reproduced during each French presidential election since 2002, is to stay at the exit of voting booths, and invite voters to participate in the experiment, following the tradition of *field experiments* in political science. We call these experiments *in situ* experiments. Voters who agree to participate are given an explanation of the alternative voting methods they will try, and are asked to fill a survey in which they give their votes. Participants generally also have to fill an additional survey about their socio-demographic profile, and sometimes answer questions regarding their preferences between the different voting methods. In most cases, the experiment is organized in order to reproduce as much as possible the conditions of a real election, using for instance actual voting urns and voting booths. Moreover, voters are generally involved in the process before and after the election: they are informed by mail that the experiment will take place in their city, they can join public meetings explaining the experiment and its objectives, and they are informed after the election about the results of the experiments (Laslier, 2019). Laslier and Van der Straeten (2004) conducted the first experiment of this kind in 2002 in three cities. Such *in situ* experiments were conducted during all French presidential elections from 2002 to 2022, but also in Germany in 2008 (Alós-Ferrer and Granić, 2012), in Benin in 2011 (Kabre et al., 2013), in Romania in 2012 (Roescu, 2014), and in Austria in 2015 and 2019 (Darmann et al., 2017; Darmann and Klamler, 2023).

The second kind of experiment is to run surveys *online* instead of going on the field to find participants. In *online* experiments, the process is the same as for *in situ* experiments: participants are first familiarized with the alternative voting methods, then they vote with these methods, and finally they are asked socio-demographic questions and questions on their preferences on the different voting methods. Since everything is happening online and anyone can freely access the website, this allows much more people to participate at the same time, and more voting methods to be tested. Blais et al. (2012) were the first to conduct an experiment of this kind during a Canadian provincial election in 2011. The first online experiment for a French presidential election was held in 2012 Blais et al. (2015); Laslier et al. (2012), and then a similar experiment was conducted in 2017 (Bouveret et al., 2018), and then again in 2022 (Delemazure and Bouveret, 2024). A similar experiment was conducted in the USA in 2016 (Igersheim et al., 2022). To test the rules presented in Chapter 5, we conducted an online experiments during the 2024 election of the French representatives to the European parliament. Another online experiment regarding the election of the European parliament members, called *EuroVote*, was conducted in 2015 by Laslier et al. (2015). In this experiment, they tested different voting methods than the ones tested in our 2024 experiment. Finally, we also conducted an online experiment during the 2022 Italian parliamentary election (Marsilio and Delemazure, 2022).

The participants of these online experiments were generally recruited through social media, mailing lists, and sometimes traditional medias. Thus, they are generally quite interested in the experiment and in the topic of alternative voting methods, and thus fill the form conscientiously and sometimes give precious feedback.[12] However, this often leads to very unrepresentative samples of participants, from a demographic and political viewpoint: they are generally more educated, younger, and more left-leaning than the general population. Thus, for some experiments, the

[12] In our 2024 experiments for the European election, we received more than 400 interesting comments on the experiment and the voting methods that were proposed.

Year	Type	City or name	Rules tested	Weighted	n	m
2002	In situ	*Gy-Les-Nonains	AV	✗	365	16
2002	In situ	*Orsay	AV	✗	2 220	16
2007	In situ	*Louvigny	AV, RV	✗	1 063	12
2007	In situ	*Cigné	AV, RV	✗	233	12
2007	In situ	*Illkirch	AV, RV	✗	1 540	12
2007	In situ	Faches-Thumesnil	IRV	✗	960	12
2007	In Situ	Orsay	MJ	✗	1 733	12
2012	In situ	*Strasbourg	AV, RV	✓	1 023	10
2012	In situ	*Louvigny	AV, RV	✓	930	10
2012	In situ	*Saint-Etienne	AV, RV	✓	387	10
2012	Online	*Vote pluriel*	AV, Plurality, IRV	✓	8 044	10
2012	Online	*Vote de valeur*	RV	✓	11 539	10
2017	In situ	Fort-de-France	Borda, IRV	✓	508	11
2017	In situ	*Allevard	RV	✓	836	11
2017	In situ	*Strasbourg	AV, RV	✓	1 071	11
2017	In situ	*Grenoble	AV, Continuous	✓	1 069	11
2017	In situ	*Hérouville-Saint-Clair	AV, RV	✓	711	11
2017	In situ	*Crolles	AV, AVR, RV	✓	2 617	11
2017	Online	**Voter autrement*	AV, RV, Borda, IRV	✓	37 462	11
2022	In situ	*Strasbourg	AV, RV, MJ	✓	938	12
2022	Online	*Representative	AV, RV, MJ	✓	2 122	12
2022	Online	**Un autre vote*	AV, RV, MJ, Borda, IRV	✓	2 307	12

Table 2.6: Experiments conducted during French presidential elections. Datasets preceded by "*" are part of the *Voter Autrement* project.

sample of participants was recruited by a polling institute in order for it to be more representative of the general population. This is what we call *representative* experiments. The drawback of recruiting participants this way is that they are less interested in the topic, and thus sometimes fill the form less conscientiously. Moreover, such experiments are generally quite expensive (count several thousand of euros for 1 000 participants). Another way to reduce the representation bias is to assign weights to the voters, as we will see at the end of this section.

Finally, a third popular kind of experiment in political science and social choice, is *laboratory* experiments (see for instance Forsythe et al. (1993); Blais et al. (2007)). These experiments do not necessarily take place during a large-scale election, and they generally have a very precise protocol and controlled environment: the candidates are fictitious (while we use the actual candidates of the election in *in situ* and *online* experiments), and the participants are paid. The goal of these experiments is often to study the strategic behavior of voters with different voting rules.

A History of Experiments During French Presidential Elections

Table 2.6 summarizes the different experiments that were conducted in parallel to French presidential elections since 2002. The first experiment of this series was an *in situ* experiment conducted in two cities (*Gy-Les-Nonains* and *Orsay*) by Laslier and Van der Straeten (2004), and only tested the approval voting rule. In addition to computing the approval winner, the authors also used the preferences to describe the relationships between the candidates (in particular, how much co-approved was each pair of candidates) and used this information to embed them in a 2-dimensional space, representing the political space.

Baujard and Igersheim (2009) conducted similar *in situ* experiments during the 2007 presidential election, in three other cities (*Cigné*, *Louvigny*, *Illkirch*). This time, the voters could vote using the approval voting rule, and a range voting rule with possible scores $Z = \{0, 1, 2\}$ (see also Laslier (2010b)). Baujard and Igersheim (2010) conducted an analysis of the experiments from 2002 and 2007 and provided extensive details on the protocol, the material and the surveys that were shown to the participants. Baujard et al. (2011) used the preferences of voters to embed the candidates in a 2-dimensional space. The same year, Farvaque et al. (2009) and Balinski and Laraki (2010) conducted similar experiments outside of the *Voter Autrement* project. The first one took place in the city of Faches-Thumesnil and focused on IRV and the simplicity criterion (i.e., how easy it is to understand the rule). The second one took place in the city of *Orsay* and tested the Majority Judgement rule.

In 2012, Baujard et al. (2013) conducted *in situ* experiments again, in three cities (*Louvigny*, *Saint-Etienne*, *Strasbourg*), this time testing different range voting score scales in each city: respectively $\{-1, 0, 1\}$, $\{0, 1, 2\}$ and $\{0, 1, 2, \dots, 20\}$ (see also Baujard et al. (2014)). This allowed to show that voters adapt their preferences to the scale they are given (Baujard et al., 2018). Lebon et al. (2017) used the approval preferences to construct a political axis of the candidates in the 2012 French presidential election based on which candidates were often approved by the same voters (inspiring our model in Chapter 6). Igersheim et al. (2016) compared the voters' behavior in this experiment to what could be observed in lab experiments with the same voting methods. In parallel to these *in situ* experiments, two *online* experiments (that were not officially part of the *Voter Autrement* project) were conducted in 2012. The first one, called *Vote Pluriel* (Van der Straeten et al., 2013) tested various rules, including approval voting, plurality and IRV. The second one, called *Vote de Valeur* (Laslier et al., 2012), tested range voting with $Z = \{-2, -1, 0, 1, 2\}$.

In 2017, Bouveret et al. (2019) conducted *in situ* experiments in five cities, testing different rules in each city. In *Allevard*, voters tested range voting with one of three possible scales (either $\{-1, 0, 1\}, \{-0.5, 0, 1\}$ or $\{-2, 0, 1\}$), in *Crolles* they tested approval voting, approval with runoff (see Chapter 3), and range voting with $Z = \{0, 1, \dots, 19, 20\}$. In *Grenoble*, they tested approval voting, and a form of range voting with continuous values: for each candidate, voters had to put a mark on a continuous line to indicate how much they like (or dislike) this candidate. In *Hérouville-Saint-Clair*, they tested approval voting, and range voting with either $Z = \{0, 1, 2, 3\}$ or $Z = \{0, 1, 2, 3, 4, 5\}$. Finally, in *Strasbourg*, they tested approval voting and range voting with one of $\{0, 1, 2\}$, $\{0, 1, 2, 3\}$, $\{-1, 0, 1\}$ and $\{-1, 0, 1, 2\}$. In addition to these *in situ* experiments, Bouveret et al. (2018) also conducted an online experiment in which more than 40 000 people participated, and could vote with approval voting, range voting with the same scales as in Strasbourg, IRV, and a variant of Borda asking to rank exactly 4 candidates. They could also give their opinion on each candidate using an almost continuous slider going from 0 to 100. Baujard et al. (2021) studied how the choice of the score scale in range voting impacted the voters' behavior. The approval preferences of the *in situ* experiments were used by Baujard and Lebon (2022) to construct a political axis of the candidates in this election. In parallel to these experiments from the *Voter Autrement* project, Kamwa et al. (2020) conducted another *in situ* experiment, testing IRV and Borda (also asking to rank exactly 4 candidates) in the city of *Fort-de-France*, in Martinique.

Finally, the most recent presidential election took place in 2022, and once again, similar experiments were conducted. An *in situ* experiment was conducted in the city of *Strasbourg*, where participants tested approval voting, range voting with $Z = \{-1, 0, 1, 2, 3, 4\}$ and the majority judgement rule (Baujard et al., 2025b). The same voting rules were tested on an online experiment with a representative sample of participants, recruited through a poll institute. Moreover, we reproduced the 2017 *online* experiment that tested a wide range of voting rules, and added the majority judgement rule to the set of tested rules (Delemazure and Bouveret, 2024).

Note that in addition to these experiments that took place during French presidential elections,

Party	LREM	FN	LR	LFI	PS	DLF	R	NPA	UPR	LO	SP
Actual	24.01	21.3	20.01	19.58	6.36	4.7	1.21	1.09	0.92	0.64	0.18
Sample	23.53	2.99	8.96	31.75	16.06	1.59	0.75	0.75	0.65	0.28	0.09
Weight	1.02	7.13	2.23	0.62	0.40	2.96	1.62	1.46	1.41	2.28	1.93

Table 2.7: Shares of votes of the candidates to the 2017 French presidential election (actual results) and the share of votes they received from the participants of the 2017 experiment in Strasbourg. The last line gives the weights assigned to each candidate's supporters in order to correct the biases.

we also conducted very similar *online* experiments for the 2022 Italian parliamentary election (Marsilio and Delemazure, 2022) with a representative sample of participants, and for the 2024 European election in France (Delemazure and Bouveret, 2024).

Assigning Weights to the Votes

As already mentioned, in most cases the participants in these experiments did not constitute a representative sample of the population, and there are important selection biases. In particular, the political distribution of the participants is generally very different from that of the general population (in general, it is more left-leaning). To observe this, participants were sometimes asked to give their vote for the official election, allowing us to compare the distribution of votes among the participants in the experiments to the official results.

In order to have more interpretable results in our experiments and to partially correct these biases, we will assign weights to the voters (when it is possible). These weights will be based on the actual votes of the participants at the election, such that the share of weights assigned to each candidate is the same as the share of votes they received at the actual election. More specifically, for each candidate, the weight of a participant $i \in V$ who voted for this candidate is set to the ratio between the share of votes this candidate received in the actual election $S_{\text{actual}}(c)/n_{\text{actual}}$, divided by its share of votes in the experiment $S_{\text{expe}}(c)/n_{\text{expe}}$, i.e. $w_i = (S_{\text{actual}}(c) \cdot n_{\text{expe}})/(S_{\text{expe}}(c) \cdot n_{\text{actual}})$. For instance, Table 2.7 gives the weights obtained in the experiment that took place in Strasbourg during the 2017 election (the candidates are identified by their party's name). Observe in particular that the weights of participants who voted for the far-right party FN are quite high (7.13), as they are underrepresented in the sample, while the weights of the participants who voted for the center-left party PS are quite low (0.40), as they are overrepresented in the sample. Note also that we assign a weight 0 to the voters who did not indicate their vote at the official election, or said they voted (or will vote) blank, or that they abstained (or will abstain).[13] Finally, note that the weights need to be (re)computed *for each sample of participants*. For instance, if we only keep the participants who tested one particular voting rule, we need to recompute the weights for this subset of participants.

Conclusion

We now have seen different ballot formats that can be used to express preferences, as well as several types of outcomes that we can obtain using these preferences. We defined the most common

[13] We could have included abstaining voters in the weighting process, but since so few participants in the experiments said they abstained or will abstain, and so much of the population is actually abstaining, this would cause this small group of abstaining voters in our experiment to decide the whole election.

single-winner and multi-winner voting rules, and we also briefly discussed the problem of deriving an ordering of candidates (or axis) from voters' preferences. We introduced tools that we use to evaluate and compare the rules we introduce. The first tool is axiomatic analysis, which allows us to study the properties of the rules on different aspects of their behavior: symmetry properties, reaction to adding or removing candidates, reaction to voters' strategic behaviors, etc. The second tool is experimental analysis, in which we evaluate the behaviors of the rules, the quality of their outcomes (for instance, using distortion), and how similar they are, using datasets of real or synthetic preferences. We discussed various models to generate synthetic data, and how they relate to each other with the map of elections. Finally, we saw that we could also use real data, and we discussed the particular case of experiments in which voters are given the opportunity to try alternative voting methods in parallel to actual large-scale political elections, with the example of the *Voter Autrement* series of experiments. We are now ready to define and model new (voting) problems, propose new rules, and compare these rules using axiomatic and experimental analyses.

Part I

Expressive Ballots for Voting Systems

Chapter 3

Approval with Runoff Voting Systems

3.1 Introduction

The first part of this thesis is dedicated to the ways we can solve issues of existing voting systems by allowing voters to give more expressive ballots. It is only natural for me to start with the voting system I know the most, the cornerstone of French democracy: *plurality with runoff* (known in France as the *scrutin uninominal majoritaire à deux tours*). This voting system works in two rounds: in the first round, voters vote for one (and only one) of the candidates. If one candidate receives more than half of the votes, they are declared the winner. Otherwise, a second round is organized between the two candidates who obtained the most votes in the first round.[1] Reducing the election to two candidates is a way to ensure that the winner will be elected with an absolute majority of the (expressed) votes. This voting system is so important in the French political landscape that it is written in the Constitution of the Fifth Republic.[2]

Plurality with runoff, which was used for large-scale elections for the first time in France during the 19th century, is now the most used voting system for electing heads of state around the world, with 84 countries using it, as shown in Figure 3.1. In France and a few other countries, it is also the voting system used for parliamentary elections.

Given its wide use, plurality with runoff is the topic of quite a number of studies in political science. In particular, the rule tends to allow the viability of more political parties than simple plurality, since voters' rational behavior does not force them to concentrate on two candidates only (Duverger, 1951). On the other hand, numerous drawbacks of plurality with runoff as a voting system were pointed out by scholars and citizens in the recent decades. In particular, plurality with runoff as a voting rule fails many desirable properties, including *independence of clones* and *monotonicity*. These issues are not only theoretical: failing these axioms has a very concrete impact on the outcome of the elections.

[1] There exist variants of this rule where the second round may involve more than two candidates, such as French parliamentary elections.

[2] *"Le Président de la République est élu à la majorité absolue des suffrages exprimés. Si celle-ci n'est pas obtenue au premier tour de scrutin, il est procédé, le quatorzième jour suivant, à un second tour. Seuls peuvent s'y présenter les deux candidats qui, le cas échéant après retrait de candidats plus favorisés, se trouvent avoir recueilli le plus grand nombre de suffrages au premier tour."*.

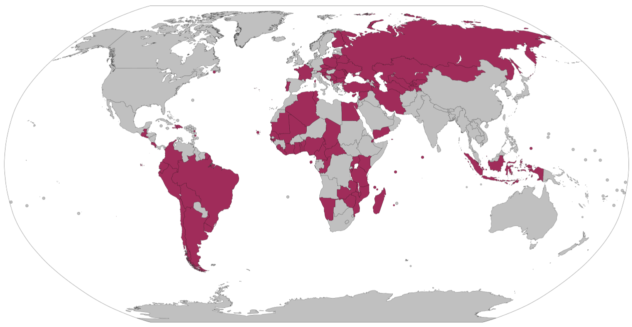


Figure 3.1: Countries in which plurality with runoff is used for electing the head of state.

Sensitivity to Clones

Consider first the *independence of clones* property (which we introduced in Section 2.4.4), with the following example of an election with three candidates Ann, Bob and Cora, having the following scores in the first round:

Ann	Bob	Cora
35%	40%	25%

With these scores, Bob and Ann go to the second round. Let us say that most of Cora's supporters decide to vote for Bob in the second round, such that Ann receives 45% of the votes, and Bob wins with 55%. Let us now assume that we add a new candidate Bobby in the election. Bobby is a *clone* of Bob: every voter who likes Bob also likes Bobby, and conversely. For instance, this could be the case if Bob and Bobby are two candidates from the same party. *Independence of clones* says that if we keep the same voters as in the first election, the winner should not change, and be either Bob or Bobby. If we assume that Bob's supporters from the first election are equally divided between Bob and Bobby, these are the results of the first round in this new election:

Ann	Bob	Bobby	Cora
35%	20%	20%	25%

The two candidates reaching the second round are now Ann and Cora, and neither Bob nor Bobby can win the election. Thus, plurality with runoff fails independence of clones.

This scenario may look oversimplified and purely theoretical, but it can be linked to actual elections that used plurality with runoff. Consider the 2002 presidential election in France, in which 16 candidates competed in the first round. The two candidates who reached the second round were the incumbent president *Jacques Chirac* and the far-right candidate *Jean-Marie Le Pen* with respectively 19.88% and 16.86% of the vote share: a second round that no polling institute predicted. The candidate for the socialist party, *Lionel Jospin*, came third with 16.18% of the votes. In this election, Jospin has the role of Bob from our toy example, as he had to share the votes from left-wing voters with 7 other candidates (3 from the far-left, 4 from the left). Together, these 7 candidates and Lionel Jospin actually got around 43% of the votes (32.45% for the left and 10.44% for the far-left), but as the supporters from the left were divided between a lot of candidates, none of them were able to get more votes than the leader of the far-right Jean-Marie Le Pen, who only had one *clone*, who received only 2.34% of the votes. In the second round, Jacques Chirac won by a large majority of the votes (82.21%) against Jean-Marie Le Pen. However, in a second round between Jacques Chirac and Lionel Jospin, the results could have been different, as the last polls were predicting a quasi-split election (with each getting around 50% of the votes).

Failure of Monotonicity

Another property failed by plurality with runoff is the *monotonicity* property (that we introduced in Section 2.4.5). Informally, this property states that if a candidate is winning in an election and we increase the number of people supporting this candidate, then it should still be winning. Conversely, a failure of monotonicity implies that it is possible to make a candidate win an election that they were not winning by *decreasing* their number of supporters (and in that sense, it is also a failure of strategyproofness). Consider for instance another election between Ann, Bob and Cora:

Ann	Bob	Cora
37%	32%	31%

In this election, Ann and Bob go to the second round. Now, let us assume that many of Cora's supporters really do not like Ann, so in the second round most of them vote for Bob, who ends up winning the election. However, let us assume that in this election, if the second round was between Ann and Cora, Bob's supporters would be evenly divided between them and Ann would actually be the winner. Ann's supporters know this, so some of them might vote for Cora instead of Ann in the first round, so that the second round is against Cora instead of Bob. Thus, we obtain the following scores:

Ann	Bob	Cora
35%	32%	33%

Here, the score of Ann decreased from the previous election, but it actually made her a winner, while she was losing before, because the second round is now against Cora. This is an example of a failure of monotonicity.

This flaw is well-known by politicians, who know how to use it to their advantage. Even if this does not correspond exactly to a violation of monotonicity, they can organize their campaigns such that they are 'helping' a candidate who will be easier to beat in the second round. Consider for instance the case of the 1988 French presidential election. The main candidate for the left was François Mitterand, and there were two main candidates for the right: Raymond Barre and Jacques Chirac. Chirac was more conservative than Barre, so Mitterand had more chance to win a second round against him (Barre would have got some of the centrist votes received by Mitterand). This might be one of the reasons that lead him to focus his attacks against Jacques Chirac, making him *de facto* his opponent, and the leader of the right (Giesbert, 1996, p. 520).

Because of these drawbacks (sensitivity to clones, failure of monotonicity), a lot of people are voting strategically when plurality with runoff is used. For instance, in France, many people are torn between voting sincerely and voting strategically (which is sometimes called "useful voting"), creating a lot of frustration: voters who voted sincerely might think they could have changed the result by being more strategic, and the ones who vote strategically without succeeding would have preferred to support their actual favorite party.

The Runoff

Luckily, there exists a method that solves both of these issues, and even more: *approval voting* (see Section 2.3.1). The principle is very simple: voters can vote for as many candidates as they want, and the candidate who receives the most votes is declared the winner. This rule is widely discussed in the scientific literature, and even promoted by some organizations (see for instance the *Center for Election Science* in the US). It seems however that this change would be hard to accept for countries that are used to runoff systems, in particular in France, where every major election has a runoff (except the election of the members of the EU Parliament, see Chapter 5).

There exists some theoretical benefits of the runoff (for instance, it prevents the election of a Condorcet loser), but the main arguments in favor of having a second round are probably more cultural and political: people are used to it, it focuses the debate between two candidates only, and it ensures that the winner is supported by an absolute majority of voters, giving in principle more legitimacy to the elected candidate.

This lead us to ask the question of whether it is possible to keep the benefits of the two-round protocol, without having to bear the drawbacks of using plurality in the first round. Clearly, if the answer to this question is positive, the format of the ballots in the first round must no longer be uninominal. As detailed in Chapter 2, several possibilities exist, like ordinal ballots, cardinal ballots or approval ballots. As we just discussed, voting with approval ballots has several advantages when there is only one round, so we can expect to keep some of the benefits if we add a second round. Moreover, approvals are simple and easy to express: changing the ballot of the first round from uninominal to approval is probably the simplest change in the voting system that can be done while still having a major positive impact on the voting process. We explore this possibility seriously in this chapter.

Approval with Runoff

We define an *approval with runoff* election as a two-round protocol: In the first round, voters cast approval ballots. We use these ballots to select two finalists and in the second round, voters cast votes for one of them. The candidate with the majority of the votes wins the overall election. This voting system was actually used in the municipal elections of St. Louis, Missouri in 2021 and 2025 after St. Louis voters passed a proposition called *Proposition D* in 2020.[3] Approval voting with runoff was also used in several cantons in Switzerland for multi-winner elections. The precise rules vary from one canton to the other so that the second round is sometimes almost unused, as in the canton of Zurich (Laslier and Van der Straeten, 2016; Van der Straeten et al., 2018).

In this chapter, we model approval with runoff as a *voting rule*, with a one-shot input, and study its properties in a similar way as we would study the properties of plurality with runoff or any single-winner voting rule. Then, two major questions arise: (1) What should the input of the rule consists of? and (2) Which rule should be used to determine the two finalists?

For the first question, the answer becomes clear once we remark that we need the approval preferences for computing the finalists, and the pairwise comparisons between the finalists for computing the winner. As we cannot know in advance who the two finalists will be, we need to be able to compare each pair of candidates, even if in the end we use pairwise comparisons for only one pair of candidates. We will thus assume that we have access to each voter's ranking of the candidates. Combining this ranking with approval preferences, and assuming that the two are consistent with each other, this means that the rankings come with a threshold that separates approved candidates from disapproved candidates. We call this kind of data structure *approval-ordinal* ballots. It was first introduced by Brams and Sanver (2009) under the name *approval-preference*. They studied two rules based on this kind of input (*preference approval voting* and *fallback voting*). Subsequent works studied possibility and impossibility results in the approval-ordinal setting. Approval-ordinal ballots have also been defined for other kinds of output, such as ranking with approval information (Kruger and Sanver, 2021), or consensus measures (Erdamar

[3] https://stlapproves.org/

et al., 2014). Erdamar et al. (2017), studied the manipulability of *resolute* rules based on approval-ordinal profiles, showing an impossibility result.

Approval with runoff was first introduced in the literature by Sanver (2010), who compared it to other rules based on approval-ordinal profiles. Approval with runoff was shown to satisfy monotonicity and to fail independence of irrelevant alternatives. More recently, Green-Armytage and Tideman (2020) compared approval with runoff, plurality with runoff and several other runoff rules for selecting the finalists, with varying input formats (ordinal, approval, valued). For approval ballots, they assume that every voter i votes sincerely according to a utility function u_i, and they approve a candidate if and only if the utility they receive from this candidate is larger than the average utility over all candidates running (for this voter). They evaluate these rules along four numerical criteria: expected utility of the winner, expected utility of the runoff loser, representativeness and resistance to strategic behavior. Their results come both from using real data and from simulations. Among other conclusions, plurality with runoff scores particularly badly, and approval with runoff, slightly better.

Notice that the approval-ordinal framework does not allow taking into account the evolution of candidates' campaigns and of voters' preferences between the two rounds. For instance, in a real election, candidates tend to move to the center in ideology and rhetorical complexity between the first and the second round (Di Tella et al., 2023), and voters might change opinions after the debate, which generally takes place between the two rounds. However, we leave these problems to further research, and as it is often the case in social choice theory, we assume a fixed electorate and we consider that the ballots cast by this electorate are *sincere*, meaning that they correspond to the true preferences of the voters, and are not influenced by strategic considerations (although several of our observations do not rely on this assumption).

For the second question (which rule should be used to determine the finalists?), things are more complex because there is not a unique way to select two candidates from the approval ballots. Indeed, the selection of two finalists from approval preferences is actually a restriction of the general setting of *approval-based committee voting* that we discussed more thoroughly in Section 2.3.2. As a reminder, for a committee size $k \in \mathbb{N}$, approval-based committee rules (ABC rules) take as input an approval profile and output a set of k candidates. In our case, we want to select $k = 2$ candidates. An extensive survey on ABC voting is Lackner and Skowron (2023), and we now have a series of results that tell us which properties the various possible rules satisfy and in which contexts they are suitable. Most importantly, the choice of the rule used in the first round has strong implications about the very nature of the two-stage rule, both from a normative point of view and from a political science point of view: should we send to the second round the two most-approved candidates? Should we guarantee that the most-approved candidate is in the second round? Should we ensure some diversity in the second round to be sure that enough voters feel represented? Should we pay attention to proportionality issues? Therefore, in this chapter we will consider *approval with runoff* not as just one rule, but *a family of rules*. Our aim is then to explore the reasons that may guide us towards the choice of one of the rules in this family.

Outline of the Chapter

This chapter is organized as follows. We start by introducing the approval with runoff model and we define several rules in Section 3.2. Then, we study the properties of these rules under the axiomatic lens (Section 3.3), the statistical lens with an analysis of the behavior of the rules in the 1-dimensional Euclidean space (Section 3.4) and finally the experimental lens (Section 3.5). We conclude and discuss further work in Section 3.6.

3.2 Approval With Runoff Rules

In this section, we introduce the approval with runoff setting and define several rules.

Approval-Ordinal Profiles

Informally, an approval with runoff rule works in two steps. In the first step, we use the approval preferences of the voters to select the two finalists. In the second step, we do a majority vote between these two finalists to select the winner. Since we cannot know in advance who the two finalists will be, we need to be able to compare each pair of candidates. We thus assume that we have access to voters' rankings of the candidates. We call this structure an *approval-ordinal* ballot (in the literature, Brams and Sanver (2009) introduced this data structure as *approval-preferences*).

More formally, we have an approval profile $P_A = (A_1, \dots, A_n)$ containing approval ballots of the voters, and an ordinal profile $P_\succ = (\succ_1, \dots, \succ_n)$ containing the rankings of the voters. We denote $P = (P_A, P_\succ)$ the approval-ordinal profile. For simplicity, we will assume *ballot consistency*, that is, consistency between the approval preferences and the ordinal preferences of the voters. Formally, this means that $x \succ_i y$ holds for all $x \in A_i$ and $y \notin A_i$. This implies that each voter $i \in V$ has a threshold in their ranking $\succ_i$ such that every candidate above the threshold is approved and every candidate below is not. Thus, we will use the following notation from Brams and Sanver (2009): $x_1 \dots x_j | x_{j+1} \dots x_m$ represents $(\succ_i, A_i)$ with $x_1 \succ_i \dots \succ_i x_m$ and $A_i = \{x_1, \dots, x_j\}$.

Ballot consistency may seem a mild behavioral requirement and is often taken for granted in the approval voting literature. However, counterexamples can be found showing that it does not necessarily hold if voters are strategic (Dutta et al., 2006) and observations show that it may fail empirically when voters, in a single-winner election, take the opportunity of a multinominal ballot to signal that they support a candidate who has no chance of winning (Baujard et al., 2021). In our case a violation of the ballot consistency requirement would be observed when a voter approves a candidate c_i, disapproves a candidate c_j but votes for c_j when the second round is between c_i and c_j. However, in this chapter, all results hold without this assumption, by slightly adapting the definitions of the properties.

Approval-Based Committee Rules

The first step of approval with runoff rules relies on approval-based committee rules (ABC rules) to select the two finalists. We already discussed ABC rules in Section 2.3.2, but we will recall here the definitions of these rules in the particular case of committee size $k = 2$. We show that in this case, most ABC rules can be defined using a simple formula. Note that we consider here *irresolute* rules, that is, rules that can output several (tied) committees.

Given an approval profile P_A, $S(c) = |\{i : c \in A_i\}|$ is the approval score of a candidate $c \in C$, and the candidates maximizing the score S are the approval winners. By extension, the approval score of a set of candidates $J \subseteq C$ is the number of ballots that contain all candidates from J, i.e. $S(J) = |\{i : J \subseteq A_i\}|$. For simplicity, we will use the lighter notation $S(abc)$ instead of $S(\{a, b, c\})$. We will also use this notation for writing profiles. For instance, $(2 \times a, 4 \times bcd, 1 \times ad)$ is the approval profile containing 2 ballots $\{a\}$, 4 ballots $\{b, c, d\}$ and 1 ballot $\{a, d\}$.

The most intuitive rule is probably (multi-winner) *approval voting* (AV), which simply selects the two candidates with the highest approval scores.

Approval Voting (AV)

$$\mathrm{AV}(P_A) = \operatorname*{argmax}_{x \neq y \in C} S(x) + S(y)$$

Note that this rule is the only rule that has been used in actual *approval with runoff* elections. In particular, it was used in the 2021 and 2025 municipal elections of St. Louis, Missouri. Moreover, the classical "approval with runoff" rule studied in the literature is based on this rule (Sanver, 2010; Green-Armytage and Tideman, 2020).

Thiele Rules

With AV, if a group of voters is larger than the other groups and if it has at least two favorite candidates, it is likely that the two finalists will be among the favorite candidates of this group, leaving the other groups unrepresented in the second round. To bring some diversity to the set of selected finalists, other rules such as Proportional Approval Voting (PAV) and Approval Chamberlin-Courant (CCAV) discount the satisfaction of voters who are already satisfied by one of the two finalists. For instance, under PAV a voter approving ℓ candidates of a committee W gives a score $s = \sum_{j=1}^{\ell} 1/j$ to the committee. For a committee of size $k = 2$, this means that a voter approving one candidate gives a score of 1, and a voter approving both candidates a score of 1.5 to the committee. If we used AV, voters approving both candidates would give a score of 2. Under CCAV, a voter approving both candidates gives a score of 1 to the committee. This rule therefore favors committees that cover as many voters as possible: it maximizes the number of voters who approve at least one of the finalists. As we mentioned in Section 2.3.2, PAV is conceptually related to proportional representation (Brill et al., 2018), while CCAV is more related to diversity.

Another way to see the PAV and CCAV scores is to say that we discard respectively 0.5 and 1 point for each voter approving both candidates of the committee from the total approval score of the pair of candidates. These rules can thus be written with simple formulas:

Proportional Approval Voting (PAV)

$$\mathrm{PAV}(P_A) = \operatorname*{argmax}_{x \neq y \in C} S(x) + S(y) - 1/2 \cdot S(xy)$$

Approval Chamberlin Courant (CCAV)

$$\mathrm{CCAV}(P_A) = \operatorname*{argmax}_{x \neq y \in C} S(x) + S(y) - S(xy)$$

In ABC voting, AV, PAV and CCAV all belong to the general family of *Thiele rules* (Orsted et al., 1894), that we discussed in Section 2.3.2. When restricted to $k = 2$, all Thiele rules select the pair of candidates $x, y \in C$ maximizing $S(x) + S(y) - \alpha S(x, y)$ for some $\alpha \in [0, 1]$. For instance, this α is equal to 0 for AV, 1/2 for PAV and 1 for CCAV. We call these rules α-AV rules.

Example 3.1
Consider for instance the following approval profile P_A:

$$2\times a \qquad 6\times ab \qquad 4\times abc \qquad 4\times cd \qquad 1\times d$$

In this profile, we have $S(a) = 12$, $S(b) = 10$, $S(c) = 8$ and $S(d) = 5$. Moreover, all of b's supporters also approve a, half of c's supporters approve a and none of d's supporters approve a. Thus, the scores of the three pairs of candidates $\{a, b\}$, $\{a, c\}$ and $\{a, d\}$ for AV, PAV and CCAV are the following:

	AV	PAV	CCAV
$\{a, b\}$	$12 + 10 = 22$	$12 + 10 - (1/2) \cdot 10 = 17$	$12 + 10 - 10 = 12$
$\{a, c\}$	$12 + 8 = 20$	$12 + 8 - (1/2) \cdot 4 = 18$	$12 + 8 - 4 = 16$
$\{a, d\}$	$12 + 5 = 17$	$12 + 5 - (1/2) \cdot 0 = 17$	$12 + 5 - 0 = 17$

It is easy to check that all other pairs of candidates get a lower score for all three rules. Thus, the winning pairs are $\{a, b\}$ for AV, $\{a, c\}$ for PAV and $\{a, d\}$ for CCAV. AV selects the pair with maximal support (22 points) while CCAV selects the pair covering the highest number of voters, as they all approve either a or d.

Sequential Rules

We saw in Section 2.3.2 that sequential versions of Thiele rules have also been defined in the literature. The candidates are then picked sequentially, and the weight of each voter is updated after a candidate is added to the committee. The main advantage in the general case is that sequential rules are easier to compute than their non-sequential counterparts. More specifically, Thiele rules are NP-hard to compute (except for AV). However, when we fix $k = 2$, Thiele rules become polynomial-time computable. The sequential versions still have some interesting properties, for instance they ensure that the first selected candidate is an approval winner.

We thus define α-seqAV rules that first select an approval winner x, then the candidate maximizing $\text{argmax}_{y \in C, y \neq x} S(y) - \alpha S(xy)$. This is equivalent to saying that the weights of voters who have approved the first finalist x are updated to $1 - \alpha$. Special cases of α-seqAV are S-PAV and S-CCAV, defined below. In the former, the weight of already satisfied voters is updated to $1/2$ ($\alpha = 1/2$), in the latter, it is updated to 0 ($\alpha = 1$). AV is equivalent to its sequential version.

Sequential Proportional Approval Voting (S-PAV)
The rule chooses the pairs $\{x, y\}$ such that x maximizes $S(x)$ and y maximizes $S(y) - 1/2 \cdot S(xy)$.

Sequential Approval Chamberlin Courant (S-CCAV)
The rule chooses the pairs $\{x, y\}$ such that x maximizes $S(x)$ and y maximizes $S(y) - S(xy)$.

Example 3.2
In the approval profile P_A from Example 3.1, observe that a is the unique approval winner. Thus, it is selected first by all α-seqAV rules. This implies that the only possible pairs of finalists are $\{a, b\}$, $\{a, c\}$ and $\{a, d\}$. Since we know that PAV selects $\{a, c\}$ and CCAV selects $\{a, d\}$, this implies that the sequential variants select the same pairs of candidates as their non-sequential counterparts.

Note that in Example 3.2, the sequential rules select the same pairs of candidates than their non-sequential counterparts, but this is not the case in general. In particular, we will see in Corollary 3.12 that the non-sequential rules might not select the approval winner among the two finalists, while sequential rules always do.

Other Sequential Rules

The Eneström-Phragmén rule (Camps et al., 2019) is another sequential ABC rule, which is not part of the family of sequential Thiele rules in general. However, in the case $k = 2$, it almost falls into the family of α-seqAV rules. Indeed, this rule first selects an approval winner x, and set the weights of voters who approve x to $\max(0, 1 - Q/S(x))$, where $Q \in [0, n/2]$ is a quota, for instance the Droop quota and the Hare quota, which are equal to respectively $n/3+1$ and $n/2$ when $k = 2$. For a fixed n, this rule is thus equivalent to the α-seqAV rule with $\alpha = \min(1, Q/S(x))$.

Eneström-Phragmén (EnePhr)
The rule chooses the pairs $\{x, y\}$ such that x maximizes $S(x)$ and y maximizes $S(y) - \min(1, Q/S(x)) \cdot S(xy)$.

Interestingly, the definition of the Eneström-Phragmén rule in the case $k = 2$ is also very close to the one of the *method of equal shares* that was introduced recently by Peters and Skowron (2020). When $k = 2$, it works as follows: the first selected candidate is an approval winner x. If $S(x) \geqslant n/2$ and there exists $y \neq x$ such that $S(y) - n/2 \cdot S(xy)/S(x) \geqslant n/2$, then select y maximizing $S(y) - n/2 \cdot S(xy)/S(x)$ (exactly as in the Eneström-Phragmén rule with the Hare quota $Q = n/2$). If the conditions on $S(x)$ and $S(y)$ are not fulfilled, there is no unique way to select the winners under the method of equal shares. One recommended way is to complete the committee using Sequential Phragmén, which we define below. However, when $k = 2$, using Sequential Phragmén to select the second candidate is equivalent to using Sequential Phragmén from the start.

Proposition 3.1
When $k = 2$, the method of equal shares is equivalent to Eneström-Phragmén with $Q = n/2$ when it returns a complete committee.

Thus, we will not consider the method of equal shares in our analysis. We now introduce Sequential Phragmén. Its formulation is quite involved in the general case, but when $k = 2$, it boils down to the following simple formula:

Sequential Phragmén (S-Phr)
The rule chooses the pairs $\{x, y\}$ such that x maximizes $S(x)$ and y minimizes:

$$\frac{1 + S(xy)/S(x)}{S(y)}$$

Example 3.3
We use the profile P_A from Examples 3.1 and 3.2. For the Eneström-Phragmén rule, the Droop quota is equal to $Q = n/3 + 1 = 20/3$ and the Hare quota to $Q = n/2 = 17/2$. Since a is the approval winner, it is selected first, and α is equal to $Q/S(a)$ which is equal to $20/36$ for the Droop quota, and $17/24$ for the Hare quota. In the first case, observe that $20/36$ is very close to $1/2$ and thus it is not surprising that the second selected candidate is c, as in S-PAV (that uses $\alpha = 1/2$). In the second case, c is also maximizing $S(c) - (17/24) \cdot S(ac) = 5.2$. Finally, c is also the candidate minimizing $(1 + S(ac)/S(a))/S(c)$ and is thus selected by the Sequential Phragmén rule. Thus, all these rules return the pair $\{a, c\}$.

Other Rules

An idea that seems good but is in general bad is to decide that each voter gives one point that they divide equally between their approved candidates. This is the idea behind *Split Approval Voting*:

Split Approval Voting (SAV)
The rule chooses the two candidates $x, y \in C$ with $x \neq y$ that maximize the split approval score $Sp(x) = \sum_{i:x \in A_i} 1/|A_i|$.

Example 3.4
In the approval profile P_A from Example 3.1, the split approval scores are $S(a) = 3+6/2+4/3 = 7+1/3$, $S(b) = 6/2+4/3 = 4+1/3$, $S(c) = 4/3+4/2 = 3+1/3$ and $S(d) = 4/2+1 = 3$. Thus, SAV returns the pair $\{a, b\}$.

We also define the trivial rule that returns all pairs of candidates. More formally, $\text{TRIV}(P_A) = \{\{x, y\} \mid x, y \in C, x \neq y\}$. Note that when we will add a runoff to the trivial rule, it will not be trivial anymore: it will output all candidates *except the Condorcet loser* whenever there is one.

The Runoff

We now define the family of approval with runoff rules (AVR rules). As we already mentioned, the idea is that we use the approval ballots in the first round to select the two finalists, and the second round consists of a majority vote between the two selected candidates. For this, we define $\text{MAJ}(P_\succ, \{x, y\})$ as the set of winners of the majority vote between x and y in the ordinal profile $P_\succ$. Note that this set of winners is a singleton except in the case of a tie between x and y.

Then, let f be an (irresolute) ABC rule, returning a non-empty set of pairs of candidates, then f^R is the (irresolute) AVR rule that applies the majority rule on every pair of candidates selected by f. More formally, for an ABC rule f we have:

$$f^R(P = (P_A, P_\succ)) = \bigcup_{\{x,y\} \in f(P_A)} \text{MAJ}(P_\succ, \{x, y\})$$

This directly gives us the definition of AV^R, PAV^R, CCAV^R, S-PAV^R, S-CCAV^R, EnePhr^R, S-Phr^R, SAV^R, and TRIV^R.

Example 3.5
Let us combine the approval profile P_A from Example 3.1 with an ordinal profile $P_\succ$, such that we obtain the following approval-ordinal profile P:

$2\times a|bcd$ $\quad$ $3\times ba|dc$ $\quad$ $3\times ab|dc$ $\quad$ $4\times bac|d$ $\quad$ $2\times cd|ba$ $\quad$ $2\times dc|ba$ $\quad$ $1\times d|bac$

Observe that voters who approve $\{a, b\}$ can have two different rankings: half of them prefer a to b, and the other half prefer b to a. In this profile, $\text{MAJ}(P_\succ, \{a, b\}) = \{b\}$ and $\text{MAJ}(P_\succ, \{a, c\}) = \text{MAJ}(P_\succ, \{a, d\}) = \{a\}$. Thus, $\text{AV}^R(P) = \text{SAV}^R(P) = \{b\}$ and for all the other ABC rules we defined (except TRIV), $f^R(P) = \{a\}$. Finally, observe that d is a Condorcet loser in this profile, and thus $\text{TRIV}^R(P) = \{a, b, c\}$.

Now that we have defined the model and some approval with runoff rules, we can study and compare them in the next sections through axiomatic, statistical, and experimental analyses.

3.3 Axiomatic Analysis

We start this axiomatic analysis with classical symmetry properties (Section 2.4.1). An AVR rule f^R is *anonymous* if for any permutation ρ of the voters and every profile P we have $f^R(\rho(P)) = f^R(P)$, and it is *neutral* if for any permutation π of the candidates and every profile P, we have $f^R(\pi(P)) = \pi(f^R(P))$. All the AVR rules defined in Section 3.2 satisfy these two properties.

Like other runoff rules (including plurality with runoff), AVR rules also satisfy the *Condorcet loser criterion* (see Section 2.4.2), which states that a candidate who loses the majority vote against every other candidate should not win the election. On the other hand, because of the runoff, we expect that no AVR rule satisfies the *reinforcement* axiom defined in Section 2.4.3. Consider for instance the two profiles $P_1 = \{10 \times a|bc, 6 \times c|ba, 5 \times b|ca\}$ and $P_2 = \{10 \times b|ca, 6 \times c|ba, 5 \times a|cb\}$, in which all approval ballots are of size 1. In these profiles, any AVR rule selects the finalists $\{a, c\}$ in P_1 and $\{b, c\}$ in P_2, and c wins the majority vote in both cases. However, in $P_1 + P_2$, the finalists are necessarily $\{a, b\}$, so c cannot be the winner.

Efficiency

For defining Pareto-efficiency, we first have to define Pareto-dominance. The following condition is a strengthening of strict Pareto (see Section 2.4.2), adapted to the approval-ordinal case. We say that candidate a dominates candidate b if

1. for every voter $i \in V$, $a \succ_i b$ and
2. for some voter $i \in V$, $a \in A_i$ and $b \notin A_i$.

Together with ballot consistency, this condition implies that every voter who approves b also approves a, and at least one voter who approves a does not approve b. Note that this also implies that $S(a) > S(b)$ and that $\text{MAJ}(P_\succ, \{a, b\}) = a$. If we did not assume ballot consistency, we would have required a third condition: (3) for every voter $i \in V$, $b \in A_i$ implies that $a \in A_i$.

Pareto-efficiency
An AVR rule f^R is *Pareto-efficient* if for any approval-ordinal profile P for which there exists $a, b \in C$ such that a dominates b, then $b \notin f^R(P)$.

Proposition 3.2
AV^R, PAV^R, S-PAV^R, S-Phr^R and SAV^R are Pareto-efficient, but not CCAV^R, S-CCAV^R, EnePhr^R, and TRIV^R.

Proof. Consider a profile $P = (P_A, P_\succ)$ in which a candidate a dominates another candidate b. Let $f \in \{$AV, PAV, S-PAV, S-Phr, EnePhr, SAV$\}$ and f^R the associated approval with runoff rule. We want to show that $b \notin f^R(P)$. We assume by contradiction that $b \in f^R(P)$. Thus, there exists a candidate $x \in C$ such that $\{x, b\} \in f(P_A)$ and $\text{MAJ}(P_\succ, \{x, b\}) = b$.

It is clear that $x \neq a$, because we know that $\text{MAJ}(\succ, \{a, b\}) = a$ since a dominates b. We are going to show that for every considered ABC rule we have a contradiction.

- Let f be an α-AV rule with $\alpha < 1$. Define $s(y, z)$ the α-AV score of a pair of candidates $\{y, z\}$:

$$s(y, z) = S(y) + S(z) - \alpha S(yz) = S(y) + (1 - \alpha)S(z) + \alpha(S(z) - S(yz))$$

Since a dominates b, we have $S(a) > S(b)$. Moreover, for any candidate $x \in C$, if voter i approves b and not x, it also approves a. Thus, $S(a) - S(ax) \geqslant S(b) - S(bx)$. We have $s(x, a) = S(a)+(1-\alpha)S(x)+\alpha(S(a)-S(xa)) > S(b)+(1-\alpha)S(x)+\alpha(S(b)-S(xb)) = s(x, b)$, thus $\{x, b\}$ cannot be a pair of finalists because the rule selects the pairs of candidates maximizing s.

- Let f be an α-seqAV rule with $\alpha < 1$. b is not the first finalist selected because $S(a) > S(b)$. Let x be the first finalist selected and define $s_x(y)$ as follows:

$$s_x(y) = S(y) - \alpha S(xy) = (1 - \alpha)S(y) + \alpha(S(x) - S(xy))$$

Again, a dominates b, so we have $S(a) > S(b)$ and for all x, $S(a) - S(ax) \geqslant S(b) - S(bx)$. Thus, $s_x(a) = (1 - \alpha)S(a) + \alpha(S(x) - S(xa)) > (1 - \alpha)S(b) + \alpha(S(x) - S(xb)) = s_x(b)$ and $\{x, b\}$ cannot be selected because it is not maximizing s_x.

- Let f be S-Phr. Since $S(a) > S(b)$, b is not the first selected candidate. Let x be the first finalist. For $y \in C$, let us define s_x as:

$$s_x(y) = \frac{1 + S(xy)/S(x)}{S(y)} = \frac{1}{S(x)}\frac{S(x) + S(xy)}{S(y)}$$

We want to prove that $\frac{S(x)+S(xa)}{S(a)} < \frac{S(x)+S(xb)}{S(b)}$. Now, observe that

$$\frac{S(x) + S(xb)}{S(b)} \geqslant \frac{S(x)}{S(b)} > 1 \geqslant \frac{S(xa) - S(xb)}{S(a) - S(b)}$$

because $S(a) - S(xa) \geqslant S(b) - S(xb)$. This gives

$$\begin{aligned}
&(S(x) + S(xb))(S(a) - S(b)) > (S(xa) - S(xb))S(b) \\
&(S(x) + S(xb))S(a) - (S(x) + S(xb))S(b) > (S(xa) - S(xb))S(b) \\
&(S(x) + S(xb))S(a) > S(xa)S(b) + S(x)S(b) \\
&\frac{S(x) + S(xb)}{S(b)} > \frac{S(x) + S(xa)}{S(a)}
\end{aligned}$$

Therefore, $s_x(a) < s_x(b)$, thus $\{b, x\}$ cannot be a pair of finalists.

- If f is SAV, we denote by $Sp(y) = \sum_{y \in A_i} \frac{1}{|A_i|}$ the split approval score. Since $b \in A_i$ implies $a \in A_i$, and there exists $i \in V$ such that $a \in A_i$ and $b \notin A_i$, it is clear that $Sp(a) > Sp(b)$. Thus $\{x, b\}$ cannot be a pair of finalists if $x \neq a$.

We showed that $\{b, x\} \notin f(P_A)$ for all x. Therefore, $b \notin f^R(P)$ for $f \in$ {AV, PAV, S-PAV, S-Phr, SAV} and these rules are Pareto-efficient.

To show that the other rules are not Pareto-efficient, consider the profile P below with $p > 3$:

$$1\times abc| \qquad 1\times ab|c \qquad 1\times a|bc \qquad p\times |bca$$

For $f \in$ {CCAV, S-CCAV}, $f(P_A) = \{\{a, b\}, \{a, c\}\}$ as both pairs cover 3 voters. However, $\text{MAJ}(\succ, \{a, c\}) = c$, so $c \in f^R(V)$. However, b dominates c. Thus, f is not Pareto-efficient. For EnePhr with quota $Q > 0$, observe that it is equivalent to S-CCAV if $Q/S(a) \geqslant 1$. Then, if $Q = \beta n = \beta(p + 3)$ with $\beta \in [0, 1]$, we choose p such that $Q/S(a) \geqslant 1$, i.e. $p \geqslant 3(1/\beta - 1)$. For this value of p, EnePhr is equivalent to S-CCAV and thus fails Pareto-efficiency.

The same profile P with $p = 0$ can be used to show that the trivial rule is not Pareto-efficient (as $\{b, c\} \in \text{TRIV}(P_A)$ and $\textsc{maj}(P_\succ, \{b, c\}) = b$, which is dominated by a). □

Note that this proof shows that all α-AV^R and α-seqAV^R rules with $\alpha < 1$ satisfy Pareto-efficiency. CCAV^R and S-CCAV^R fail Pareto-efficiency because CCAV and S-CCAV are sometimes too irresolute. Still, they satisfy a weaker notion of Pareto-efficiency, which says that if a dominates b and $b \in f^R(P)$, then $a \in f^R(P)$. We can also define a variant of CCAV, let us call it CCAV$_+$, in which ties between pairs of finalists with the same CCAV scores are broken in favor of the ones with the highest AV score. This variant is Pareto-efficient because for every candidate b dominated by a candidate a and every other candidate $x \in C$, the CCAV scores $s(x, a) \geqslant s(x, b)$, and $S(a) > S(b)$, so $\{x, b\}$ cannot be a pair of finalists. We can similarly define variants of S-CCAV^R and EnePhr^R.

Strategy-proofness

In most models, strategy-proofness notions (see Section 2.4.5) are incompatible with basic efficiency or proportionality notions. The AVR framework is no exception, as we will prove that a certain form of strategy-proofness is not compatible with our Pareto-efficiency axiom. However, we will show that a weaker strategy-proofness notion is satisfiable by some of the rules. This contrasts with the work by Erdamar et al. (2017), who studied the manipulability of *resolute* rules based on approval-ordinal and showed an impossibility result.

Let us first define strategy-proofness for AVR rules. We say that a profile P' is an i-deviation of P if for all voters $j \neq i$ we have $A_j = A'_j$ and $\succ_j = \succ'_j$. Generally, a rule is said to be strategy-proof if no voter can benefit from changing their vote. However, deciding what "benefit" means here can take many forms. In particular, we can base the notion of benefit on the ordinal preferences *or* on the approval preferences of the voters. There is little hope that any AVR rule can satisfy a strategy-proofness notion based on rankings, since only the approval preferences are used to select the finalists, so by changing their threshold, a voter can easily change the finalists. Thus, we only consider strategy-proofness notions based on the approval preferences of the voters. In particular, a weak notion would say that the manipulating voter should not be able to change the set of winners from no candidate they approve to only candidates they approve.

Strategy-proofness
An AVR rule f^R is *strategy-proof* if for every profile P there is no voter $i \in V$ and i-deviation P' of P such that $f^R(P) \cap A_i = \emptyset$ and $f^R(P') \subseteq A_i$.

Interestingly, both AV^R and CCAV^R satisfy this strategy-proofness axiom. However, no α-AV^R rule for $\alpha \notin \{0, 1\}$ satisfies it, and similarly for α-seqAV^R rules.

Proposition 3.3
AV^R, CCAV^R and TRIV^R are strategy-proof but not PAV^R, S-PAV^R, S-CCAV^R, EnePhr^R, S-Phr^R and SAV^R.

Proof. We start by proving that AV^R is strategy-proof. Assume by contradiction that it is not and that there are profiles P and P' with $b \notin A_i$ winning in P and $a \in A_i$ winning in P'. This means there is a pair $\{x, b\}$ selected in P and not in P' and a pair $\{a, y\}$ selected in P' and not in P. Thus, $S(x) + S(b) > S(a) + S(y)$. From P to P', the approval scores of a and b cannot change since a is already approved in P and b already disapproved. However, the scores of x and y can change, and they can change by 1 each.

Obviously, $x \neq y$ since the scores of a and b do not change. Thus, $x \neq y$. If only the score of x or the score of y changes, this would mean that $S(x)+S(b) < S(a)+S(y)+1$, which is not possible since $S(x)+S(b) > S(a)+S(y)$ and all approval scores are integer. For the same reason, $x \neq a$ and $y \neq b$.

Thus, the score of both x and y changes, meaning that voter i approved x and now disapproves it and disapproved y and now approves it, and we have $S(a)+S(y) < S(x)+S(b) < S(a)+S(y)+2$, which implies that $S(x)+S(b) = S(a)+S(y)+1$.

Moreover, $S(a) \leqslant S(b)$, otherwise $\{x, b\}$ would not have been a possible pair in P and $S(b) \leqslant S(a)$ otherwise $\{y, a\}$ would not have been a possible pair in P'. This means that $S(a) = S(b)$ and $S(x) = S(y)+1$.

We can also say that $S(y) \leqslant S(b)$ and $S(a) \leqslant S(x)$, otherwise $\{x, b\}$ would not have been selected in P. Thus, $S(a) \leqslant S(x) \leqslant S(a)+1$. There are now two cases: $S(x) = S(a)$ or $S(x) = S(a)+1$.

If $S(x) = S(a) = S(b) = S(y)+1$, then $\{a, b\}$ was a possible pair in P and is also a possible pair in P', but in that case we cannot have only disapproved winners before and only approved winners after (since there is a common winner in P and P'). Similarly, if $S(x) = S(a)+1 = S(y)+1 = S(b)+1$, the pair $\{x, y\}$ would have been selected in both cases. Since $x \succ_i y$ in P (as voter i approves x but not y), we cannot have y winning before and losing after, or x winning after but not before. Finally, if x is winning in P but not y, and y winning in P' but not x, this cannot fulfil the condition of a successful manipulation. AV^R is thus strategy-proof.

Let us now prove that CCAV^R is strategy-proof. We still have a pair $\{x, b\}$ selected in P and not in P' and a pair $\{a, y\}$ selected in P' and not in P. Thus, $s(x, b) = S(x)+S(b)-S(xb) > S(a)+S(y)-S(ay) = s(a, y)$ in P and $s'(x, b) = S'(x)+S'(b)-S'(xb) < S'(a)+S'(y)-S'(ay) = s'(a, y)$ in P'. Since a is already approved and b already disapproved in P, we have $S(a) = S'(a)$ and $S(b) = S'(b)$. For any x and y, the only possible change that decreases the score of $\{x, b\}$ and increases the score of $\{a, y\}$ is if x goes from approved to disapproved by the voter i. Since a is already approved by voter i, additionally approving y will not change the score of $\{a, y\}$. Thus, the score of $\{x, b\}$ would decrease by 1 and the score of $\{a, y\}$ not change, giving $S(x)+S(b)-S(xb)-1 < S(a)+S(y)-S(ay) < S(x)+S(b)-S(xb)$ which is not possible since all approval scores are integer. This proves that CCAV^R satisfies strategy-proofness.

TRIV^R is strategy-proof as a corollary of Proposition 3.6, that we prove later.

Now, let us show that the other rules are not strategy-proof. For S-CCAV^R, consider the following profile:

P :	$1\times a\vert yb$	$p+1\times a\vert by$	$p\times b\vert ay$	$p\times ay\vert b$	$p+1\times by\vert a$	$2\times\vert bay$
P' :	$1\times \underline{y\vert a}b$	$p+1\times a\vert by$	$p\times b\vert ay$	$p\times ay\vert b$	$p+1\times by\vert a$	$2\times\vert bay$

In P, the approval winner is a with $S(a) = 2p+2$ while $S(b) = S(y) = 2p+1$. Then, since $S(ay) = p$ and $S(ab) = 0$, the selected pair is $\{a, b\}$ and b is winning the runoff. In P', the approval winner is y with $S(y) = 2p+2$ while $S(b) = S(a) = 2p+1$. Then, since $S(ay) = p$ and $S(by) = p+1$, the selected pair is $\{a, y\}$ and a is winning the runoff. This is a successful manipulation from the deviating voter, so S-CCAV^R is not strategy-proof.

For α-AV^R with $0 < \alpha < 1$, consider the following profiles, with $p > 1/\alpha$:

P :	$1\times ax\vert yb$	$p\times xy\vert ab$	$p\times ax\vert by$	$p\times yb\vert xa$	$p\times ab\vert xy$	$1\times b\vert axy$	$1\times\vert bxay$
P' :	$1\times a\underline{y\vert x}b$	$p\times xy\vert ab$	$p\times ax\vert by$	$p\times yb\vert xa$	$p\times ab\vert xy$	$1\times b\vert axy$	$1\times\vert bxay$

In P, the score of the pairs are $s(x,b) = 4p+2$, $s(a,y) = 4p+1$ and $s(x_1,x_2) \leqslant 3p+(1-\alpha)p+2$ for all other pairs of candidates. Clearly $\{x,b\}$ is the only selected pair in P, and b wins the majority vote so $f^R(P) = \{b\}$. Now, consider the profile P' which is an deviation of P by the first voter who approved $A_i = \{a,x\}$ and now approves $A'_i = \{a,y\}$. In this profile, the scores are $s(x,b) = 4p+1$, $s(a,y) = 4p+1+(1-\alpha)$ and $s(x_1,x_2) \leqslant 3p+(1-\alpha)p+2$ for all other pairs of candidates. With $\alpha \notin \{0,1\}$ and $p > 1/\alpha$, $\{a,y\}$ is the only selected pair of candidates and a is winning the majority vote so $f^R(P') = \{a\}$. This is a successful manipulation and these rules are not strategy-proof, including PAV^R.

For α-seqAV rules, consider the following profiles:

$$\begin{array}{llllll} P: & 1\times ax|b & p_1\times x|ab & p_2\times ab|x & p_3\times b|ax & p_4\times |abx \\ P': & 1\times a|\underline{x}b & p_1\times x|ab & p_2\times ab|x & p_3\times b|ax & p_4\times |abx \end{array}$$

In these profiles, we will select the values of p_i such that b is the sole approval winner, and the second finalist is x in P but a in P'. In order for b to be the approval winner, we need that $p_2+p_3 > p_1+1$. In order for x to be selected in P and a in P', we need $p_1+1 > 1+(1-\alpha)p_2 > p_1$. This can be done if we choose p_1 and p_2 such that $p_1/(1-\alpha) > p_2 > p_1/(1-\alpha) - 1/(1-\alpha)$. Integer values always exist for p_1 and p_2 if $\alpha \notin \{0,1\}$ since $1/(1-\alpha) > 1$. We then choose p_3 such that $p_2+p_3 > p_1+1$, and finally we set $p_4 > p_1+p_2+p_3$ such that b is winning the runoff against x but not a. Thus, $f^R(P) = \{b\}$ and $f^R(P') = \{a\}$ and these rules are not strategy-proof. Note that by correctly setting p_4, and possibly changing the $|abx$ votes to $abx|$ votes, this result applies to EnePhr^R.

For S-Phr^R, we take the same profiles as above, with $p_1 = p_2 = p$, $p_3 = p^2$ and $p_4 = p^2+2p$, where $p \geqslant 2$. In these profiles, it is clear that b is the approval winner. Then the scores of the pairs in P are $s(b,x) = 1/(p+1)$ and $s(b,a) = (1+p/(p+p^2))/(p+1)$. In P', these are $s'(b,x) = 1/p$ and $s'(b,a) = s(b,a)$. If we want that $\{x,b\}$ is selected in P and $\{a,b\}$ in P', we need:

$$\frac{1}{p+1} < \frac{1+p/(p+p^2)}{p+1} < \frac{1}{p} \quad \Leftrightarrow \quad p+1 > \frac{p+1}{1+p/(p+p^2)} > p$$

The first inequality is obvious. Now, we prove that $(p+1)/(1+p/(p+p^2)) > p$:

$$\begin{aligned} \frac{p+1}{1+p/(p+p^2)} - p &= p\left(\frac{1}{1+p/(p+p^2)} - 1\right) + \frac{1}{1+p/(p+p^2)} \\ &= p\left(\frac{-p/(p+p^2)}{1+p/(p+p^2)}\right) + \frac{1}{1+p/(p+p^2)} \\ &= \frac{1-p^2/(p+p^2)}{1+p/(p+p^2)} > 0. \end{aligned}$$

This proves that we have S-Phr$^R(P) = \{b\}$ but S-Phr$^R(P) = \{a\}$ and thus S-Phr is not strategy-proof.

For SAV^R, take the following profiles:

$$\begin{array}{lll} P: & 1\times abcd|efg & 1\times efg|abc \\ P': & 1\times ab|\underline{cd}efg & 1\times efg|abc \end{array}$$

In P, the pairs of finalists are $\{e,f\}$, $\{e,g\}$ and $\{f,g\}$. In P', the only pair of finalists is $\{a,b\}$. This is a successful manipulation and SAV^R is not strategy-proof. □

We can also deduce the following corollary:

Corollary 3.4
AV^R and $CCAV^R$ are the only strategy-proof α-AV^R rules and AV^R is the only strategy-proof α-seqAV^R rule.

We can also slightly strengthen the strategy-proofness notion such that instead of requiring all winners in P' to be approved by the manipulating voter, we only ask that one of them is approved. Informally, this means that a voter cannot deviate from a profile in which they approve no winner to a profile in which they approve *at least* one winner. This stronger notion implies the weaker one, but it is still weaker than *Kelly strategy-proofness* for irresolute rules (Kelly, 1977) and weak orders, which states that a manipulation is successful if all winners in P' are weakly preferred by the manipulating voter to all winners in P. Kelly-strategyproofness has been used by Brandl and Peters (2022) to characterize approval voting among approval-based single-winner voting rules. Finally, the notion we define here is also equivalent to the *strategyproofness for unrepresented voters* defined by Delemazure et al. (2023a) in the approval-based multi-winner setting.

Strong Strategy-proofness
An AVR rule f^R is *strongly strategy-proof* if for every profile P, there is no voter $i \in V$ and i-deviation P' of P such that $|f^R(P) \cap A_i| = 0$ and $|f^R(P') \cap A_i| \geqslant 1$.

Unfortunately, with this slight change, AV^R and $CCAV^R$ are not strategy-proofness anymore. Actually, we can show that it is even incompatible with Pareto-efficiency.

Theorem 3.5
No AVR rule is strongly strategy-proof and Pareto-efficient.

Proof. Let f be an ABC rule and assume that the AVR rule f^R is Pareto-efficient and strongly strategy-proof. Consider the following profiles that all have $n = 13$ voters.

P^1 :		$2\times a\|bc$	$1\times c\|ab$	$10\times abc\|$
P^2 :		$2\times a\|cb$	$1\times c\|ab$	$10\times cba\|$
P^3 :		$2\times a\|bc$	$1\times c\|ab$	$10\times cab\|$
P^4 :	$1\times ab\|c$	$1\times a\|bc$	$1\times c\|ab$	$10\times abc\|$
P^5 :	$1\times a\underline{b}\|c$	$1\times a\|bc$	$1\times c\|ab$	$10\times cab\|$
P^6 :	$1\times ab\|c$	$1\times a\|bc$	$1\times c\|ba$	$10\times acb\|$
P^7 :	$1\times ab\|c$	$1\times a\|bc$	$1\times bc\|a$	$10\times bca\|$
P^8 :	$1\times ab\|c$	$1\times a\|bc$	$1\times c\underline{b}\|a$	$10\times acb\|$
P^9 :	$1\times ab\|c$	$1\times a\|bc$	$1\times cb\|a$	$10\times cab\|$
P^{10} :	$1\times ab\|c$	$1\times a\|bc$	$1\times c\|\underline{b}a$	$10\times cab\|$

Observe that $P^1_A = P^2_A = P^3_A$, $P^4_A = P^5_A = P^6_A = P^{10}_A$ and $P^7_A = P^8_A = P^9_A$.

In P^1, a dominates b, so by Pareto-efficiency, $b \notin f^R(P^1)$. Since we have $\textsc{maj}(P^1_\succ, \{b, c\}) = b$, this implies $\{b, c\} \notin f(P^1_A)$. Similarly in P^2, c dominates b and $\textsc{maj}(P^2_\succ, \{a, b\}) = b$ so $\{a, b\} \notin f(P^2_A) = f(P^1_A)$. Thus, we necessarily have $f(P^1_A) = \{\{a, c\}\}$. This gives $f^R(P^1) = \{a\}$ and $f^R(P^2) = f^R(P^3) = \{c\}$.

In P^4, a dominates b, therefore by Pareto-efficiency, and since we have $\text{MAJ}(P^4_\succ, \{b,c\}) = b$, we have $\{b,c\} \notin f(P^4_A)$. Now observe that P^5 is a deviation of P^3, and since $f^R(P^3) = \{c\}$, we have by strong strategy-proofness that $a \notin f^R(P^5)$. Since $\text{MAJ}(P^5_\succ, \{a,b\}) = a$, this implies that $\{a,b\} \notin f(P^5_A) = f(P^4_A)$. Therefore, $f(P^4_A) = \{\{a,c\}\}$. This gives $f^R(P^4) = f^R(P^6) = \{a\}$ and $f^R(P^5) = f^R(P^{10}) = \{c\}$.

In P^7, b dominates c and $\text{MAJ}(P^7_\succ, \{a,c\}) = c$, so $\{a,c\} \notin f(P^7_A)$. Now observe that P^8 is a deviation of P^6. We know that $f^R(P^6) = \{a\}$, thus by strategy-proofness, $c \notin f^R(P^8)$. Since $\text{MAJ}(P^8_\succ, \{c,b\}) = c$, this implies that $\{c,b\} \notin f(P^8_A) = f(P^7_A)$. Thus, $f(P^7_A) = \{\{a,b\}\}$, giving $f^R(P^7) = \{b\}$ and $f^R(P^8) = f^R(P^9) = \{a\}$.

Finally, P^{10} is a deviation of P^9, in which a is the winner. This implies that a should also be the sole winner in P^{10}, as the deviating voter was approving b and c in P^9. However, we know that $f^R(P^{10}) = \{c\}$. This contradicts strong strategy-proofness, and proves the theorem. □

Proposition 3.6
TRIV^R is strongly strategy-proof, but not AV^R, PAV^R, CCAV^R, S-PAV^R, S-CCAV^R, EnePhr^R, S-Phr^R, and SAV^R.

Proof. We know from Theorem 3.5 that no AVR rule is Pareto-efficient and strongly strategy-proof, giving us the result for AV^R,PAV^R, S-PAV^R, S-Phr^R and SAV^R. For CCAV^R and S-CCAV^R, consider the following profile P with $p \geqslant 4$, and a deviation of it P':

$$\begin{array}{lllll} P: & 1\times ca|b & 1\times c|ab & 1\times b|ca & p\times abc| \\ P': & 1\times a|\underline{c}b & 1\times c|ab & 1\times b|ca & p\times abc| \end{array}$$

Both rules select $\{c,b\}$ as finalists and b as a winner in P. In P', all pairs of finalists are possible, thus a is a possible winner since $\text{MAJ}(P'_\succ, \{a,c\}) = a$. This is a successful deviation as the deviating voter went from approving no winner to approving one. For EnePhr^R, the same profile works by taking p big enough so that it is equivalent to S-CCAV^R (see proof of Proposition 3.2)

Let us now show that TRIV^R is strategy-proof. Recall that it outputs all candidates except the Condorcet loser, call it l, if there is one. The only possible occurence of a manipulation is when $l \notin f^R(P)$ and a voter i with approval ballot $A_i = \{l\}$ can deviate to a profile P' such that $l \in f^R(P')$. Because this voter only approves l, it is necessarily ranked on top due to ballot consistency, and thus the Condorcet loser status of l will not change for any deviation of this voter, making it impossible to obtain $l \in f^R(P')$. □

Monotonicity

Another axiom that is closely related to strategy-proofness is monotonicity (see Section 2.4.5). Recall that we explained in the introduction of this chapter that plurality with runoff fails monotonicity, partly because of the runoff. It is then natural to ask if we can find AVR rules that satisfy monotonicity. In the approval-ordinal setting, we can adapt the notion of c-improvement as follows: $(A'_i, \succcurlyeq'_i)$ is a c-improvement of $(A_i, \succcurlyeq_i)$ if $A'_i = A_i \cup \{c\}$ or $A'_i = A_i$, and for all $x, y \in C$ with $y \neq c$, if $x \succcurlyeq_i y$, then $x \succcurlyeq'_i y$. In other words, a c-improvement is a deviation in which the only possible changes are that the position of c in i's ranking improved and it can become approved.

Then, monotonicity states that if c is a winning candidate in some profile and the support for c increases (by doing c-improvements) then c should remain a winning candidate.

Monotonicity
An AVR rule f^R is *monotonic* if for every profile P, candidate $c \in f^R(P)$ and every c-improvement P' of P, we have $c \in f^R(P')$.

We use the same definition as Sanver (2010). Note that this definition also makes sense if we do not require ballot consistency. Unfortunately, most of the rules defined in Section 3.2 fail monotonicity.

Proposition 3.7
AV^R and TRIV^R are monotonic but not PAV^R, CCAV^R, S-PAV^R, S-CCAV^R, EnePhr^R, S-Phr^R, and SAV^R.

Proof. Let us first show that AV^R is monotonic. Le $P = (P_A, P_\succ)$ be a profile and $a \in \text{AV}^R(P)$ a winning candidate. Thus, there exists $x \in C$ such that $\{a, x\} \in \text{AV}(P_A)$ and $\textsc{maj}(P_\succ, \{a, x\}) = a$. Consider $P' = (P'_A, P'_\succ)$ an a-improvement of P, and denote by S' the approval score in P'. We know that $S(a) \leqslant S'(a)$ and for all candidates $y \neq a, S'(y) = S(y)$. Therefore, $\{a, x\} \in \text{AV}(P_A)$ directly implies that $\{a, x\} \in \text{AV}(P'_A)$. Moreover, since the only change in P' rankings is that the position of a improved, it is clear that $\textsc{maj}(P'_\succ, \{a, x\}) = a$. Thus, $a \in \text{AV}^R(P')$ and AV^R satisfies monotonicity.

To show that TRIV^R satisfies the property, recall that it returns all Condorcet non-losers. If a is not a Condorcet loser in a profile P, then it is clearly still not one in any a-improvement of P. Thus, TRIV^R satisfies monotonicity.

For all the other rules, consider the following profiles:

P :	$2\times a\|bc$	$1\times b\|ca$	$1\times c\|ba$	$10\times cab\|$
P' :	$2\times a\|bc$	$1\times b\underline{a}\|c$	$1\times c\|ba$	$10\times cab\|$

One can easily check that for all rules $f \in$ {PAV, CCAV, S-PAV, S-CCAV, EnePhr, S-Phr, SAV}, we have $f(P_A) = \{\{a, b\}, \{a, c\}\}$. Additionally, $\textsc{maj}(P_\succ, \{a, b\}) = a$ so for all the rules considered, $a \in f^R(P)$. It is clear that P' is an a-improvement of P. However, for all the rules considered, this change disadvantages the pair of finalists $\{a, b\}$ as b's supporter is now already satisfied by a. With all the rules considered, we have $f(P'_A) = \{\{a, c\}\}$ and since $\textsc{maj}(P'_\succ, \{a, c\}) = c$, this means that $a \notin f^R(P')$. Thus, none of these rules satisfies monotonicity. ☐

The proof for (S-)PAV^R and (S-)CCAV^R actually works for all α-AV^R and α-seqAV^R rules, giving us the following corollary.

Corollary 3.8
AV^R is the only α-AV^R rule and the only α-seqAV^R rule that is monotonic.

The fact that approval with runoff satisfies monotonicity was already mentioned by Sanver (2010) (Theorem 20.4.1). However, the author did not provide a proof for this result. It is also, among the rules defined in Section 3.2, the only rule that satisfies both monotonicity and Pareto-efficiency. However, there exist other AVR rules satisfying the two properties. Consider for instance the rule f^R where f returns all pairs of candidates $\{a, b\}$ such that one of the following two conditions is fulfilled: (i) neither a nor b is Pareto-dominated in P_A, or (ii) a is the only candidate dominating b in P_A.

Let us show that this rule satisfies Pareto-efficiency and monotonicity. We start with Pareto-efficiency. Let b be a candidate dominated by a in some profile P. We know that if there is some

$x \in C$ such that $\{x, b\} \in f(P_A)$, then $x = a$ by the definition of the rule. Since a dominates b, it is clear that $\text{MAJ}(P_\succ, \{a, b\}) = a$ and $b \notin f^R(P)$. To show that it satisfies monotonicity, consider a profile P and a winning candidate $a \in f^R(P)$. Let P' be an a-improvement of P. Since a is winning, there is a pair $\{x, a\} \in f(P_A)$ such that $\text{MAJ}(P_\succ, \{x, a\}) = a$. This means that x is either not dominated or dominated only by a. If one new voter approves a in P', then x is still either not dominated or dominated only by a, therefore we still have $\{x, a\} \in f(P'_A)$. If no new voter approves a in P', we clearly have $\{x, a\} \in f(P'_A)$. Since P' is an a-improvement of P, it is clear that $\text{MAJ}(P'_\succ, \{x, a\}) = a$ and thus $a \in f^R(P')$. Therefore, this rule is Pareto-efficient and monotonic, as well as neutral and anonymous, so we cannot characterize AV^R with these four properties only.

Independence of Clones

Another property failed by plurality with runoff that we discussed in the introduction is independence of clones (see Section 2.4.4). Informally, it states that adding a *clone* of a candidate in an election should not alter significantly the outcome of this election (Tideman, 1987). As a reminder, we say that two candidates a and a' are clones if for every $i \in V$, (i) $a \in A_i$ if and only if $a' \in A_i$ and (ii) for all $x \in C$ such that $x \notin \{a, a'\}$, $a \succ_i x$ if and only if $a' \succ_i x$. In the following, we denote by P_{-x} the profile that is equivalent to P with the candidate x removed from the preferences. In other words, the set of candidates of P_{-x} is $C \setminus \{x\}$. Then, we have:

Independence of clones
An AVR rule f^R is *independent of clones* if for any profile P in which two candidates $a, a' \in C$ are clones, the following two conditions hold:

1. for every candidate $x \neq a$, $x \in f^R(P)$ if and only if $x \in f^R(P_{-a'})$.
2. $a \in f^R(P_{-a'})$ if and only if $f^R(P) \cap \{a, a'\} \neq \emptyset$.

Unfortunately, we can very easily show that no AVR rule satisfies this axiom. Consider the simple profile on three candidates $\{a, b, c\}$ with one ballot $P = \{abc|\}$, meaning that all candidates are approved. Without loss of generality, let us assume that for an AVR rule f^R, $\{b, c\}$ is a selected pair in P_A (if it is not, we can swap the names of the candidates). Thus, $b \in f(P)$. Since b and c are clones in P, then b should also win in P_{-c}. However, the only pair in P_{-c} is $\{a, b\}$ and a is preferred over b by the sole voter. Thus, $b \notin P_{-c}$. This contradicts independence of clones.

However, the profiles in which all or almost all rules fail independence of clones are profiles in which one candidate is approved in every non-empty approval ballot. If we restrict the domain by eliminating these pathological profiles, we can define a weaker version of independence of clones that might be satisfied by some of our rules.

Weak independence of clones
An AVR rule f^R is *weakly independent of clones* if it is independent of clones on every profile P such that no candidate $x \in C$ is approved by every non-empty ballot, i.e. for all $x \in C$ there exists $i \in V$ such that $A_i \neq \emptyset$ and $x \notin A_i$.

We can show that CCAV^R and S-CCAV^R satisfy this property, but not the other rules defined in Section 3.2. In particular, the rules satisfying monotonicity AV^R and TRIV^R fail this property. Unfortunately, this can be generalized to every neutral monotonic AVR rule:

Theorem 3.9
No AVR rule is monotonic and weakly independent of clones.

Proof. Let f^R be an AVR rule that is monotonic. Consider the following profiles:

P^1 :	$1\times a\|bc$	$1\times b\|ca$	$1\times c\|ab$	$10\times cab\|$
P^2 :	$1\times a\|bc$	$1\times b\underline{a}\|c$	$1\times c\|ab$	$10\times cab\|$
P^3 :	$1\times a\|bc$	$1\times ba\|c$	$1\times c\|ab$	$10\times cba\|$
P^4 :	$1\times a\underline{b}\|c$	$1\times ba\|c$	$1\times c\|ab$	$10\times cba\|$

We assume without loss of generality that $\{a,b\} \in f(P^1_A)$, otherwise we change the names of the candidates. Since $\textsc{maj}(P^1_\succ, \{a,b\}) = a$, we have $a \in f^R(P^1)$. Observe that P^2 is an a-improvement of P^1, so $a \in f^R(P^2)$ by monotonicity. Since $\textsc{maj}(P^2_\succ, \{a,c\}) = c$, this implies that $\{a,b\} \in f(P^2_A)$.

Observe that $P^2_A = P^3_A$ so $\{a,b\} \in f(P^3_A)$ and thus $b \in f^R(P^3)$. Moreover, P^4 is a b-improvement of P^3 so by monotonicity $b \in f^R(P^4)$ and since $\textsc{maj}(P^4_\succ, \{b,c\}) = c$, we have $\{a,b\} \in f(P^4_A)$. Observe that a and b are clones in P^4, thus a should be a winning candidate in $(P^4)_{-b}$. However, it is clear that in this profile the only possible pair of candidate is $\{a,c\}$ and thus $f^R((P^4)_{-b}) = \textsc{maj}(P^4_\succ, \{a,c\}) = \{c\}$, breaking weak independence of clones. □

Observe that the proof of this theorem also works to show that no AVR rule is strongly strategy-proof and weakly independent of clones. Indeed, P^1 is a deviation of P^2 in which the second voter is manipulating. Thus, we cannot go from neither a nor b winning in P^2 to one of them winning in P^1 by strong strategy-proofness. Since b is a Condorcet loser in both profile, this means that if a is winning in P^1, it should also win in P^2, as in the original proof. Similarly, we can show that if b is winning in P^3 it should also be a winner in P^4, and we obtain the exact same proof.

Corollary 3.10
No AVR rule is strongly strategy-proof and weakly independent of clones.

We already have a rule that satisfies monotonicity, we will now show that CCAV^R and S-CCAV^R both satisfy weak independence of clones. All the other rules we study fail it. Note that this implies that CCAV^R satisfies both weak independence of clones and strategy-proofness.

Proposition 3.11
CCAV^R and S-CCAV^R are weakly independent of clones but not AV^R, PAV^R, S-PAV^R, EnePhr^R, S-Phr^R, SAV^R, and TRIV^R.

Proof. We first show that CCAV^R and S-CCAV^R satisfy weak independence of clones. Let P be a profile in which there is no candidate approved in every non-empty ballot, and let a and a' be clones in P. Let us denote $P' = P_{-a'}$. In this proof, we will write $s(x,y) = S(x) + S(y) - S(xy)$ and $s'(x,y) = S'(x) + S'(y) - S'(xy)$ the CCAV score of a pair $\{x,y\}$ in respectively P_A and P'_A.

Since a and a' are clones, for all $x \neq a, a'$ we have $S(xa) = S(xa') = S'(xa)$. Therefore, for all $x, y \notin \{a,a'\}$, $s(x,y) = s'(x,y)$ and $s(x,a) = s(x,a') = s'(x,a)$. Finally, $s(a,a') = S(a) + S(a') - S(aa') = S(a)$ since $S(a) = S(a') = S(aa')$.

We now prove that $\{a,a'\} \notin f(P_A)$ for $f \in \{$CCAV, S-CCAV$\}$. If a is not an approval winner in P_A, let w be one approval winner. We have $s(w,a) \geqslant S(w) > S(a) = s(a,a')$, thus $\{a,a'\} \notin f(P_A)$. Let us now assume that a is an approval winner in P_A. Since no candidate is approved in every non-empty ballot, there exist $x \in C$ and $i \in V$ such that $x \in A_i$ and $a \notin A_i$. Thus $s(a,x) = S(a) + S(x) - S(xa) \geqslant S(a) + 1 > S(a) = s(a,a')$, thus $\{a,a'\} \notin f(P_A)$.

In both cases, $\{a,a'\} \notin f(P_A)$ and all the other pairs of candidates have the same CCAV score. Thus, for $x, y \neq a'$, $\{x,y\} \in f(P_A)$ if and only if $\{x,y\} \in f(P'_A)$ and for all x, $\{x,a'\} \in f(P_A)$ if and only if $\{x,a\} \in f(P'_A)$. Since the majority relations are unchanged between $P_\succ$ and $P'_\succ$, then we have that for all $x \neq a'$, $x \in f^R(P)$ if and only if $x \in f^R(P')$ and $a' \in f^R(P)$ if and only if $a \in f^R(P')$. This proves that CCAV^R and S-CCAV^R are weakly independent of clones.

We now give a counterexample for all the other rules. Consider the following profile P for $p \geqslant 2$:

$$1{\times}b|aa' \qquad p{\times}aa'|b \qquad p \times baa'|$$

One can easily check that for $f \in \{$AV, PAV, S-PAV, EnePhr, S-Phr, SAV, TRIV$\}$, we have that $\{a,a'\} \in f(P_A)$ and thus $a \in f^R(P)$. However, in $P_{-a'}$ the only possible pair of finalists is $\{a,b\}$ and $\textsc{maj}((P_\succ)_{-a'}, \{a,b\}) = b$ so $f^R(P_{-a'}) = \{b\}$. This proves that all these rules fail weak independence of clones. □

Note that the counterexample used in the second part of the proof can be used to show that α-AV^R rules and α-seqAV^R rules also fail independence of clones for $\alpha < 1$, using a big enough value of p in the profile P. This gives the following corollary:

Corollary 3.12
CCAV^R and S-CCAV^R are respectively the only α-AV^R and α-seqAV^R rules that satisfy weak independence of clones.

Favorite consistency

Finally, we want to study one last property of the rules. Unlike the other properties discussed in this section, this one is about the first round of the election, i.e. it is a property of the ABC rule used to select the finalists, and not the AVR rule.

To motivate this property, consider the following example profile:

$$1{\times}\{b\} \qquad 49{\times}\{a,b\} \qquad 49{\times}\{a,c\} \qquad 1{\times}\{c\}$$

In this profile, 98% of voters approve a, and the other candidates are approved each by 50% of the voters. However, if we use CCAV, the two finalists would be b and c, as they form the only pair of candidates that covers all voters (every voter approves either b or c). In a real-world scenario, it seems undesirable that an approval winner such as a, with an approval score of 98%, is not selected for the second round of the election. This motivates the following axiom:

Favorite-consistency
An ABC rule f is *favorite-consistent* if every winning committee contains an approval winner, i.e. for all $W \in f(P_A)$, we have $|W \cap \operatorname{argmax}_{x \in C} S(x)| \geqslant 1$.

Note that this property can also be defined for committees of size $k > 2$. By their definition, all sequential rules satisfy this property. However, any α-AV rule (and thus Thiele rules in the general case) breaks favorite-consistency for $\alpha > 0$. We can prove it by adapting our motivating example. Set $p > 1/\alpha$ and consider the following profile:

$$p-1{\times}\{b\} \qquad p{\times}\{a,b\} \qquad p{\times}\{a,c\} \qquad p-1{\times}\{c\}$$

	AV	PAV	CCAV	S-PAV	S-CCAV	EnePhr	S-Phr	SAV	TRIV
Pareto-efficiency	✓	✓	✗	✓	✗	✗	✓	✓	✗
strategy-proofness	✓	✗	✓	✗	✗	✗	✗	✗	✓
strong strategy-proofness	✗	✗	✗	✗	✗	✗	✗	✗	✓
monotonicity	✓	✗	✗	✗	✗	✗	✗	✗	✓
weak ind. of clones	✗	✗	✓	✗	✓	✗	✗	✗	✗
approval-efficiency	✓	✗	✗	✓	✓	✓	✓	✗	✗

Table 3.1: Summary of the axiomatic properties of the AVR rules.

In this profile, the approval winner is a but the score of the pairs $\{a, b\}$ and $\{a, c\}$ is $4p - 1 - \alpha p$, while the score of $\{b, c\}$ is $4p - 2$, which is strictly greater by our hypothesis on p. Thus, the selected committee is $\{b, c\}$, contradicting favorite-consistency. This implies that PAV and CCAV fail favorite-consistency. Using the same profile and $p \geqslant 3$, we can show that SAV also fails it.

Proposition 3.13
AV, S-PAV, S-CCAV, EnePhr, S-Phr satisfy favorite consistency while PAV, CCAV, SAV, and TRIV fail it.

We can also turn this property of ABC rules into a property of AVR rules using an efficiency axiom. Informally, this property states that a candidate cannot be a winner if it is not an approval winner, or winning the majority vote against any approval winner.

Approval-efficiency
An AVR rule f^R is *approval-efficient* if for all profiles P and all candidate $a \in f^R(P)$, either a is an approval winner $a \in \operatorname{argmax} S$ or there exists an approval winner $b \in \operatorname{argmax} S$ such that $\textsc{maj}(P_\succ, \{a, b\}) = a$.

It is clear that if an ABC rule f satisfies favorite-consistency then the AVR rule f^R satisfies approval-efficiency. For α-AV^R rules with $\alpha > 0$ such as PAV^R and CCAV^R, as well as for SAV^R and TRIV^R, the approval profile P_A used as a counter example for favorite-consistency can be used to construct an approval-ordinal profile P in which the approval winners a and b are winning the majority vote against both c and d. Because $\{c, d\}$ is a possible pair of finalists for all these rules, they fail approval-efficiency.

Corollary 3.14
AV^R, S-PAV^R, S-CCAV^R, EnePhr^R, S-Phr^R satisfy approval-efficiency while PAV^R, CCAV^R, SAV^R, and TRIV^R fail it.

Corollary 3.15
AV^R is the only α-AV^R rule that satisfies approval-efficiency.

An open problem is whether AV^R is the only neutral and anonymous AVR rule that satisfies approval-efficiency and either strategy-proofness or monotonicity.

Summary of the Results

We conclude this section with a summary of the axiomatic results. Table 3.1 summarizes the properties of the AVR rules, depending on the ABC rule they are based on. By additionally taking into account the fact that some rules are easier to understand and explain than others, the rules

	strategy-proof	monotonic	weak independence of clones
Pareto-efficient	✗(Theorem 3.5)	✓(AV^R)	✓(CCAV_+^R)
strongly strategy-proof		✓(TRIV^R)	✗(Corollary 3.10)
monotonic			✗(Theorem 3.9)

Table 3.2: Summary of the incompatibilities between pairs of axioms.

that have the most advantages are probably AV^R and S-CCAV^R (or S-CCAV_+^R, that we recall below), though they should be used in different contexts. With AV^R, the two finalists will be the most liked ones, but they might be very similar to each other. Thus, the choice in the second round might not be interesting for a large part of the voters, that may not feel represented by any of the finalists. On the contrary, S-CCAV^R will select a diverse set of finalists. However, the second finalist might not be as liked as the first one, making the victory too easy for the first candidate.

Table 3.2 summarizes the incompatibilities between pairs of axioms, and if they are compatible, we give a rule satisfying both properties. Note that this is not a characterization. In this table, CCAV_+^R corresponds to the rule in which the pairs of finalists are the one maximizing the CCAV scores, and ties between pairs with the same CCAV score are broken according to the AV score. Note that the sequential version S-CCAV_+^R also satisfies both axioms.

3.4 Statistical Analysis

We now want to explore the spectrum between ABC rules that select the most popular candidates, typically AV, and rules that favour diversity in the set of finalists, typically CCAV and S-CCAV or, to a lesser extent, PAV and S-PAV. We also want to study the difference between sequential and non-sequential rules. We will proceed to two different analyses with these goals. The first one, in this section, is based on statistical distributions over 1-dimensional Euclidean preferences and provides theoretical insights, while the second one, in the next section, is based on experiments and simulations using real data.

In this section, we study the behavior of α-AV and α-seqAV rules in the 1-dimensional Euclidean model, described in Section 2.5.1. As a reminder, in this model there is a function pos that maps voters and candidates to positions in $\mathbb{R}$, and preferences of voters depend on their distances to the candidates. For the approval votes, we use a threshold d such that every voter i approves all candidates $c \in C$ with $|\mathsf{pos}(c) - \mathsf{pos}(i)| \leqslant d$.

Note that this model ignores that voters might vote strategically. However, classical political science Chapman (1955); Cox (1984, 1997) argues that in the first round of a runoff, voters concentrate on which candidate is going to arrive second and third (this is in practice the more pressing question) and vote on this basis. For instance, voters might individually adapt their approval thresholds such that they do not approve all three candidates that are expected to arrive first. More precisely, Laslier and Van der Straeten (2016) put forward theoretical arguments that imply putting the threshold at the utility level of the candidate supposed to arrive second.

We will now study how the choices of the threshold d and of the parameter α of α-AV and α-seqAV rules have an impact on the positions of the candidates selected for the runoff, depending on the density of voters at each point in the space. For simplicity, we assume that the positions of the voters follow a continuous density function ϕ such that a proportion $\int_p^{p'} \phi(x)dx$ of voters are between positions p and p', and $\int_{-\infty}^{+\infty} \phi(x)dx = 1$. In a sense, this can be seen as an Euclidean preference profile with an infinite number of voters, drawn at random according to the density

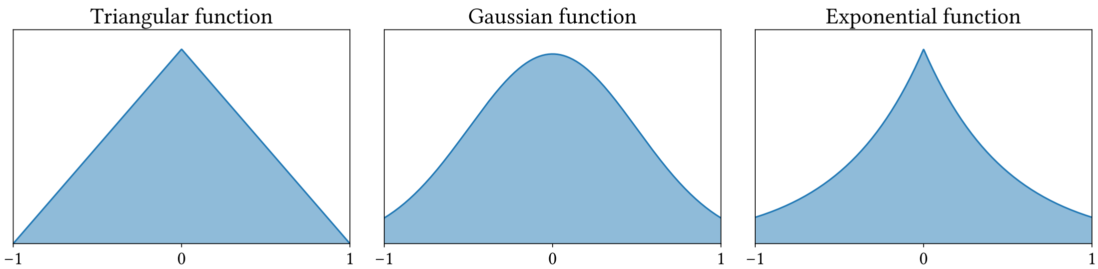


Figure 3.2: Density functions considered in this section

function ϕ. This approach is very similar to the analysis conducted by Tomlinson et al. (2024), in which the authors use this 1-dimensional Euclidean model with density functions to highlight the moderating effect of Instant Runoff Voting (but using ordinal preferences).

In this section, we assume for simplicity that the density function is symmetric and single-peaked, i.e. for all $p \in \mathbb{R}$, $\phi(p) = \phi(-p)$ and for all $p, p' \in \mathbb{R}$ such that $|p| < |p'|$, $\phi(p) \geqslant \phi(p')$. Figure 3.2 shows examples of such density functions, that will be analyzed later in this section.

α-AV rules

We start by computing the positions of the two finalists for α-AV rules. We can first show that the positions of the finalists are symmetric. This makes sense since the distribution is symmetric and the role of the two candidates are symmetric in the computation of the α-AV score.

Then, there are two cases: either the finalists have common supporters, which means that the distance between them is less than $2d$, or they have no common supporters. In that case, if the density function is strictly decreasing on $\mathbb{R}_+$, the positions of the finalists are such that $|\operatorname{pos}(x)| = |\operatorname{pos}(y)| = d$. In the former case, the positions that maximize the α-AV score are given by a formula depending on the density function ϕ, the approval radius d, and the parameter α. We will see later how we can reduce this formula for some specific density functions.

Proposition 3.16
For α-AV rules and a symmetric single-peaked density function ϕ, there are positions of the finalists $\{x, y\}$ such that $p = \operatorname{pos}(x) = -\operatorname{pos}(y) > 0$ and:

- If $\phi(2d)/\phi(0) \leqslant 1 - \alpha$, then p is such that $\phi(d + p)/\phi(d - p) = 1 - \alpha$,
- Otherwise, $p = d$.

Proof. We first show that there exists a pair of finalists $\{x, y\}$ such that $\operatorname{pos}(x) = -\operatorname{pos}(y)$. In what follows, we will write $p_x = \operatorname{pos}(x)$ (similarly for all other candidates). Assume by contradiction that all pairs of finalists are such that $|p_x| \neq |p_y|$. It is clear that there is a pair of finalists such that the sign of p_x and the one of p_y are opposite, otherwise we reverse the position of one of them and it will reduce the value of $S(xy)$ without changing the value of $S(x)$ and $S(y)$.

Without loss of generality let us assume that $p_x \leqslant 0 \leqslant p_y$ and $|p_x| < |p_y|$. We denote the function $h(t_1, t_2)$ that represent the density of voters between t_1 and t_2 as:

$$h(t_1, t_2) = \int_{t_1}^{t_2} \phi(t)dt$$

Observe that for all $c \in C$, we have $S(c) = h(p_c - d, p_c + d)$, and we have $S(xy) = h(p_y - d, p_x + d)$. The α-AV score of the pair (x, y) is:

$$s(x, y) = S(x) + S(y) - \alpha S(xy) = h(p_x - d, p_x + d) + h(p_y - d, p_y + d) - \alpha h(p_y - d, p_x + d)$$

We know that (x, x') with $p_x = -p_{x'}$ is not a possible pair of finalists so $s(x, y) > s(x, x')$ and:

$$\begin{aligned} &h(p_x - d, p_x + d) + h(p_y - d, p_y + d) - \alpha h(p_y - d, p_x + d) \\ &> h(p_x - d, p_x + d) + h(-p_x - d, -p_x + d) - \alpha h(-p_x - d, p_x + d) \end{aligned}$$

which can be reduced to:

$$\begin{aligned} h(p_y - d, p_y + d) - \alpha h(p_y - d, p_x + d) &> h(-p_x - d, -p_x + d) - \alpha h(-p_x - d, p_x + d) \\ (1 - \alpha)h(p_y - d, p_x + d) + h(p_x + d, p_y + d) &> (1 - \alpha)h(-p_x - d, p_x + d) + h(p_x + d, -p_x + d) \end{aligned}$$

and we obtain $h(-p_x + d, p_y + d) > (1 - \alpha)h(-p_x - d, p_y - d)$.

Similarly, $\{y, y'\}$ such that $p_{y'} = -p_y$ is not a possible pair of finalists, so $s(x, y) > s(y, y')$:

$$\begin{aligned} &h(p_x - d, p_x + d) + h(p_y - d, p_y + d) - \alpha h(p_y - d, p_x + d) \\ &> h(p_y - d, p_y + d) + h(-p_y - d, -p_y + d) - \alpha h(-p_y - d, p_y + d) \end{aligned}$$

which can be reduced to:

$$\begin{aligned} h(p_x - d, p_x + d) - \alpha h(p_y - d, p_y + d) &> h(-p_y - d, -p_y + d) - \alpha h(p_y - d, -p_y + d) \\ (1 - \alpha)h(p_y - d, p_x + d) + h(p_x - d, p_y - d) &> (1 - \alpha)h(p_y - d, -p_y + d) + h(-p_y - d, p_y + d) \end{aligned}$$

which gives $(1 - \alpha)h(-p_y + d, p_x + d) > h(-p_y - d, p_x - d)$. Because ϕ is symmetrical, we have $\phi(-t) = \phi(t)$ and for all t_1, t_2, $h(t_1, t_2) = h(-t_2, -t_1)$, this gives:

$$(1 - \alpha)h(-p_x - d, p_y - d) > h(-p_x + d, p_y + d)$$

Which contradict what we previously obtained. This proves that there are no x, y with positions $|p_x| \neq |p_y|$ and $s(x, y) > \max(s(x, x), s(y, y))$ and thus there exists a position p_x such that the maximum of s is obtained for x, x' with $p_{x'} = -p_x$.

Now, observe that the score of a pair x, x' such that $p_{x'} = -p_x$ (assuming without loss of generality that $p_x > 0$) and $p_x \leqslant d$ is equal to:

$$\begin{aligned} s(x, x') &= 2h(p_x - d, p_x + d) - \alpha h(p_x + d, -p_x + d) \\ &= (2 - \alpha)h(p_x - d, -p_x + d) + 2h(-p_x + d, p_x + d) \\ &= 2(2 - \alpha)h(0, d - p_x) + 2h(d - p_x, d + p_x) \\ &= (4 - 2\alpha)(\Phi(d - p_x) - \Phi(0)) + 2(\Phi(d + p_x) - \Phi(d - p_x)) \end{aligned}$$

with Φ the integral of ϕ. Note that we used symmetry in the first line. The function reaches its maximum when $\frac{\partial s(x,x')}{\partial p_x} = 0$, which happens when

$$(4-2\alpha)(-\phi(d-p_x)) + 2\phi(d+p_x) + 2\phi(d-p_x) = 0$$
$$(2\alpha-2)\phi(d-p_x) + 2\phi(d+p_x) = 0$$
$$\phi(d+p_x)/\phi(d-p_x) = 1-\alpha$$

Note that $\phi(d+p_x)$ is decreasing with p_x and $\phi(d-p_x)$ is increasing with p_x. Thus, $\phi(d+p_x)/\phi(d-p_x)$ is decreasing with p_x. If $\phi(2d)/\phi(0) \leqslant 1-\alpha$, the equation has at least one solution p_x such that $|p_x| \leqslant d$. If no position satisfies this equation, this means that the score is increasing until $p_x = d$ is reached, and the positions maximizing the score are $p_x = d$ and $p_{x'} = -d$. After these positions, $s(x,x') = 2\int_{p_x-d}^{p_x+d}\phi(t)dt$ and $\phi(t)$ is decreasing when t increases, so $s(x,x')$ is necessarily decreasing. This concludes the proof. □

Based on the formula obtained in Proposition 3.16, it is clear that for any symmetric single-peaked density function, the positions of the finalists move further from the center as α increases. In particular, if $\alpha = 0$, the position $p_x = 0$ satisfies the condition, since $\phi(d+p_x)/\phi(d-p_x) = \phi(d)/\phi(d) = 1$. Note that in this proposition and in the next one, we give positions of *some* pairs of finalists and not *all* pairs. Indeed, for some density functions, there can be much more than one possible pair of finalists. Consider for instance the case of the uniform function in $[-1,1]$. With an approval radius $d = 0.2$, all pairs of candidates between -0.8 and 0.8 such that their positions are at distance at least $|\operatorname{pos}(x) - \operatorname{pos}(y)| \geqslant 0.4$ are possible finalists. However, for the three functions shown in Figure 3.2, there is always a unique pair of finalists.

α-seqAV rules

We now study the positions of the finalists for α-seqAV rules. Clearly, the first candidate is at the position maximizing the approval score, and for any symmetric single-peaked density functions, the position $\operatorname{pos}(x) = 0$ satisfies this condition. Moreover, if the function is strictly monotonic on $\mathbb{R}_+$, this is the only possible position for the first finalist.

Again, there are two main cases for the second finalist: either it shares common supporters with the first one, and thus has position $|\operatorname{pos}(y)| \leqslant 2d$, or it shares no common supporters, and in that case a possible position maximizing the α-seqAV score for the second finalist is such that $|\operatorname{pos}(y)| = 2d$. Moreover, if the function is strictly monotonic on $\mathbb{R}_+$, this is the only possible position. In the former case, we can again distinguish two cases: either the distance between the two finalists is lower than one approval radius d, or it is between d and $2d$. In both cases, the positions need to satisfy a formula depending on the density function ϕ, the approval radius d, and the parameter α. We will later see how to reduce these formulas for some specific density functions.

Proposition 3.17

For α-seqAV rules and symmetric single-peaked density function ϕ, there is a pair of finalists $\{x,y\}$ such that the position of the first finalist x is $\operatorname{pos}(x) = 0$ and the one of the second finalist y given α and d is $p = |\operatorname{pos}(y)|$ given by:

- If $\phi(2d)/\phi(0) \leqslant 1-\alpha$, then p is such that $\phi(d+p)/\phi(d-p) = 1-\alpha$.
- Otherwise, if $\min_{d\leqslant p\leqslant 2d}\phi(p+d)/\phi(p-d) \leqslant 1-\alpha$, then p is such that $\phi(d+p)/\phi(d-p) = 1-\alpha$.

- Otherwise, $p = 2d$.

Proof. We know that the distribution of voters follows the density function $\phi(t)$. The position of the first finalist x is clearly $\mathsf{pos}(x) = 0$ as the density function is single-peaked and symmetric (so it reaches its peak at 0). Let y be a candidate at position $\mathsf{pos}(y) = p$. Without loss of generality we assume $p \geqslant 0$ (as the density function is symmetric).

First, consider positions $p \leqslant d$. Note that $-d \leqslant p - d \leqslant 0$. For a voter $i \in V$, the value of $A_i \cap \{x, y\}$ where A_i is their approval ballot is given by:

$$\begin{aligned}
\mathsf{pos}(i) &< -d && \{\} \\
-d \leqslant \mathsf{pos}(i) &< p - d && \{x\} \\
p - d \leqslant \mathsf{pos}(i) &\leqslant d && \{x, y\} \\
d < \mathsf{pos}(i) &\leqslant p + d && \{y\} \\
p + d < \mathsf{pos}(i) & && \{\}
\end{aligned}$$

As in the proof of Proposition 3.16, we will use the function $h(t_1, t_2) = \int_{t_1}^{t_2} \phi(t)dt$. The α-seqAV score of y is given by:

$$\begin{aligned}
s(y) =& S(y) - \alpha S(xy) = h(p - d, p + d) - \alpha h(p - d, d + p) \\
=& (1 - \alpha)h(p - d, 0) - \alpha h(0, d) + h(0, d - p) + h(d - p, d + p) \\
=& (2 - \alpha)h(0, d - p) + h(d - p, d + p) + K
\end{aligned}$$

by symmetry, with K a constant independent of p. Let Φ be the integral of ϕ. We have:

$$\begin{aligned}
s(y) =& (2 - \alpha)(\Phi(d - p) - \Phi(0)) + (\Phi(d + p) - \Phi(d - p)) + K \\
=& (1 - \alpha)\Phi(d - p) + \Phi(d + p) + K'
\end{aligned}$$

with K' a constant independent of p. This reaches its maximum when $\frac{\partial s(y)}{\partial p} = 0$. This gives:

$$\begin{aligned}
(\alpha - 1)\phi(d - p) + \phi(d + p) = 0 \\
\phi(d + p)/\phi(d - p) = 1 - \alpha
\end{aligned}$$

Now, observe that since $p \leqslant d$ and ϕ is increasing on $\mathbb{R}_{\geqslant 0}$ (by single-peakedness). Thus, $\phi(d+p)$ is decreasing with p, $\phi(d-p)$ is increasing with p and $\phi(d+p)/\phi(d-p)$ is decreasing. For $p = 0$, we have $\phi(d+p)/\phi(d-p) = 1$, so there exists a maximum with $p \leqslant d$ if and only if $\phi(2d)/\phi(0) \leqslant 1-\alpha$. This gives the first part of the formula.

Let us now assume that $d \leqslant p \leqslant 2d$. The intervals corresponding to each approval set are the same as in the previous case.

The score of y is now given by

$$\begin{aligned}
s(y) =& S(y) - \alpha S(xy) = h(p - d, p + d) - \alpha h(d + p, p - d) \\
=& (1 - \alpha)h(p - d) + h(d, p + d) \\
=& (1 - \alpha)(\Phi(d) - \Phi(p - d)) + [\Phi(p + d) - \Phi(d)]
\end{aligned}$$

This reaches a maximum when $(\alpha - 1)\phi(p - d) + \phi(p + d) = 0$, i.e. $\phi(p + d)/\phi(p - d) = 1 - \alpha$. Note that this happens only if the minimum of $\phi(p + d)/\phi(p - d)$ between d and $2d$ is lower than

$1-\alpha$. Moreover, this can happen for several values of p, but this at least restrict the set of possible positions.

If the score does not reach a maximum in the first two cases, then $p \geqslant 2d$. We now have $s(y) = \int_{p-d}^{p+d} \phi(t)dt$ which can only decrease when p increases since $\phi(t)$ decreases with t. Thus, if no maximum was found before, then the position with maximal score is $p = 2d$. This concludes the proof. □

Note that the first case of the proposition gives $p \leqslant d$, and the second case $p \leqslant 2d$.

Examples of functions

To get a better understanding of what these results mean, we will consider three different density functions that satisfy the required conditions: the triangular function, the Gaussian function, and the exponential function. They are displayed in Figure 3.2.

Triangular function

Let us start with the triangular density function $\phi(t) = (1/K)(1-t)$ for $t \in [-1, 1]$, where K is a normalization constant.

For α-AV rules, we need $\phi(d+p)/\phi(d-p) = 1-\alpha$ for $0 \leqslant p \leqslant d$. This gives:

$$\begin{aligned}(1-d-p)/(1-d+p) &= 1-\alpha \\ 1-d-p &= (1-\alpha)(1-d+p) \\ -p &= p - \alpha p - \alpha(1-d) \\ p(\alpha-2) &= -\alpha(1-d) \\ p &= (1-d)\alpha/(2-\alpha).\end{aligned}$$

We have $p \leqslant d$ if $(1-d)\alpha \leqslant (2-\alpha)d$, which gives $\alpha \leqslant 2d$. If $\alpha > 2d$, then we have $p = d$. To summarize, we have for α-AV rules:

$$p = \begin{cases} (1-d)\alpha/(2-\alpha) & \text{if } \alpha \leqslant 2d \\ d & \text{if } \alpha > 2d \end{cases}$$

For α-seqAV rules, we also have $p = (1-d)\alpha/(2-\alpha)$ if $\alpha \leqslant 2d$, as the formula is the same than for α-AV rules. For positions such that $d \leqslant p \leqslant 2d$, we need $\phi(p+d)/\phi(p-d) = 1-\alpha$, giving:

$$\begin{aligned}(1-p-d)/(1-p+d) &= 1-\alpha \\ 1-p-d &= (1-\alpha)(1-p+d) \\ -d &= d + \alpha p - \alpha(1+d) \\ -p\alpha &= -\alpha(1+d) + 2d \\ p &= 1 + d - 2d/\alpha.\end{aligned}$$

We have $d \leqslant p \leqslant 2d$ if $0 \leqslant 1 - 2d/\alpha \leqslant d$ and thus $2d \leqslant \alpha \leqslant 2d/(1-d)$. If $\alpha > 2d/(1-d)$, then

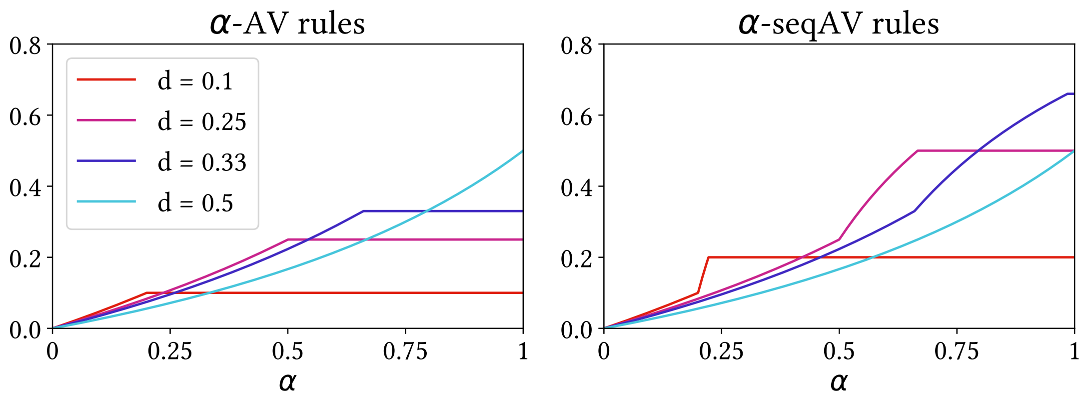


Figure 3.3: Distance between the center of the distribution and the positions of the finalists (resp. the second finalist) for α-AV rules (resp. α-seqAV rules) and the triangular density function.

the position of the second finalist is $p = 2d$. To summarize, we have for α-seqAV rules:

$$p = \begin{cases} (1-d)\alpha/(2-\alpha) & \text{if } \alpha \leqslant 2d \\ 1 + d - 2d/\alpha & \text{if } 2d \leqslant \alpha \leqslant 2d/(1-d) \\ 2d & \text{if } \alpha > 2d/(1-d) \end{cases}$$

Figure 3.3 shows the functions that describe these positions for $d \in \{0.1, 0.25, 0.33, 0.5\}$. We can clearly see that after some α is reached, the positions of the finalists remain unchanged and at distance d (for α-AV rules) or $2d$ (for α-seqAV rules). Moreover, for α-seqAV rules, we can easily distinguish the three different cases for the position of the second finalist.

Normal function

Let us now consider the normal density function (with variance σ and centered in $\mu = 0$ for symmetry), which is frequently used in the 1-dimensional Euclidean model. We have for all $t \in \mathbb{R}$, $\phi(t) = (1/K)e^{-t^2/(2\sigma^2)}$ where K is a normalization constant.

For α-AV rules, the condition $\phi(d+p)/\phi(d-p) = 1 - \alpha$ for $0 \leqslant p \leqslant d$ becomes:

$$\begin{aligned} e^{-(1/2\sigma^2)((d+p)^2-(d-p)^2)} &= 1 - \alpha \\ e^{-(1/2\sigma^2)(4dp)} &= 1 - \alpha \\ e^{-2dp/\sigma^2} &= 1 - \alpha \\ 2dp/\sigma^2 &= \log(1/(1-\alpha)) \\ p &= (\sigma^2/2d)\log(1/(1-\alpha)). \end{aligned}$$

We have $p \leqslant d$ if $\log(1/1-\alpha) \leqslant 2d^2/\sigma^2 \Leftrightarrow 1 - \alpha \geqslant e^{-2d^2/\sigma^2} \Leftrightarrow \alpha \leqslant 1 - e^{-2d^2/\sigma^2}$. Otherwise, $p = d$ is the optimal position. To summarize, we have for α-AV rules:

$$p = \begin{cases} (\sigma^2/2d)\log(1/(1-\alpha)) & \text{if } \alpha \leqslant 1 - e^{-2d^2/\sigma^2} \\ d & \text{if } \alpha > 1 - e^{-2d^2/\sigma^2} \end{cases}$$

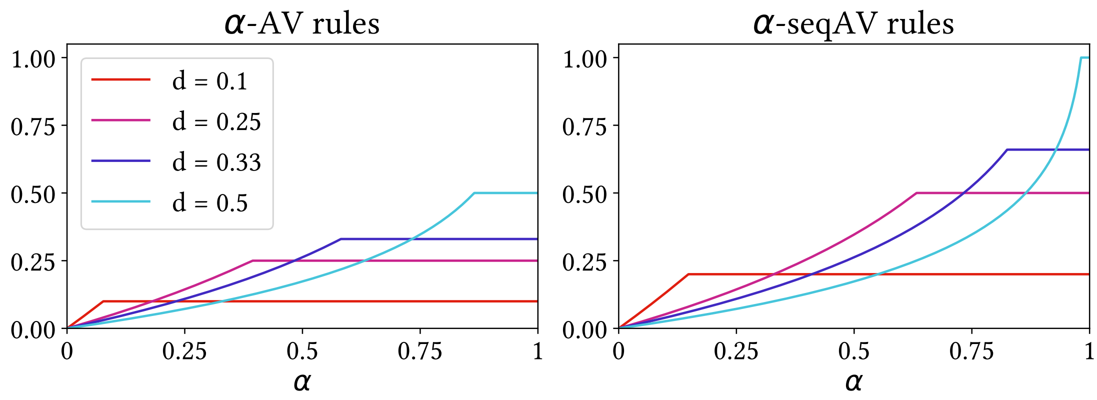


Figure 3.4: Distance between the center of the distribution and the positions of the finalists (resp. the second finalist) for α-AV rules (resp. α-seqAV rules) and the normal density function.

For α-seqAV rules, the first case is based on the same formula, so if $\alpha \leqslant 1 - e^{-2d^2/\sigma^2}$, we also have $p = (\sigma^2/2d)\log(1/(1-\alpha))$. In the case $d \leqslant p \leqslant 2d$, we need $\phi(p+d)/\phi(p-d) = 1-\alpha$ but observe that this gives the same condition than in the first case, as $\phi(p-d) = \phi(d-p)$. Thus, $p = (\sigma^2/2d)\log(1/(1-\alpha))$ and $p \leqslant 2d$ if $\alpha \leqslant 1 - e^{-4d^2/\sigma^2}$. If α is greater than this value, then the position of the second finalist is $p = 2d$. To summarize, we have for α-seqAV rules:

$$p = \begin{cases} (\sigma^2/2d)\log(1/(1-\alpha)) & \text{if } \alpha \leqslant 1 - e^{-4d^2/\sigma^2} \\ 2d & \text{if } \alpha > 1 - e^{-4d^2/\sigma^2} \end{cases}$$

The obtained functions are shown on Figure 3.4. In comparison to the triangular distribution, there are here only two cases for the position of the second finalist with α-seqAV rules.

Exponential function

Finally, consider the exponential density function given by $\phi(t) = (1/K)e^{-\beta|t|}$ for $t \in \mathbb{R}$, with K the normalization constant and $\beta > 0$ some parameter.

For α-AV rules, the condition for $p \leqslant d$ is $\phi(d+p)/\phi(d-p) = 1-\alpha$ and gives:

$$\begin{aligned} e^{-\beta((d+p)-(d-p))} &= 1-\alpha \\ 2\beta p &= \log(1/(1-\alpha)) \\ p &= (1/2\beta)\log(1/(1-\alpha)) \end{aligned}$$

Note that this is independent of the approval radius d. This position satisfies the condition $p \leqslant d$ if $\alpha \leqslant 1 - e^{-2\beta d}$. Otherwise, we have $p = d$. To summarize, we have for α-AV rules:

$$p = \begin{cases} (1/2\beta)\log(1/(1-\alpha)) & \text{if } \alpha \leqslant 1 - e^{-2\beta d} \\ d & \text{if } \alpha > 1 - e^{-2\beta d} \end{cases}$$

Again, the first case of α-seqAV rules is based on the same condition as the one for α-AV rules. For the case $d \leqslant p \leqslant 2d$, we have $\phi(p+d)/\phi(p-d) = 1-\alpha$, which is true if and only if $\alpha = 1-e^{-2\beta d}$ and in that case *all* positions between d and $2d$ maximize the score for the second finalist. Finally,

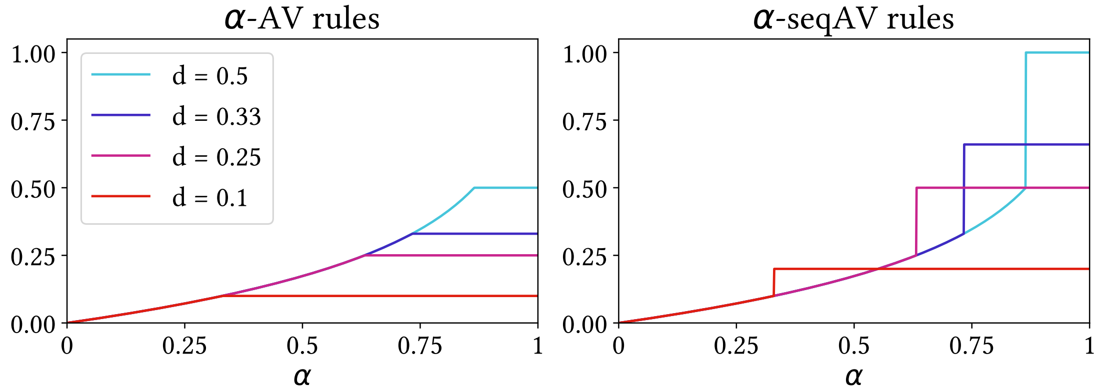


Figure 3.5: Distance between the center of the distribution and the positions of the finalists (resp. the second finalist) for α-AV rules (resp. α-seqAV rules) and the exponential density function.

if $\alpha > 1 - e^{-2\beta d}$, the position of the second finalist is $p = 2d$. To summarize, we have:

$$p = \begin{cases} (1/2\beta)\log(1/(1-\alpha)) & \text{if } \alpha < 1 - e^{-2\beta d} \\ d \leqslant p \leqslant 2d & \text{if } \alpha = 1 - e^{-2\beta d} \\ 2d & \text{if } \alpha > 1 - e^{-2\beta d} \end{cases}$$

The obtained functions are shown on Figure 3.5. We can very clearly see that for the case in which the two finalists share common supporters (i.e., the position satisfies $p < d$), their positions do not depend on the approval radius d (the different curves are superimposed). Moreover, observe that the curves for α-AV rules and α-seqAV rules are the same for $\alpha \leqslant 1 - e^{-2\beta d}$. However, in the first case the other finalist has a symmetrical position while it has position 0 in the second case. This implies that for the same α, the distance between the two finalists will be greater for α-AV rules than for α-seqAV rules. Note that this observation actually holds for any symmetric single-peaked density function, and α such that $\phi(2d)/\phi(0) \leqslant 1 - \alpha$.

3.5 Experimental Analysis

Finally, we evaluate the different rules with experimental analyses. In particular, we want to see how the rules behave on real data and how they compare to each other.

Datasets

To conduct these experiments, we used approval preference datasets collected from various sources:

- Datasets from the *Voter Autrement* collection, gathered during political elections: participants were asked what they would vote at the election if the voting rule was different. We refer to Section 2.5.4 for an extensive discussion of these datasets. In this analysis, we used datasets that include approval preferences. The only dataset in which voters tested approval *with runoff* (and not simple approval) is the dataset from the 2017 election in *Crolles*. We additionally consider the dataset from the 2022 Italian parliamentary election (Marsilio and

year	type	place	rule	weighted	n	m
2002	In Situ	Gy-Les-Nonains	AV	✗	365	16
2002	In Situ	Orsay	AV	✗	2220	16
2007	In Situ	Louvigny	AV	✗	1063	12
2007	In Situ	Cigné	AV	✗	233	12
2007	In Situ	Illkirch	AV	✗	1540	12
2012	In Situ	Strasbourg	AV	✓	1071	11
2012	In Situ	Louvigny	AV	✓	930	11
2012	In Situ	Saint-Etienne	AV	✓	387	11
2017	In Situ	Strasbourg	AV	✓	1023	11
2017	In Situ	Grenoble	AV	✓	1069	11
2017	In Situ	Hérouville-Saint-Clair	AV	✓	711	11
2017	In Situ	Crolles	AV	✓	1321	11
2017	In Situ	Crolles	AVR	✓	2122	11
2017	Online	-	AV	✓	20076	11
2022	In Situ	Strasbourg	AV	✓	931	12
2022	Representative	-	AV	✓	931	12
2022	Online	-	AV	✓	1379	12
2022	Representative	Italy	AV	✓	1021	14

Table 3.3: Datasets of the *Voter Autrement* collection used in this chapter.

Delemazure, 2022). Note that in some of these datasets, weights are assigned to the votes to make the samples more representative of the general population (see Section 2.5.4).

- Datasets from Preflib (Mattei and Walsh, 2013) containing few voters (less than 100). Two datasets are about song selection at a summer camp and the other two are votes in a best poster competition during a summer school in San Sebastian.

- The dataset of the approval votes for the 2023 election of the representatives of the Society for Social Choice and Welfare. Note that this was a multi-winner election.

- Datasets gathered from the platform *TierMaker.com*, as described in Section 2.5.3. These datasets contain preferences of voters on a variety of topics. Voters assigned a category to each candidate in their *tierlists*, and we convert the set of candidates ranked in the best category to approval ballots. To get datasets with a lot of voters, we selected popular topics. The number of voters ranges from 74 to 5 031 and the number of candidates from 8 to 51.

Table 3.3 summarizes the political datasets of the *Voter Autrement* collection that will be used in this chapter. We provide for each dataset the number of participants n and the number of candidates m. Table 3.4 summarizes the information about the non-political datasets. Overall, we have 38 datasets, 18 of which are political datasets from the *Voter Autrement* collection. In our analyses, we will often separate the political datasets from the other datasets.

Similarities between rules

We ran our rules on the different datasets, and Figure 3.6 shows the percentage of datasets on which each pair of rules agrees on the finalists, for both the *Voter Autrement* datasets, and the other datasets. For the former, we also compare the finalists selected by approval-based rules to the actual finalists of the election (using plurality).

source	name	n	m
PrefLib	poster-1	65	17
PrefLib	poster-2	60	17
PrefLib	song-1	39	8
PrefLib	song-2	56	10
SSCW	2023 election	53	32
TierMaker	Colors	286	33
TierMaker	European Countries	1126	51
TierMaker	Harry Potter Movies	512	8
TierMaker	Months	296	12
TierMaker	Numbers	74	10
TierMaker	Planets	83	9
TierMaker	Pokemon Starters	2265	27
TierMaker	Pokemon Types	1359	18
TierMaker	School Courses	323	15
TierMaker	Spider-man Movies	431	10
TierMaker	Star Wars Movies	5031	11
TierMaker	Zodiac Signs	406	12
TierMaker	Pixar Movies	843	27
TierMaker	One Piece Arcs	493	31
TierMaker	Taylor Swift's Albums	201	12

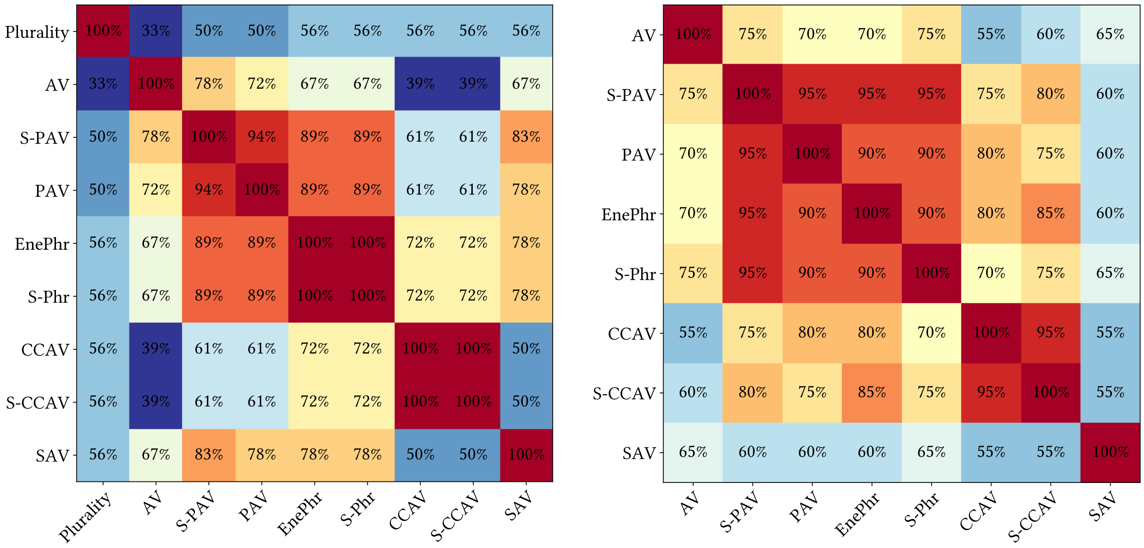
Table 3.4: Datasets used in this chapter that are not part of the *Voter Autrement* collection.

First, we observe on Figure 3.6 (a) that approval-based rules very often select different finalists than the plurality rule. This is especially true for AV, which coincides with plurality with runoff on only 33% of the *Voter Autrement* datasets.

We also observe that α-AV rules and their sequential equivalents agree quite frequently: around 95% of the datasets for PAV and 97% for CCAV (100% for the *Voter Autrement* datasets). However, we have a clear divide for α-AV and α-seqAV rules based on the value of α, especially for the *Voter Autrement* datasets. For instance, AV agrees with CCAV and S-CCAV on only 39% of the datasets (respectively 55% and 60% for non-political datasets). PAV lies in between the two as it agrees with AV 67% of the time (70% for non-political datasets) and with CCAV 61% of the time (80% for non-political datasets). As they share the same proportionality objective, sequential Phragmén and Eneström-Phragmén very often agree with PAV and S-PAV (89% for *Voter Autrement* datasets), and with each other (95%). Note that in these experiments, we use the Droop quota for Eneström-Phragmén, and that the associated α values are between 0.6 and 0.8 for *Voter Autrement* datasets and between 0.4 and 1 for other datasets. Finally, SAV is more isolated, as it is not particularly similar to any other rule (on the political datasets, it has a similarity of 83% with S-PAV, but this drops down to 60% for non-political datasets). Finally, we found that on 27% of *Voter Autrement* datasets and 50% of the other datasets, all rules agree on the same pair of finalists.

Highlights from the data

We will now discuss some specific differences between the rules, using examples from the datasets. In particular, we will focus on three aspects: the effect of α for α-AV rules and α-seqAV rules, the difference between sequential and non-sequential rules, and the particular case of SAV.



(a) *Voter Autrement* datasets. (b) Non-political datasets.

Figure 3.6: Percentage of agreement on the pairs of finalists between the rules on subsets of the datasets.

The effect of α

Let us first analyze the impact of the parameter α for α-AV rules and α-seqAV rules. We have already seen its importance in the statistical analysis (Section 3.4), which showed that rules with small values of α choose similar candidates (often more centrist), while rules with high values of α choose more diverse pairs of candidates. This is also what we observe in our datasets.

Let us look at the example of the Grenoble dataset of the 2017 *Voter Autrement* experiment. We will focus on the five main candidates: the left-wing candidate *Jean-Luc Mélenchon* (JLM), the candidate for the Socialist Party *Benoît Hamon* (BH), who is considered as center-left (though Hamon represented the left branch of the party), the center/center-right candidate *Emmanuel Macron* (EM), the right-wing candidate *François Fillon* (FF) (who represented the more conservative branch of the party), and the far-right candidate *Marine Le Pen* (MLP). The table below shows the approval scores of the different candidates in our dataset (after assigning weights to the votes as described in Section 2.5.4). Even though he did poorly at the actual election, Benoît Hamon is here the approval winner.

Candidate	Approval score	Actual score at the election
Jean-Luc Mélenchon (JLM)	40,67%	19,58%
Benoît Hamon (BH)	42,24%	6,36%
Emmanuel Macron (EM)	41,07%	24,01%
François Fillon (FF)	32,67%	20,01%
Marine Le Pen (MLP)	28,80%	21,30%

The pairs of candidates selected by AV, PAV, CCAV, S-PAV and S-CCAV are the following:

AV	BH	EM			
S-PAV	BH	EM	S-CCAV	BH	FF
PAV	JLM	EM	CCAV	BH	FF

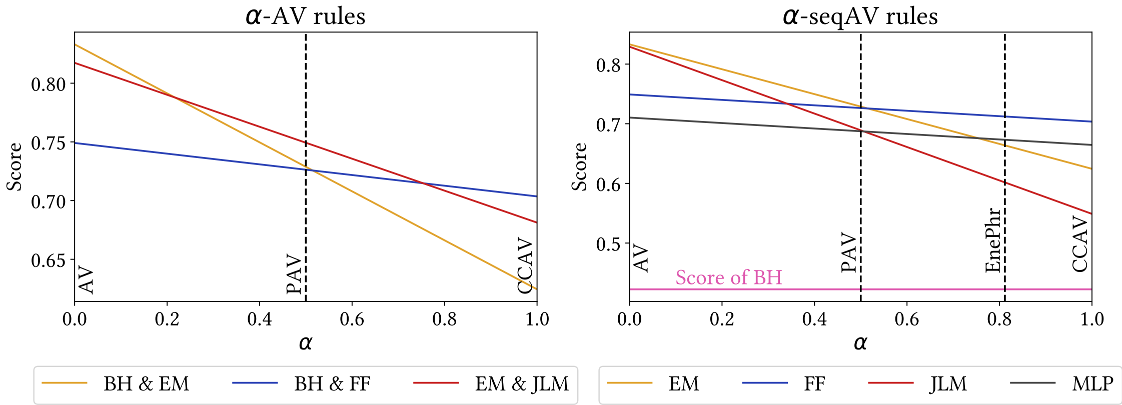


Figure 3.7: Scores of pairs of candidates (resp. of potential second finalist) for α-AV rules (resp. α-seqAV rules) for the Grenoble 2017 dataset of the *Voter Autrement* collection.

We observe that AV selects two candidates that are both considered close to the center (center-left BH and center-right EM) and that are similar in some aspect (both were members of the same party one year before the election). PAV and CCAV bring some diversity by replacing the center-left candidate BH by the left-wing candidate JLM (in the case of PAV), or the center-right candidate EM by the right-wing candidate FF (in the case of CCAV). One could argue that CCAV returns the most diverse pair of candidates in this case.

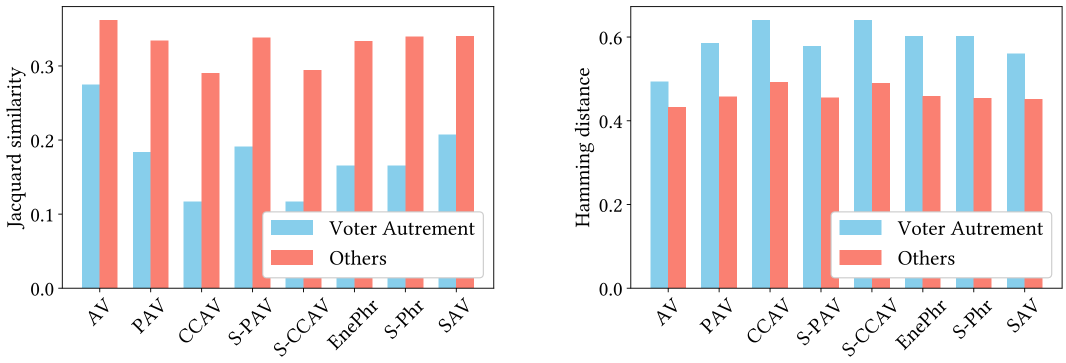
Let us now look at sequential rules. They all select BH as their first finalist, since it is the approval winner in this dataset. S-PAV selects the same finalists as AV, but S-CCAV replaces the center-right candidate EM by the right-wing candidate FF, as CCAV.

To better understand these results, let us look at the co-approval of pairs of candidates, i.e. the percentage of voters approving both candidates $S(xy)/n$. These co-approval values are given in the following table. The diagonal values are the approval scores of the candidates.

	JLM	BH	EM	FF	MLP
JLM	40,7%	28%	13,6%	3,7%	7,6%
BH	28%	42,2%	20,9%	4,5%	4,6%
EM	13,6%	20,9%	41,1%	14,1%	5,8%
FF	3,7%	4,5%	14,1%	32,7%	14,1%
MLP	7,6%	4,6%	5,8%	14,1%	28,8%

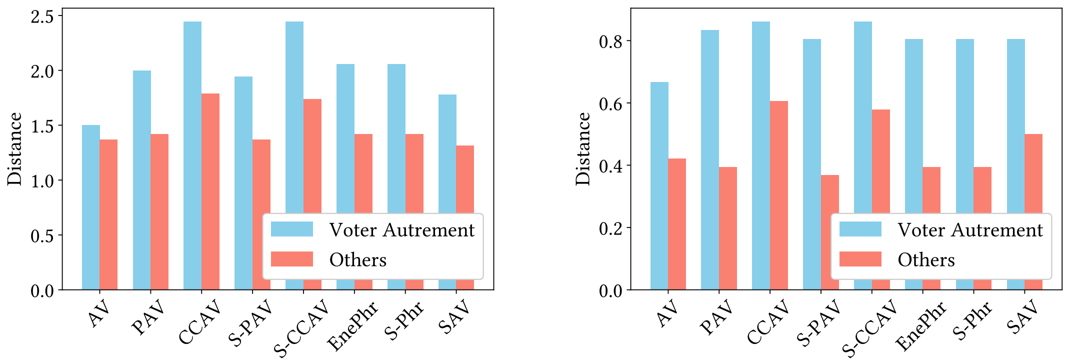
We observe that BH has a lot of common supporters with JLM and EM (but not necessarily the same ones). By their definitions, α-AV rules and α-seqAV rules with high values of α select candidates sharing less supporters than rules with low values of α. We can see this on Figure 3.7 which shows the scores of pairs of candidates (or of possible second finalists for sequential rules) for different values α between 0 and 1. The stronger the slope of the line for a pair of candidates, the more supporters they share and the more similar they are.

To see this effect more generally, we computed for each rule the average Jaccard similarity and Hamming distance between the approval vectors of the two finalists (Figure 3.8). If we denote $A(c) = \{i \in V \mid c \in A_i\}$ the set of supporters of a candidate $c \in C$, the Jaccard similarity between two candidates a and b is equal to $|A(a) \cap A(b)|/|A(a) \cup A(b)|$, i.e., the size of the intersection of their supporters divided by the size of their union. The Hamming distance is equal to the number of voters who approved one candidate but not the other $|A(a) \setminus A(b)| + |A(b) \setminus A(a)|$. We



(a) Average Jaccard similarity. (b) Average Hamming distance.

Figure 3.8: The average Jaccard similarity (Figure a) and Hamming distance (Figure b) between the two finalists over all datasets.



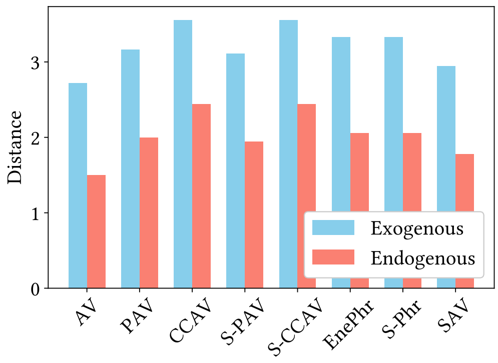
(a) Distance between them. (b) Distance to the center.

Figure 3.9: The average distance between the two finalists (Figure a) and between each finalist and the center (Figure b), over all the datasets, using the Forbidden Triples method from Chapter 6.

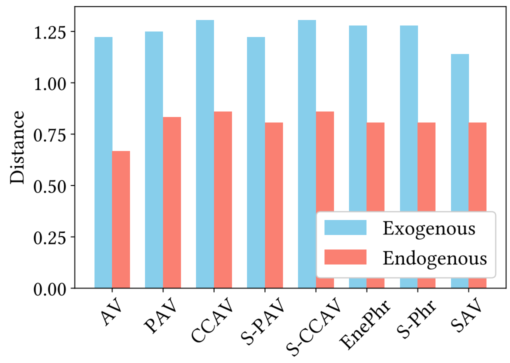
observe on Figure 3.8 that these metrics are tied to the value of α: while rules with low α (like AV) select pairs of candidates with high Jaccard similarities and low Hamming distances, rules with high α (like CCAV and S-CCAV) select pairs of candidates with low Jaccard similarities and high Hamming distances. Actually, both metrics can be defined using the AV and CCAV scores. Indeed, if we denote by $S_{\text{AV}}(a,b) = S(a) + S(b)$ and $S_{\text{CCAV}}(a,b) = S(a) + S(b) - S(ab)$ respectively the AV and CCAV scores of $\{a,b\}$, then the Jaccard similarity is equal to $(S_{\text{AV}}(a,b) - S_{\text{CCAV}}(a,b))/S_{\text{CCAV}}(a,b) = S_{\text{AV}}(a,b)/S_{\text{CCAV}}(a,b) - 1$, and the Hamming distance is equal to $(2 \cdot S_{\text{CCAV}}(a,b) - S_{\text{AV}}(a,b))/n$.

The impact of α is more important on the *Voter Autrement* datasets, in which the two finalists are generally less similar to each other than in the other datasets, likely because political datasets are less consensual and more divided than other ones. Interestingly, we do not observe any strong "clone effect" in our datasets, as the maximum Jaccard similarity between the finalists selected by AV over all datasets is of 64 % (and of 53 % if we only consider the political datasets).

We also wanted to check the results from the 1-dimensional Euclidean setting that was introduced in Section 3.4. For this, we used the *axis rules* that are discussed in Chapter 6. These axis



(a) Distance between them.



(b) Distance to the center.

Figure 3.10: The average distance between the two finalists (Figure a) and between each finalist and the center (Figure b), over *Voter Autrement* datasets using endogenous and exogenous axes.

rules take as input an approval profile P_A, and return an ordering (or axis) of the candidates such that candidates that are close to each other on the axis are often approved together. For instance, in a political setting, the axis can be interpreted as the left-right political spectrum. Using these axis rules, we can then look at the positions of the selected finalists on the ordering, and look at the distance between them, as well as their distance to the center of the axis. Following the theoretical observations from Section 3.4, we should observe that the α-AV and α-seqAV rules with low α return finalists close to each other and to the center, and that rules with high α return finalists which are further from each other and from the center. For this experiment, we used the *Forbidden Triples* rule, and we restricted the preference profile to the 8 most approved candidates of each dataset (as the computational complexity of the rule grows exponentially with the number of candidates). We assume that the distance between consecutive candidates on an axis is always equal to 1. Thus, the center of the axis is between the fourth and the fifth candidate (at position 4.5). Figure 3.9 shows the results of the experiments, and we can see that our hypotheses are verified on these datasets. This is especially true for the *Voter Autrement* datasets, in which the average distance between the two finalists is of 1.4 for AV and 2.5 for CCAV and S-CCAV.

One criticism of using axis rules to build the axes is that both the axes *and* the finalists are computed from the same profiles of approval preferences. In that sense, we say that the axes are *endogenous*. Moreover, the axes returned by the Forbidden Triples rule are not always consistent with the actual left-right political axes. In particular, this method often pushes the candidates with high approval scores towards the center of the axis. To address this issue, we also computed the same metrics using *exogenous* axes for the political datasets. We built these axes using the orderings of the candidates on the Wikipedia pages relative to the polls conducted for each election, which are more faithful to the actual left-right political spectrum. In Figure 3.10, we compare the results for the political datasets using the exogenous and the endogenous axes. We observe that the distances between the finalists and to center of the axis are generally higher for the exogenous axes (because popular candidates are pushed towards the center on the endogenous axes), but the relative performances of the rules regarding the distance between the two finalists are similar for the two kinds of axes. However, with the exogenous axes, the distance between the finalists and the center of the axis seems to be stable across the different rules.

Sequential or not

The second difference we want to discuss is the difference between rules that assign a score to pairs of candidates (α-AV rules and SAV) and sequential rules (such as α-seqAV rules). As shown on Figure 3.6, α-AV rules often return the same results as their sequential counterparts, but it is not always the case. In particular, PAV did not select the approval winner as a finalist in one dataset, and CCAV in two of them, illustrating the violation of favorite consistency. Note however that the results on these datasets do not mean that PAV selects the approval winner less often than CCAV in general. Interestingly, SAV returns the approval winner for all of our datasets, despite not satisfying favorite consistency in theory.

As one can expect from the statistical analysis in Section 3.4, the differences appear when sequential rules select a very centrist and consensual candidate, while non-sequential rules select more diverse pairs of candidates, for instance one from the left and one from the right in a political context. This is what we observe if we combine the three datasets of the 2007 presidential election from the *Voter Autrement* collection. We will show that in this dataset PAV and CCAV disagree with their sequential counterparts, as they do not select the approval winner.

We will focus on three candidates: the left-wing candidate *Ségolène Royal* (SR), the centrist *François Bayrou* (FB), and the right-wing candidate *Nicolas Sarkozy* (NS). For this dataset, we do not have access to the actual votes of the participants at the election, so we did not assign weights to the voters. The table below shows the approval scores of these candidates. In this dataset, Bayrou is the approval winner.

Candidate	Approval score	Actual score at the election
Ségolène Royal (SR)	41.5%	25.9%
François Bayrou (FB)	47.2%	18.6%
Nicolas Sarkozy (NS)	42.9%	31.2%

In this dataset, François Bayrou has a lot of common supporters with both Ségolène Royal (with 20.3% of participants approving both) and Nicolas Sarkozy (with 20.7% of participants approving both). However, only 8.1% of participants approved both Ségolène Royal and Nicolas Sarkozy. For this reason, this pair will be returned by the non-sequential rules that favor diversity, while sequential rules will always return a pair involving François Bayrou. Figure 3.11 shows the score of each pair for α-AV rules with α varying between 0 and 1. While the AV rule, as all sequential rules, selects the approval winner François Bayrou and Nicolas Sarkozy, the PAV and CCAV rules both select Nicolas Sarkozy and Ségolène Royal.

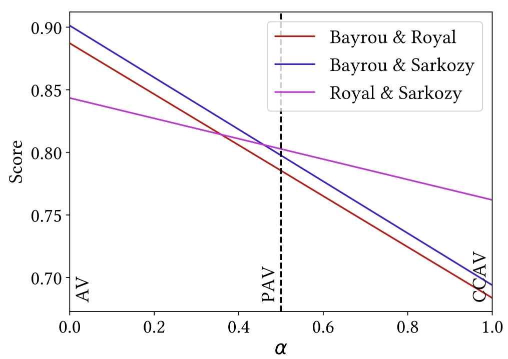


Figure 3.11: Scores of pairs of candidates for α-AV rules for the combined 2007 dataset of the *Voter Autrement* collection.

Moreover, according to the polls, François Bayrou would have won a runoff against both Ségolène Royal and Nicolas Sarkozy, so here sequential rules and non-sequential rules not only differ on the pair of finalists, but also on the winner of the election.

The case of the SAV rule

Finally, let us look at SAV. As voters divide their vote between the candidates they approve, the ones who approve more candidates actually have less power. In that sense, SAV heavily suffers from non-independence of clones, like plurality with runoff.

In the *Voter Autrement* datasets, this gives a disadvantage to left-wing candidates, as they are generally more numerous, and thus co-approved with more candidates. In particular, in the 7 *Voter Autrement* datasets on which AV and SAV disagree, they disagree on one candidate, and in all cases SAV chooses a candidate more right-wing than AV.

Consider for instance the *Strasbourg* dataset of the 2022 presidential election. We focus on three candidates: the center-right candidate *Emmanuel Macron* (EM), the left-wing candidate *Jean-Luc Mélenchon* (JLM) and the far-right candidate *Marine Le Pen* (MLP). The table below shows the approval scores of these different candidates after weighting the votes, as well as the average number of candidates approved by the supporters of each candidate.

Candidate	Approval score	Average number of approved candidates
Jean-Luc Mélenchon (JLM)	40.1%	3.2
Emmanuel Macron (EM)	44.1%	2.8
Marine Le Pen (MLP)	32.9%	2.7

As we can see, Marine Le Pen's supporters approved on average less candidates than Jean-Luc Mélenchon's supporters. Thus, SAV selects Marine Le Pen as the finalist against Emmanuel Macron, even though she has much less approval votes than Jean-Luc Mélenchon.

3.6 Discussion

In this chapter, our starting point was to consider that *approval with runoff* is not *one* rule but a *family of rules*, parameterized by the ABC rule chosen for determining the finalists. Our axiomatic, statistical and experimental results show that this choice does make a big difference on the selection of the finalists, and thus on the winner of the election.

Which Rule to Choose?

If approval with runoff has to be used in political elections, the choice of the ABC rule would be crucial, and far from easy, but our results already give some useful pieces of information. What we can conclude from both our theoretical and experimental results is that AV tends to select more centrist and consensual finalists (possibly clones of each other), while CCAV tends to select very different candidates, for instance a left-wing one and a right-wing one. PAV lies in between the two and could be seen as a good tradeoff, however in most cases it is actually more interesting to go for one of the extremes (AV or CCAV), as they are easier to interpret. To this, we add that non-sequential rules, in addition to being more complicated to explain, might not select the candidate that received the most approval support as a finalist. For this reason, the two rules that seem reasonable to use in most contexts are the AV rule and the S-CCAV rule. On the other hand, with sequential rules such as S-CCAV, the two finalists have a different status: one is the approval winner, while the other is selected *based on who is the approval winner*, and could be seen as a "contender". Finally, SAV does not seem like a reasonable choice. It relies on the false idea that splitting the vote of each voter between approved candidates ensures that all voters have the same voting power (since they all give the same total of points), while the opposite is true: voters who approve more candidates have less power.

If we compare these rules to the one we aim to replace, *plurality with runoff*, we observe that they have several advantages. First, the common advantage between all approval with runoff rules is that voters can give more expressive preferences, and do not have to decide between different candidates they like. Moreover, the approval with runoff rules do not suffer from all of the issues that we can encounter with plurality with runoff. On the one hand, AV^R satisfies the monotonicity property, which is failed by plurality with runoff. On the other hand, S-CCAV^R is independent of clones, while plurality with runoff is not. AV^R is not independent of clones either, but in a less problematic way than plurality with runoff (with AV^R, having a clone does not hurt a candidate, but may help it).

Further work

Our work opens several perspectives for further work. Some have already been done since the publication of the conference paper in 2021. In particular, Ebadian et al. (2023) study approval with runoff under the lens of worst-case distortion in the utilitarian framework (see Section 2.5.2). They assume that each candidate $x \in C$ provides some utility $u_i(x)$ to each voter $i \in V$, with the utilities summing up to 1 for each voter. Then they assume that there is a threshold $\tau \in [0, 1]$ such that every voter approves candidates who provide them with a utility $u_i(x) \geqslant \tau$. They show that for deterministic AVR rules, the worst-case distortion is at least $\Theta(m^2)$, and this optimal value is reached by AV^R for $\tau = 1/m$. Moreover, they show that in the same model approval voting without runoff has worst-case distortion $\Theta(m)$, implying that the runoff might incur a welfare loss. Additionally, they discuss how adding randomization (for the selection of the finalists, and for the choice of the winner in the runoff) can decrease the worst-case distortion to $O(m)$. However, like most works on distortion, they focus on the *worst-case* distortion. An interesting question would be to study the average distortion of these rules for several preference models.

It would also be interesting to look deeper into strategic aspects of these rules. In particular, the simple 1-dimensional Euclidean model used in Section 3.4 could be extended to be more realistic. For instance, we could adapt theoretical (Laslier and Van der Straeten, 2016) and empirical (Van der Straeten et al., 2018) results saying that the approval thresholds of voters depend on which candidates are expected to arrive in the first three positions.

Another interesting path for further work is to extend the study to approval rules with a *conditional* runoff. In such setting, the runoff happens only if there is no clear winner in the first round. In that sense, it is similar to the way plurality with runoff is implemented in most countries, in which there is no second round when one candidate gets the majority of the votes in the first round. One could then think of similar rules for the approval setting, in which we do not do a second round in the case one candidate is approved by a quota of the voters (for instance half of them), or/and if the first candidate's score is much higher than the score of the second one.

Implementing Approval with Runoff

One of the main advantages of plurality with runoff is that this rule is widely used, well-known, and easy to implement, while approval with runoff rules are probably harder to explain and to implement. In that sense, an important question is: will citizens understand and accept such rules? In particular, many people intuitively believe that allowing people to approve several candidates

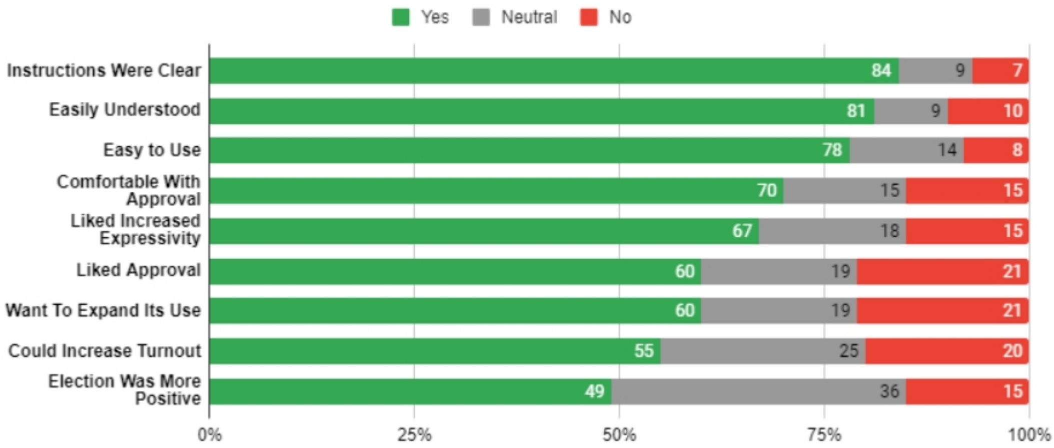


Figure 3.12: St. Louis voters' opinions on approval voting.

give more power to voters approving more candidates, breaking the *one person, one vote* principle.

Another natural objection would be that this is an unnecessary change: people might actually not use the opportunity to vote for several candidates, and just vote for their favorite. However, real-life scenarios and all of our experiments show the opposite. In the St. Louis, Missouri municipal elections of 2021 that used approval with runoff, people approved on average 1.6 candidates out of 4. In the *Voter Autrement* experiments, the average number of candidates approved by the participants that provided approval ballots ranges from 2.1 to 3.4.

Another question is whether people who used approval voting or approval with runoff in real-life or in our experiments liked this rule, and whether they are willing to change for these new rules. A poll conducted by *Change Research* and *The Center for Election Science* on St. Louis voters after the approval with runoff election shows that a large majority liked the change of voting system (Figure 3.12[4]). In particular, 78% said that they found approval ballots easy to use, and 60% that they liked the new system. We also note that 67% said they liked the increased expressivity. In the *Voter Autrement* experiments, participants were sometimes asked to evaluate the methods they just tried, including the approval rule (without runoff). Results vary a lot from one dataset to another. In 2017, participants in Strasbourg were asked if they *liked* the voting rules, and in Hérouville-Saint-Clair if they thought it was *possible to adopt* the voting rules for the presidential election. While 49% of participants liked approval voting in Strasbourg, only 33% of participants thought it was possible to adopt approval voting for the presidential election in Hérouville-Saint-Clair. In the *online* experiments, we asked participants to evaluate the rules they just tried. Here, the results are more positive. In the 2022 dataset, 74% of participants indicated being "satisfied" or "very satisfied" by approval voting, which is the highest score over all rules tested. In the 2017 dataset, approval voting comes second after Borda with 65% of "satisfied" or "very satisfied" participants. Note however that these results are not representative of the general population, since most participants on these online experiments are already interested in the topic of voting rules. A survey conducted in 2022 on a representative sample showed that a majority of people (56%) would prefer keeping the current voting system for the presidential election over a new one (approval and the Majority Judgement were proposed as alternatives).

[4] Source of the figure: `https://le.utah.gov/interim/2021/pdf/00002622.pdf` (page 13)

Chapter 4

Instant Runoff Voting with Indifferences: Approval-IRV

4.1 Introduction

In the previous chapter, we discussed how we could replace the plurality with runoff voting method by an approval with runoff method in which voters are allowed to vote for several candidates in the first round of the election. Plurality with runoff is not the only voting rule that is used for high-stakes single-winner elections. One of its most popular alternatives is *Instant Runoff Voting*, which goes under various names depending on the place and the time: *Alternative Vote* in the UK, *Preferential Voting* in Australia, or *Ranked Choice Voting* in the US. It is also sometimes called *Hare System*, from the name of the British barrister Thomas Hare, who is often credited for the conception of the rule.

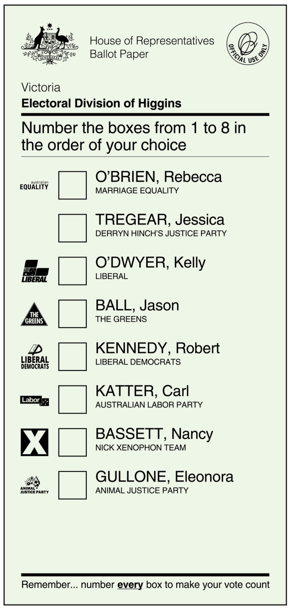


Figure 4.1: Ballot from 2016 Australian federal election.

Instant Runoff Voting

We first recall the process of the *Instant Runoff Voting* (IRV) rule, that we introduced in Section 2.3.1. As a reminder, IRV is a single-winner voting rule based on the voters' rankings of the available candidates. It works in rounds, by sequentially eliminating candidates: at each round, we use voters' rankings to associate them to their favorite candidate among the ones that are not yet eliminated, and the candidate with the fewest associated voters is eliminated. The last round is a majority vote between the last two remaining candidates. As in plurality with runoff and approval with runoff, there is the idea of doing several rounds of vote, with a reduced number of candidates at each round. However, because voters provide a full ranking, runoffs can be done automatically, without the need for voters to cast a new ballot at each round like with plurality with runoff and approval with runoff.

While a wide variety of ranking-based voting rules have been studied over the centuries, IRV is one of the few rules that has been adopted for high-stakes elections to political offices. Australia was the first country to adopt it and uses this rule to elect its House of Representatives since 1918 (see Figure 4.1). Ireland uses it to elect its President, and it is used in Alaska and Maine for several offices. It is also in use at the local level in many jurisdictions, as well as within some societies and political parties. Electoral reform advocates such as *FairVote*[1] in the US and *Electoral Reform Society*[2] in the UK are pushing for further adoption, arguing that IRV and its ranking-based input leads to election outcomes that better reflect voters' preferences.

In particular, in addition to enabling voters to give more expressive preferences, IRV is known to satisfy independence of clones, in contrast to plurality and plurality with runoff. However, because it is based on the principle of *runoffs*, IRV does not satisfy monotonicity, and is prone to various manipulations. To see the violation of monotonicity, consider the example used in the introduction of Chapter 3 for plurality with runoff: since it contains only three candidates, the two rules are equivalent.

Another drawback of IRV is a phenomenon which is sometimes called "favorite betrayal" (Small, 2010), or "compromise strategic voting" (Green-Armytage, 2014; Graham-Squire and McCune, 2023). Consider an election with three candidates Ann, Bob and Cora and the following votes:

$$32\%\ \text{Ann} \succ \text{Bob} \succ \text{Cora} \qquad 33\%\ \text{Cora} \succ \text{Ann} \succ \text{Bob} \qquad 35\%\ \text{Bob} \succ \text{Cora} \succ \text{Ann}$$

In this election, Ann is eliminated first, and Bob wins the majority vote against Cora. However, Cora's supporters would rather have Ann elected than Bob, and they know that Bob wins the majority vote against Cora, but not against Ann. Thus, some of them might decide to put Ann first instead of Cora, so that Cora is eliminated first, allowing Ann to win the election. For instance, we could have:

$$32\%\ \text{Ann} \succ \text{Bob} \succ \text{Cora} \qquad 31\%\ \text{Cora} \succ \text{Ann} \succ \text{Bob} \qquad 35\%\ \text{Bob} \succ \text{Cora} \succ \text{Ann}$$
$$2\%\ \underline{\text{Ann} \succ \text{Cora}} \succ \text{Bob}$$

By *betraying* their favorite candidate Cora and voting for Ann, the 2% of voters who changed their vote ensured that Ann was elected instead of Bob, thus being more satisfied than if they had voted sincerely. This phenomenon is often used as an argument against IRV.

Another important drawback of IRV is the burden it imposes on voters who may need to rank a large number of candidates. This is particularly severe in Australia, where voting is compulsory and a ballot is invalid if it fails to rank every candidate. As we can see in Figure 4.2, more than half of Australian voters need to rank 8 or more candidates. Most other jurisdictions allow voters to submit a truncated ranking, where the voter may rank only some of the candidates, and the vote is not taken into account (or "exhausted") after all the candidates that were ranked have been eliminated. However, voters who wish to rank some disfavored candidates in low positions must rank the whole field. It also forces voters to clearly distinguish all candidates that they favor, even when they may not have sufficient information to do so. In particular, voters cannot just submit a simple "approval vote" where they indicate several candidates as acceptable.

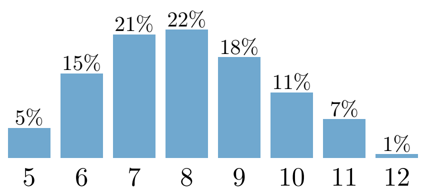
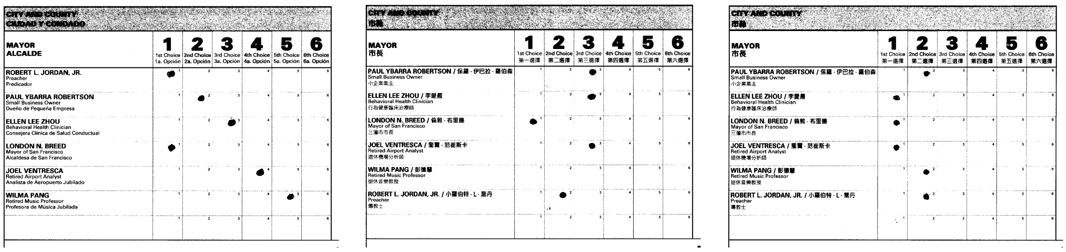


Figure 4.2: The fraction of voters in the 2022 Australian federal election that needed to rank each number of candidates between 5 and 12.

[1] https://fairvote.org/
[2] https://www.electoral-reform.org.uk/



(a) Two top choices (b) Non-ranked candidate (c) An approval vote

Figure 4.3: Examples of ballots that can be interpreted as weak orders (2019 mayoral election in San Francisco).

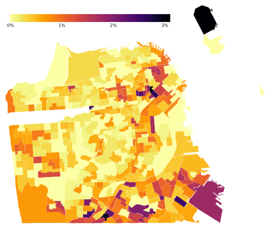
Benefits of Weak Orders

A possible solution to these issues is to use more expressive preferences, by allowing voters to express indifferences, that is, to assign several candidates an equal rank, thus providing a weak order instead of a ranking. There are several other arguments for moving from standard IRV to a system that allows for weak orders.

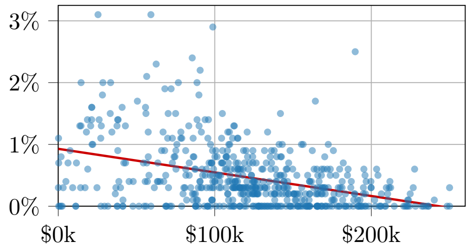
First, allowing equal rankings gives voters more ways to express their preferences. For example, some people might be truly indifferent between candidates, and forcing them to rank such candidates would be distorting. For this reason, some have argued that Australia's compulsory voting forces voters to lie (Rydon, 1968; Orr, 1997). This would also mitigate the issue of favorite betrayal. For instance, in the previous example, voters would be able to put both Ann and Cora first, thus not betraying their favorite candidate for another one. Weak orders also allow voters to give a high ranking to candidates they like and a low ranking to the ones they dislike, without having to rank all candidates, and in particular the ones whom they are not familiar with. This also makes it possible to submit a simple approval vote, which is a particular kind of weak order. The additional expressive power is also useful in U.S. jurisdictions that allow voters to only use 3–5 ranks. Weak orders allow voting for more than 3–5 candidates within the same number of ranks.

Second, allowing weak orders can reduce the amount of invalid ballots. Indeed, compared to elections using plurality or plurality with runoff, voting in an IRV election is more difficult, and the voting instructions are more complicated to follow. In particular, some voters may not realize that they must only place one candidate in each rank, submitting a ballot which encodes a weak order will be counted as invalid. In the American context, this mistake is known as an "overvote", and such ballots are either rejected or partially counted (e.g., if the top ranks contain unique choices, the ballot is counted as a vote for them, but once those candidates are eliminated and a rank with several candidates is reached, the vote is ignored in the following rounds). Figure 4.3 shows examples of such ballots from the 2019 mayoral election in San Francisco that used IRV and that can be interpreted as weak orders. For how many ballots would this make a difference, in practice? One can try to quantify this via the number of overvotes reported by election officers. McCune (2023) reports that in a very close 2021 City Council election in Portland, Maine, a different treatment of overvotes would have changed the winner. One issue with reported overvotes is that not all ballots containing overvotes encode valid weak orders. In particular, they might assign multiple ranks to the same candidate and are therefore definitely invalid. Conveniently, San Francisco makes very detailed vote data publicly available in a CVR JSON format, and even complements these with image scans of every ballot.[3] We analyzed this data to see how many

[3]https://www.sf.gov/results



(a) Map of San Francisco election precincts, colored by the fraction of votes that could be interpreted as a weak orders.



(b) Among election precincts, median household income (horizontal axis) is negatively correlated with percent of ballots showing a weak order (vertical axis; $r = -0.4$, $p < 0.001$).

Figure 4.4: Ballot data from the 2019 mayoral election in San Francisco.

ballots could be interpreted as a weak order with at least one indifference between candidates. For the 2019 mayoral election in San Francisco, we found 899 such ballots out of 206 117 ballots submitted (0.4%); see Figure 4.3 for some examples. Voters casting such weak order ballots are geographically concentrated (Figure 4.4 (a)) and the fraction of ballots that encode weak orders in specific election precincts correlates negatively with median income (Figure 4.4 (b)). This mirrors previous findings of higher rates of invalid ballots in precincts with lower incomes in San Francisco (Neely and McDaniel, 2015) and New York City (Cormack, 2023), as well as of the frequency of overvotes (Pettigrew and Radley, 2023). In Scotland's 2017 local elections, 1.6% of ballots were rejected because of multiple top choices (Electoral Commission, 2017).

Split-IRV

Because IRV is only defined for *linear orders* (rankings without indifferences), to implement this solution we need to generalize IRV to *weak orders* (rankings with indifferences), and the right way to generalize it is not obvious. In a series of articles in the journal *Voting Matters*, Meek (1994, Section 6), Warren (1996), and Hill (2001) developed a way in which IRV (and in fact, every ranking-based voting rule) can be generalized to weak orders. Their idea was to replace every weak order by several (weighted) ranking votes for all possible ways in which the indifferences can be broken. For example, a voter who has a complete ranking, except for an indifference between a and b, would be replaced by a vote of weight $1/2$ with a ranked above b and another vote of weight $1/2$ with b above a. Similarly, a voter who reports indifference between all candidates would be replaced by $m!$ votes each with weight $1/m!$. After this replacement operation, linear-order IRV is applied.

Aziz and Lee (2020, footnote 8) note that, as described, this leads to an algorithm that may take exponential time. However, there is an equivalent description of this rule that is polynomial-time and easy to understand: for a voter who currently ranks t candidates on top (if we ignore candidates that have been eliminated), the voter assigns $1/t$ points to each of these candidates, and the rule repeatedly eliminates the candidate with the fewest points. This rule has some intuitive appeal since it encodes the idea that every voter has a "single vote" and can only assign a total of one point, which has to be split between all their top-ranked candidates. This follows the same principle as *Split Approval Voting*, that we discussed in Section 3.2 (and in Chapter 3). Accordingly, we call this rule *Split-IRV*. The multi-winner version of this rule *Split-STV*, generalizing *Single Transferable Vote* to weak orders, is implemented in practice: Mollison (2023) reports that Split-STV "was first

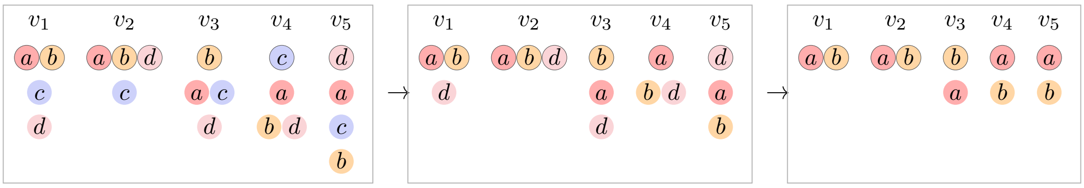


Figure 4.5: An example of Approval-IRV with voters $v_1, \ldots, v_5$. The first eliminated candidate is c, which is ranked on top only once. Then d is eliminated, and finally a wins the majority vote against b. Thus, a is the winner.

used by the *John Muir Trust* (for Trustee elections) in 1998, and by the *London Mathematical Society* in 1999", and both still use Split-STV today. It is also implemented in the `vote` package for R (Raftery et al., 2021, p. 682).

Approval-IRV

In this chapter, we argue that *Split-IRV* is not the correct generalization of IRV to weak orders, and we suggest another generalization that we call *Approval-IRV*. This rule interprets each weak order as an approval vote for the highest-ranked candidates that have not yet been eliminated. It then repeatedly eliminates the candidate with the lowest approval score (i.e., the candidate who is top-ranked by the fewest voters), until only one candidate remains and is declared the winner. Figure 4.5 shows an example of how the rule works. Note that this rule is different from Split-IRV: in Approval-IRV, each voter gives one point to all the top-ranked candidates, while in Split-IRV, each voter divides one point between the top-ranked candidates. For instance, in the example shown in Figure 4.5, a is the winner under Approval-IRV, but under Split-IRV, a is the first candidate to be eliminated, as it received only $1/2 + 1/3$ points, and b is the final winner.

To our knowledge, the only previous scholarly discussion of Approval-IRV (and its multi-winner equivalent Approval-STV) is by Janson (2016, Section 18.2). He discovered the possibility of generalizing IRV and STV via approval, based on his historical analysis of voting methods used in Sweden in the early 1900s. He called this approach "Phragmén's principle" after the Swedish mathematician Lars Edvard Phragmén (1863–1937) who, around 1903, had proposed a similar approach to extend certain voting rules to work with votes that have two levels of approval. Janson did not analyze the properties of Approval-IRV, but wrote that *"it would be interesting to compare the two ways to handle weakly ordered lists. It seems that Phragmén's principle [...] might have some advantages over splitting the vote between total orderings as described [by Meek and Hill]."* In this chapter, we take up Janson's research program and confirm his prediction: Approval-IRV is a better generalization than Split-IRV.

Outline of the Chapter

This chapter is organized as follows. In Section 4.2, we formally define generalizations of IRV, in particular Approval-IRV and Split-IRV. In Section 4.3, we study the axiomatic properties of these rules. In particular, we prove two characterizations of Approval-IRV: First, as the only rule of a particular family to satisfy the natural generalization to weak orders of the main properties of IRV (independence of clones and the majority criterion), and second, as the only generalization of IRV of a particular family to satisfy a weak monotonicity property. In Section 4.4, we present the results of several experiments on synthetic and real data comparing Approval-IRV and Split-IRV. Finally, in Section 4.5, we conclude and discuss further work.

4.2 Runoff Scoring Rules for Weak Orders

In this section, we introduce the model and the rules. We first recall the notations related to the notion of weak orders, and then we define the family of voting rules we study, namely runoff scoring rules, with a special focus on two of them: Approval-IRV and Split-IRV.

Weak Orders

We assume in this chapter that we have a profile of weak orders $P = (\succcurlyeq_1, \dots, \succ_n)$, where $\succcurlyeq_i$ is the weak order of voter i. As a reminder, we denote $a \succ b$ if $a \succcurlyeq b$ and $b \not\succcurlyeq a$ (strict preference) and $a \sim b$ if both $a \succcurlyeq b$ and $b \succcurlyeq a$ (indifference). More generally, for sets $A, B \subseteq C$ we say that $A \succ B$ if $a \succ b$ for all $a \in A$ and $b \in B$. We will often write weak orders $\succcurlyeq$ in the following format: $\{c_1, c_2\} \succ \{c_3\} \succ \{c_4\}$. More formally, every weak order partitions C into *indifference classes* $C_1, \dots, C_k$ such that $C_1 \succ \cdots \succ C_k$ and whenever $a, b \in C_j$ for some j, then $a \sim b$. We will sometimes refer to C_1 as the top indifference class of the weak order, and C_k the bottom one.

The *order type* of a weak order $C_1 \succ \cdots \succ C_k$ is the ordered list $\tau = (|C_1|, \dots, |C_k|)$ of the sizes of its indifference classes. We usually denote an order type using the notation $\tau_{|C_1|\dots|C_k|}$ if this causes no ambiguity; for example τ_{121} for the order type $(1, 2, 1)$. Figure 4.6 shows some examples of order types. A weak order with order type $(1, 1, \dots, 1)$ (i.e., with no indifferences) is called a *linear order* and corresponds to a ranking, and a weak order with an order type τ with $|\tau| = 2$ (e.g., τ_{23}) is called *dichotomous* and corresponds to an approval ballot.

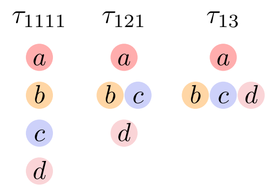


Figure 4.6: Examples of weak orders with different order types.

Runoff Scoring Rules

In this chapter, we will study *irresolute* single-winner voting rules f that take as input a profile P and output a non-empty set of tied winners $f(P) \subseteq C$ (usually a singleton). We will focus on *runoff scoring rules* (Smith, 1973, Section 4) that we discussed in Section 2.3.1 for the case of rankings. The principle can easily be generalized to weak orders, for which runoff scoring rules work as follows: at each step, every voter assign a score to every candidate depending (i) on the order type of their weak order and (ii) on the position of the candidate in that order. The candidate with the lowest total score is eliminated from all orders, and we repeat the process with the remaining candidates. As IRV is part of this family for linear orders, it is natural to focus on this family of rules for defining generalizations of IRV to weak orders.

To define more formally these runoff scoring rules, we need the notion of (positional) scoring systems for weak orders. A *scoring system* is a function s that maps order types τ to scores $s(\tau) = (s(\tau)_1, \dots, s(\tau)_{|\tau|}) \in \mathbb{R}^{|\tau|}_{\geqslant 0}$ where $s(\tau)_{|\tau|} = 0$ without loss of generality. Given a scoring system s and a weak order $C_1 \succ \cdots \succ C_k$ with order type τ, a candidate $c \in C_j$ is assigned a score of $S_{\succcurlyeq,s}(c) = s(\tau)_j$. Given a profile $P = (\succcurlyeq_1, \dots, \succcurlyeq_n)$, the *score* of the candidate c is $S_{P,s}(c) = \sum_{i \in V} S_{\succcurlyeq_i,s}(c)$. We often drop the subscript if P and s are clear from the context, and write $S(c)$ for $S_{P,s}(c)$. In this chapter, we will only consider *monotone* scoring systems, i.e., such that $s(\tau)_1 \geqslant \ldots \geqslant s(\tau)_{|\tau|}$ for all order types τ.

A runoff scoring rule associated to a scoring system s is then defined inductively as follows. Consider a profile P, defined on a candidate set C. If the candidate set is a singleton $C = \{c\}$, we set $f(P) = \{c\}$. Otherwise, if $|C| \geqslant 2$, let $L = \{l \in C : S_{P,s}(l) \leqslant S_{P,s}(c) \text{ for all } c \in C\}$ be the set of lowest-scoring candidates in P. Then we set $f(P) = \bigcup_{l \in L} f(P_{-l})$ where P_{-l} refers to the profile obtained from P by restricting all the weak orders in it to the candidate set $C \setminus \{l\}$ where l has been deleted. Note that this way of breaking ties is known as *parallel-universe tie-breaking* (Conitzer et al., 2009, Section 7), and leads to an axiomatically well-behaved rule. Deciding whether a candidate is a winner under this tie-breaking is NP-hard (Boehmer et al., 2023, Theorem 6.2), but this should not be a problem in political elections with moderately low m since the problem can be solved using an $O(2^m \cdot nm^2)$ time algorithm (Boehmer et al., 2023, Theorem 6.1).

If we restrict ourselves to profiles of linear orders, we fall back on the definition of runoff scoring rules for rankings introduced in Example 2.2. For profiles of weak orders, we are particularly interested in two runoff scoring rules that generalize IRV in natural ways. All generalizations of IRV need to use the plurality scores for linear orders $s(\tau_{1\dots1}) = (1, 0, \dots, 0)$. Following the idea that with IRV we only give a positive score to the top-ranked candidate of every order, our generalizations of IRV will both only assign a positive score only to the candidates of the first indifference class of every weak order.

The first generalization of IRV that we consider is *Approval-IRV*. In Approval-IRV, all the top-ranked candidates will get one point, whatever the type of the weak order.

Approval-IRV
The *Approval-IRV* rule is the runoff scoring rule based on the approval scoring system:

$$s(\tau) = (1, 0, \dots, 0) \text{ for all order types } \tau$$

The second generalization is *Split-IRV*, and this time each voter assigns a total of one point, which is split evenly among the top-ranked candidates of the weak order. For example, if there are three candidates in the top indifference class, each will receive a score of 1/3.

Split-IRV
The *Split-IRV* rule is the runoff scoring rule based on the split scoring system:

$$s(\tau) = (1/\tau(1), 0, \dots, 0) \text{ for all order types } \tau$$

4.3 Axiomatic Analysis

Let us now axiomatically evaluate these two rules. In particular, we will characterize Approval-IRV in two ways: first, as the only runoff scoring rule that satisfies two axioms characteristic of IRV, and second, as the only runoff scoring rule that generalizes IRV and satisfies a weak monotonicity axiom.

Generalizing Properties of IRV

The IRV rule is known to satisfy several desirable properties that are failed by many other voting rules. In particular, it satisfies independence of clones (Section 2.4.4) and the majority criterion (Section 2.4.2). We will show that among runoff scoring rules for weak orders, Approval-IRV is the only rule that satisfies natural generalizations of these two properties to weak orders.

Independence of Clones

Among social choice theorists, IRV stands out from most other ranking-based voting rules because it satisfies independence of clones (Tideman, 1987). As a reminder, this axiom requires that adding new candidates to an election who are very similar to existing candidates (so that all voters rank

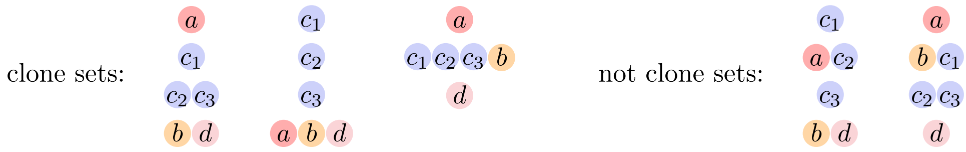


Figure 4.7: Examples of $T = \{c_1, c_2, c_3\}$ being a clone set or not being a clone set.

them in adjacent positions) should not change the outcome. Formally, given a profile P, a set of candidates $T \subseteq C$ is a *clone set* if for every voter $i \in V$ and every candidate $x \notin T$, we have either

$$c \succ_i x \text{ for all } c \in T, \qquad \text{or} \qquad c \sim_i x \text{ for all } c \in T, \qquad \text{or} \qquad x \succ_i c \text{ for all } c \in T.$$

See Figure 4.7 for examples and non-examples of clone sets.

We recall the independence of clones axiom defined in Section 2.4.4:

Independence of Clones

A voting rule f satisfies *independence of clones* if for all profiles P with clone set $T \subseteq C$, letting $\hat{P}$ be the profile obtained by removing all but one candidate $\hat{c}$ from T, it holds that

1. for every $x \notin T$, $x \in f(P)$ if and only if $x \in f(\hat{P})$, and
2. $\hat{c} \in f(\hat{P})$ if and only if there exists $c \in T$ such that $c \in f(P)$.

Approval-IRV satisfies independence of clones. We can show this by adapting the standard proofs that linear-order IRV satisfies the axiom (Tideman, 1987; Freeman et al., 2014a), using the fact that cloning a candidate does not change the score of any non-clone candidate in any round of Approval-IRV. The formal proof is due to my coauthor Dominik Peters and can be found in Delemazure and Peters (2024).

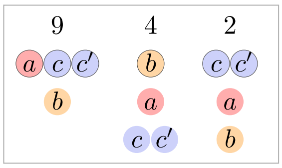


Figure 4.8: Split-IRV fails independence of clones.

However, Split-IRV fails independence of clones. A simple counterexample is shown in Figure 4.8, where a is eliminated with a score of 3 while other candidates each have a score of 4; then finally one of the clones c and c' wins. However, when removing the clone c' of c from the profile, we get a profile where the score of a is 4.5, the score of b is 4, and the score of c is 6.5. Thus, b gets eliminated, after which a wins the majority vote against c. Thus, the winner changed from c or c' to a, which is not allowed by independence of clones.

Proposition 4.1

Approval-IRV is independent of clones, but Split-IRV is not.

Respecting Cohesive Majorities

Another characteristic property of linear-order IRV, often known as the *majority criterion*, is that if a majority of voters places some candidate first in their ranking, then that candidate should win (see Section 2.4.2). There are several ways to define such a condition for weak orders. A weak version is *respect for unanimous majorities* (Brandl and Peters, 2022), which says that if a majority of voters rank the same set T of candidates as their first indifference class, then only candidates in T should be among the winners. This is satisfied by both Approval-IRV and Split-IRV.

	Approval-IRV	Split-IRV
Respect for unanimous majorities	✓	✓
Respect for cohesive majorities	✓	✗
Select some majority candidate	✗	✗

Table 4.1: Comparison of majority properties satisfied by the rules.

47%	4%	25%	24%
a b	a	c	d
c	b	b	b
d	c	d	c
	d	a	a

Figure 4.9: A problem with electing majority candidates.

An axiom that is intuitive but undesirably strong is *select some majority candidate*, which demands that if some candidate $c \in C$ is a majority candidate (i.e., is in the top indifference class of a majority of voters), then *all* winning candidates must be majority candidates (noting that there could be several). To see why this might be undesirable, consider the profile in Figure 4.9 where this axiom would demand that a is the winner. However, candidate b is strictly preferred over a by 49% of voters, while only 4% of voters have the opposite strict preference. Thus, there is a good argument that b should be the winner, and indeed it is the winner under Approval-IRV (it is also the Condorcet winner). In contrast, Split-IRV selects a. In general, both rules fail this axiom.

We propose an axiom called "respect for cohesive majorities" that logically lies between these two axioms (see Table 4.1). It says that if there is a majority of voters who rank some candidate c on top (so they are "cohesive"), possibly among others, then the winning candidate must be ranked top by at least one member of that majority. For instance, in the profile in Figure 4.9, the axiom would demand that the winning candidate is either a or b.

Given a profile P, write $\text{top}_i = \{c \in C : c \succcurlyeq_i d \text{ for all } d \in C\}$ for the top candidates of voter i.

Respect for Cohesive Majorities

A voting rule f *respects cohesive majorities* if for all profiles P and all subsets of voters $S \subseteq V$ such that $|S| > n/2$ and $\bigcap_{i \in S} \text{top}_i \neq \emptyset$, we have $f(P) \subseteq \bigcup_{i \in S} \text{top}_i$.

Intuitively, if a group of $n/2 + t$ voters is a cohesive majority that jointly ranks c on top, then the group has a "default claim" that c should be elected. If a rule wants to override this claim, the axiom says that it must choose some candidate x that at least t voters in that group think is at least as good as c (and that they hence rank on top). Making this choice justifiably overrides the claim because at most $n/2$ voters remain in the group, who do not form a majority on their own.

Figure 4.10 shows an example profile where a majority of voters (19 out of 37) are cohesive as they agree on candidate a. Therefore, the axiom demands that the winning candidates are either a, b, or c, but not d. On the shown profile, d is the winner under Split-IRV, which therefore fails our axiom. Approval-IRV selects a, consistently with our axiom.

Respect for cohesive majorities is inspired by proportionality axioms, and is actually a special case of the "generalized PSC" axiom for multi-winner rules introduced by Aziz and Lee (2020) (adapted to the single-winner setting). Our axiom also implies the "weak defensive strategy criterion" (Ossipoff, 2000), which says that if a majority of voters strictly prefers a over b, then there should be votes that they can submit that ensure that b loses. The submitted votes should be *sincere* (they only differ from the true preferences by including extra indifferences). Respect for cohesive majorities implies this property, since each member of the majority can report all candidates they strictly prefer to b as their top candidates, and they would thus form a cohesive group.

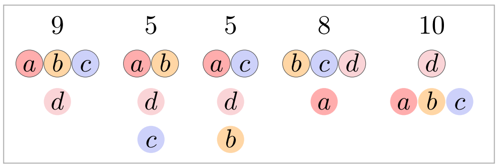
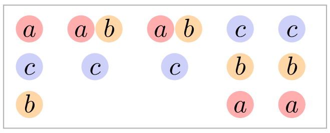


Figure 4.10: Split-IRV violates respect for cohesive majorities because it first eliminates a, then b and c, and elects d.

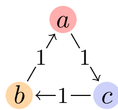
As we mentioned, Split-IRV fails to respect cohesive majorities. In addition, all standard Condorcet extensions fail the axiom, as the following result implies (see Section 2.3.1 for definitions):

Proposition 4.2
No C2 voting rule respects cohesive majorities.

Proof. Consider the following profile:



with weighted majority graph



Without loss of generality, because the voting rule is C2, we may assume c is a winner in this profile (otherwise we can rename candidates in the profile but retain the same weighted majority graph). However, a majority of voters places a in top position and none of these voters puts c in top position. Therefore, by respect for cohesive majorities, c must not be a winner, a contradiction. □

On the other hand, Approval-IRV does satisfy the axiom.

Proposition 4.3
Approval-IRV respects cohesive majorities, but not Split-IRV.

Proof. Let S be a subset of more than $n/2$ voters who all rank candidate c top. We need to show that the winner under Approval-IRV is ranked top by some voter in S. If c is the winner, we are done. Otherwise, consider the round at which c gets eliminated. At that time, a majority of voters (at least S) ranks c on top, so it has approval score of more than $n/2$. Yet because it is about to be eliminated, this is the lowest score. Therefore, every remaining candidate x is ranked top by a majority of voters. Since two strict majorities must intersect, it follows that there is some voter $i \in S$ who ranks x top at this time, and because c is not yet eliminated, i must have ranked x top from the start (as x and c are in the same indifference class). Therefore, at this time all remaining candidates are top candidates for some voter in S, and hence this will also be true of the eventual winner. □

Characterization

We have seen that Approval-IRV satisfies independence of clones and respects cohesive majorities, two properties that are characteristic of IRV in the linear-order context. Split-IRV fails both axioms. We will now prove that Approval-IRV is actually characterized by these two properties within the class of runoff scoring rules.

Freeman et al. (2014a, Theorem 1) previously characterized linear-order IRV to be the only linear-order runoff scoring rule satisfying independence of clones. However, we cannot use their result since they interpreted IRV as a social welfare function that outputs a *ranking* of candidates (the elimination order) and not a single winner. In this setting, independence of clones is a much stronger property. Indeed, Freeman et al. (2014a) write that they "do not know whether other nontrivial runoff scoring rules would satisfy the property" when looking at social choice functions, because their "proofs relied heavily on being able to alter *some* position in the ranking", while in our characterization we need to reason about the set of final winners.

Theorem 4.4
When there are at least $m \geqslant 4$ candidates, Approval-IRV is the only runoff scoring rule that satisfies independence of clones and respects cohesive majorities.

The proof is quite involved, since we need to characterize the scoring vector for every possible order type. The goal is to show that if a runoff scoring rule satisfies independence of clones and respects cohesive majorities, then every order type τ is associated to the score vector $(1, 0, \ldots, 0)$. We first show that it is true for all linear order types $\tau_{1\ldots1}$. Then, we show it for all order types on 3 and 4 candidates. We then use a variety of induction steps to deduce the result for all dichotomous order types. Finally, another induction characterizes the score vector for all remaining order types.

Proof. Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. We want to show that for all order types τ, their scoring vector is $s(\tau) = (1, 0, \ldots, 0)$. We first prove it for linear order types $\tau = (1, 1, \ldots, 1, 1)$, then for all other order types.

In this proof, if a ballot is indifferent between all candidates, i.e., its order type is of length $|\tau| = 1$, we say that $s(\tau) = (0)$ and no candidates get points. Moreover, in every profile, when we use some $q \in \mathbb{N}$, we assume that q is large enough (e.g., $q > 100$), such that what matters most is the coefficient of q (for instance, we will consider that $2q > q + 5$).

Because there is only one linear order type of length m, we denote $s(m) = s(\tau)$ for $|\tau| = m$. We know that $s(m)_1 > 0$ otherwise $s(m) = (0, 0, \ldots, 0, 0)$ and a is not the only winner in the profile with one ranking where a is first, contradicting respect for cohesive majorities. Thus, we can assume without loss of generality that for all m, $s(m)_1 = 1$ (as there is only one linear order type for each number of candidates). In Lemma 4.5, we prove that $s(\tau) = (1, 0, \ldots, 0)$ for all linear order types τ.

Lemma 4.5
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then any linear order type τ is associated to the scoring vector $s(\tau) = (1, 0, \ldots, 0)$.

Proof. We prove it by induction on the number of candidates m. For $m = 2$ it is true because $\tau_{11} = (1, 1)$ is the only order type, so we can assume it has scoring vector $(1, 0)$. For $m = 3$, we will use two order types: $\tau_{111} = (1, 1, 1)$ and $\tau_{21} = (2, 1)$. Denote x and y such that $s(\tau_{111}) = (1, x, 0)$ and $s(\tau_{21}) = (y, 0)$ with $x \in [0, 1]$ and $y \geqslant 0$. We want to prove that $x = 0$.

We will prove (1) $y \leqslant 1$, (2) if $y < 1$ then $x = 0$ and (3) if $y = 1$ then $x = 0$. This will prove that $x = 0$. Note that in this lemma we will not determine the exact value of y.

Let us first prove that (1) $y \leqslant 1$. Assume for contradiction that $y > 1$. Let $q \in \mathbb{N}$ with $q > 1/(y - 1)$ and consider the following profile P:

$$q + 1 : \{a\} \succ \{b\} \succ \{c\} \qquad\qquad q : \{b, c\} \succ \{a\}$$

By respect for cohesive majorities, a should be the winner. But the scores are $S(a) = q + 1$ and $S(b) \geqslant S(c) = qy$. Since $q > 1/(y - 1)$, a is eliminated first, a contradiction.

Let us now prove that (2) if $y < 1$, then $x = 0$. Assume for contradiction that $y < 1$ and $x > 0$. Let $q \in \mathbb{N}$ such that $q > 1/x$ and $q > 1/(1-y)$ and consider the profile P:

$$q : \{c\} \succ \{b\} \succ \{a\} \qquad q : \{a, b\} \succ \{c\} \qquad 1 : \{a\} \succ \{c\} \succ \{b\}$$

In this profile, a is ranked top by more than half of the voters, so $f(P) \subseteq \{a, b\}$ by respect for cohesive majorities. However, the scores are $S(a) = qy + 1$, $S(b) = qy + qx$ and $S(c) = q + x$. Because $q > 1/x$ we have $S(b) > S(a)$ and because $q > 1/(1 - y)$, we have $S(c) > S(a)$. Thus, a is eliminated first. The scores are now $S(c) = q + 1$ and $S(b) = q$, so c wins, a contradiction. Thus, if $y < 1$, then $x = 0$.

Let us now prove that (3) if $y = 1$, then $x = 0$ again. Assume for contradiction that $y = 1$ but $x > 0$. Let $q \in \mathbb{N}$ such that $q > 1/x$ and $q > 2$ and consider the profile P:

$$q+1 : \{b, c\} \succ \{a\} \qquad q : \{a\} \succ \{c\} \succ \{b\} \qquad 1 : \{a, b\} \succ \{c\} \qquad 1 : \{a\} \succ \{b\} \succ \{c\}$$

In this profile, a is ranked top by more than half of the voters, so $f(P) \subseteq \{a, b\}$ by respect for cohesive majorities. However, the scores are $S(a) = q + 2$, $S(b) = q + 2 + x > S(a)$ and $S(c) = q + 1 + qx$. Since $q > 1/x$, we have $S(c) > S(a)$ and a is eliminated first. Then the scores are $S(c) = q$ and $S(b) = 2$, so c wins, a contradiction.

Therefore, we must have $x = 0$ and $s(\tau_{111}) = (1, 0, 0)$. Let us now prove it for $m = 4$ (i.e. $\tau = (1, 1, 1, 1)$). We already know that $s(4)_1 = 1$. Assume for contradiction that $s(4)_2 = x > 0$. Let $q \in \mathbb{N}$ such that $q > 1/x$ and consider the following profile:

$$q+1 : \{b\} \succ \{c\} \succ \{a\} \succ \{a'\} \qquad q : \{c\} \succ \{a\} \succ \{a'\} \succ \{b\}$$
$$q : \{a\} \succ \{a'\} \succ \{c\} \succ \{b\} \qquad q : \{a'\} \succ \{a\} \succ \{c\} \succ \{b\}$$

In this profile, the scores are $S(c) \geqslant q + (q+1)x$, $S(b) = q+1$ and $S(a) \geqslant S(a') \geqslant q + qx$. Because $q > 1/x$, we have $S(c) > S(b)$ and $S(a') > S(b)$, so b is eliminated first. The scoring vectors are now $(1, 0, 0)$, so the scores are $S(c) = 2q + 1$ and $S(a) = S(a') = q$ so one of a or a' is eliminated, and c wins the majority vote in the final round. Note that in this profile a and a' are clones, so c should also win in the profile in which we remove the clone a' of a. However, in this profile, the scores are $S(b) = q + 1$, $S(c) = q$ and $S(a) = 2q$, so c is eliminated first, a contradiction. This proves that $s(4)_2 = 0$ and thus $s(4) = (1, 0, 0, 0)$.

We now prove by induction that for all m, $s(m) = (1, 0, \dots, 0)$. Assume it is true for some $m \geqslant 4$, let us prove it for $m + 1$. Assume for contradiction that $s(m+1)_2 > 0$. Let $q \in \mathbb{N}$ with $q > 1/s(m+1)_2$ and consider the profile P of linear orders on $C = \{a, a', b, c_1, \dots, c_{m-2}\}$ which contains:

- For each $i \in [1, m-2]$, q linear orders $\{c_i\} \succ \{b\} \succ \{a\} \succ \{a'\} \succ \{c_{i+1}\} \succ \cdots \succ \{c_{i+m-3}\}$ where the subscripts of c_i have to be taken modulo $m - 2$ (for instance, the linear order starting with c_2 ends with c_1).

- $q - 1$ linear orders $\{b\} \succ \{a\} \succ \{a'\} \succ \{c_1\} \succ \cdots \succ \{c_{m-2}\}$.

- q linear orders $\{a\} \succ \{a'\} \succ \{b\} \succ \{c_1\} \succ \cdots \succ \{c_{m-2}\}$.

- q linear orders $\{a'\} \succ \{a\} \succ \{b\} \succ \{c_1\} \succ \cdots \succ \{c_{m-2}\}$.

Note that in this profile, each c_i is ranked last in at least q orders. The scores are

$$\begin{aligned}
S(a) \geqslant S(a') &= q + qs(m+1)_2 + qs(m+1)_3 + q(m-2)s(m+1)_4 \\
S(b) &= (q-1) + q(m-2)s(m+1)_2 + 2qs(m+1)_3 \\
S(c_i) &\leqslant q + (3q-1)s(m+1)_4 + q(m-4)s(m+1)_5
\end{aligned}$$

We know that $s(m+1)_2 \geqslant s(m+1)_3 \geqslant s(m+1)_4 \geqslant s(m+1)_5$, so we can make the following observation for all candidates c_i:

$$\begin{aligned}
S(b) &\geqslant (q-1) + q(m-2)s(m+1)_2 + 2qs(m+1)_3 \\
&= q - 1 + qs(m+1)_2 + q(m-3)s(m+1)_2 + 2qs(m+1)_3 \\
&\geqslant q - 1 + qs(m+1)_2 + q(m-3)s(m+1)_3 + 2qs(m+1)_3 \\
&= q - 1 + qs(m+1)_2 + q(m-1)s(m+1)_3 \\
&= q - 1 + qs(m+1)_2 + q(m-4)s(m+1)_3 + 3qs(m+1)_3 \\
&\geqslant q - 1 + qs(m+1)_2 + q(m-4)s(m+1)_5 + 3qs(m+1)_4 \\
&\geqslant q - 1 + qs(m+1)_2 + q(m-4)s(m+1)_5 + (3q-1)s(m+1)_4 \\
&> q + q(m-4)s(m+1)_5 + (3q-1)s(m+1)_4 \qquad \text{(since } q > 1/s(m+1)_2\text{)} \\
&\geqslant S(c_i).
\end{aligned}$$

Moreover, we have:

$$\begin{aligned}
S(a) \geqslant S(a') &\geqslant q + q(m-2)s(m+1)_4 + (q-1)s(m+1)_3 + qs(m+1)_2 \\
&= q + q(m-4)s(m+1)_4 + 2qs(m+1)_4 + (q-1)s(m+1)_3 + qs(m+1)_2 \\
&\geqslant q + q(m-4)s(m+1)_4 + (3q-1)s(m+1)_4 + qs(m+1)_2 \\
&> q + q(m-4)s(m+1)_5 + (3q-1)s(m+1)_4 \qquad \text{(since } s(m+1)_2 > 0\text{)} \\
&\geqslant S(c_i).
\end{aligned}$$

Therefore, one of the c_i is eliminated first. By induction hypothesis, all scoring vectors are now $(1, 0, \dots, 0)$. The scores are now $S(a) = S(a') = q$, $S(b) = 2q - 1$ and $S(c_i) = q$ for all c_i that are not eliminated. Moreover, the score of any candidate other than b is upper bounded by q as long as b is not eliminated, except for a (or a'), which obtains a score of $2q$ once its clone a' (or a) is eliminated. Therefore, the c_i are successively eliminated until a (or a') and b remain. The scores are $S(a) = 2q$ and $S(b) = q - 1 + q(m-2)$. Because $m \geqslant 4$, $S(b) > S(a)$ and b wins.

Note that in this profile, a and a' are clones. Therefore, by independence of clones, b should also win in the profile in which we remove the clone a' of a. However, in this profile, all scoring vectors are $(1, 0, \dots, 0)$ by our induction hypothesis. The scores are $S(a) = 2q$, $S(c_i) = q$ for all c_i and $S(b) = q - 1$. Thus, b is eliminated first, a contradiction.

This proves that $s(m+1)_2 = 0$. Because we know that $s(m+1)_i \leqslant s(m+1)_2$ for all $i \geqslant 3$, we have $s(m+1) = (1, 0, \dots, 0)$ and the induction hypothesis concludes the proof. □

We now prove the result for all other order types. For this, we proceed in several steps. We first show that some specific order types with length $|\tau| \leqslant 3$ or defined for $m \leqslant 5$ candidates have approval scores as their scoring vector. Then, we proceed by induction to prove the result for all other order types.

First, we prove the results for τ_{21} and τ_{31}.

Lemma 4.6
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau_{21}) = s(\tau_{31}) = (1, 0)$.

Proof. We will show that (1) $s(\tau_{21})_1 \leqslant 1$, (2) $s(\tau_{21})_1 \in [0, 1/2] \cup \{1\}$, (3) $s(\tau_{21})_1 = s(\tau_{31})_1$ and (4) $s(\tau_{21})_1 \geqslant 5/9$. Combining these gives $s(\tau_{21}) = s(\tau_{31}) = (1, 0)$.

We first prove that (1) $s(\tau_{21})_1 \leqslant 1$ using respect for cohesive majorities. Assume for a contradiction that $s(\tau_{21})_1 > 1$. Let $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{21})_1 - 1)$ and consider the profile P:

$$q : \{c, b\} \succ \{a\} \qquad q + 1 : \{a\} \succ \{c\} \succ \{b\}$$

Here, a is in the top indifference class of more than half voters, but the scores are $S(a) = q + 1$ and $S(c) = S(b) = qs(\tau_{21})_1$. Since $q > 1/(s(\tau_{21})_1 - 1)$, $S(c) = S(b) > S(a)$ and a is eliminated first, which contradicts respect for cohesive majorities.

We now prove that (2) $s(\tau_{21})_1 \in [0, 1/2] \cup \{1\}$ using respect for cohesive majorities. Assume for a contradiction that $1/2 < s(\tau_{21})_1 < 1$ and let $q \in \mathbb{N}$ be such that $q > \max(1/(1 - s(\tau_{21})_1), 1/(2s(\tau_{21})_1 - 1))$. Consider the following profile P:

$$q + 1 : \{a\} \succ \{c\} \succ \{b\} \qquad 3q : \{a, b\} \succ \{c\}$$
$$2q : \{b, c\} \succ \{a\} \qquad 2q : \{c\} \succ \{b\} \succ \{a\}$$

In this profile, a is in the top indifference class of more than half of the voters. Respect for cohesive majorities imposes that $f(P) \subseteq \{a, b\}$. The scores are $S(a) = q+1+3qs(\tau_{21})_1$, $S(b) = 5qs(\tau_{21})_1$ and $S(c) = 2q+2qs(\tau_{21})_1$. Since $q > 1/(1-s(\tau_{21})_1)$, we have $S(a) < S(c)$ and since $q > 1/(2s(\tau_{21})_1-1)$ we have $S(a) < S(b)$. Thus, a is eliminated first and c wins the majority vote against b, which contradicts respect for cohesive majorities.

We now prove that (3) $s(\tau_{21})_1 = s(\tau_{31})_1$ using independence of clones. First assume that $s(\tau_{21})_1 < s(\tau_{31})_1$. Let $q, q' \in \mathbb{N}$ such that $qs(\tau_{21})_1 < q' < qs(\tau_{31})_1$. Consider the profile P:

$$q' + 2 : \{c\} \succ \{c'\} \succ \{b\} \succ \{a\} \qquad 2q' : \{b\} \succ \{a\} \succ \{c'\} \succ \{c\}$$
$$q' + 1 : \{c'\} \succ \{c\} \succ \{b\} \succ \{a\} \qquad q' : \{a\} \succ \{b\} \succ \{c\} \succ \{c'\}$$
$$q : \{c, c', a\} \succ \{b\}$$

In this profile, the scores are $S(c) > S(c') > S(a) = q' + qs(\tau_{31})_1 > 2q'$ and $S(b) = 2q'$. Thus, b is eliminated first. The new scores are $S(c) > S(c') = q' + 1$ and $S(a) = 3q'$. Thus, c' and c are eliminated, and a wins. Now, observe that c and c' are clones in P. Therefore, independence of clones imposes that a should also be winning in the profile P' in which we remove c'. In P', the scores are $S(c) > S(a) = qs(\tau_{21})_1 + q' < 2q'$ and $S(b) = 2q'$. Therefore, a is eliminated first, which contradicts independence of clones.

We now assume that $s(\tau_{21})_1 > s(\tau_{31})_1$. Let $q, q' \in \mathbb{N}$ such that $qs(\tau_{21})_1 > q' > qs(\tau_{31})_1$. Using the same profiles P and P' as above, we have that a is eliminated first in P, but in P', b is eliminated first, and a is the winner. This contradicts again independence of clones. We conclude that $s(\tau_{21})_1 = s(\tau_{31})_1$.

We now prove that (4) $s(\tau_{21})_1 \geqslant 5/9$. Assume for a contradiction that $s(\tau_{21})_1 < 5/9$ and consider the following profile P.

$$\begin{aligned} 4 :&\{a,b\} \succ \{d\} \succ \{c\} & 5 :&\{a,c\} \succ \{d\} \succ \{b\} \\ 5 :&\{b,d\} \succ \{c\} \succ \{a\} & 4 :&\{c,d\} \succ \{b\} \succ \{a\} \\ 1 :&\{b\} \succ \{d\} \succ \{c\} \succ \{a\} & 1 :&\{c\} \succ \{d\} \succ \{b\} \succ \{a\} \\ 10 :&\{d\} \succ \{c\} \succ \{b\} \succ \{a\} & 13 :&\{a,b,c\} \succ \{d\} \end{aligned}$$

In this profile, a is on the top indifference class of more than half of the voters, therefore by respect for cohesive majorities we have $f(P) \subseteq \{a,b,c\}$. The scores are $S(a) = 9s(\tau_{211})_1 + 13s(\tau_{31})_1$, $S(c) \geqslant S(b) = 9s(\tau_{211})_1 + 13s(\tau_{31})_1 + 1 > S(a)$ and $S(d) \geqslant 10 + 9s(\tau_{211})_1$. Note that we are only lower bounding as we do not know if $(\tau_{211})_2 = 0$. Since $s(\tau_{31})_1 = s(\tau_{21})_1 < 5/9 < 10/13$, we have that $S(a) < S(d)$. Therefore, a is eliminated first. The new scores are $S(b) = 18s(\tau_{21})_1 + 5$, $S(c) = 17s(\tau_{21})_1 + 6$ and $S(d) = 10 + 9s(\tau_{21})_1$, since $s(\tau_{21})_1 < 5/6$, we have $S(b) < S(d)$, and $S(b) < S(c)$. So b is eliminated. Finally, d wins the majority vote against c. This contradicts respect for cohesive majorities.

Thus, since $s(\tau_{21})_1 \in [0,1/2] \cup \{1\}$ and $s(\tau_{21})_1 \geqslant 5/9 > 1/2$, we can conclude that $s(\tau_{21}) = (1,0)$. Moreover, $s(\tau_{31}) = (1,0)$. □

We now prove the following lemma that will be useful for the next steps.

Lemma 4.7
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau_{211})_1 = 1$.

Proof. We know from Lemma 4.6 that $s(\tau_{21}) = s(\tau_{31}) = (1,0)$, and from Lemma 4.5 that the scoring vector of a linear order type is $(1,0,\dots,0)$.

Let us first prove that $s(\tau_{211})_1 \leqslant 1$. Assume that $s(\tau_{211})_1 > 1$ and take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{211})_1 - 1)$. Consider the following profile P:

$$\begin{aligned} 2q+1 :&\{b,d\} \succ \{c\} \succ \{a\} & 3q :&\{a,b,c\} \succ \{d\} \\ 2q-1 :&\{c,d\} \succ \{b\} \succ \{a\} & q+4 :&\{a\} \succ \{d\} \succ \{c\} \succ \{b\} \end{aligned}$$

In this profile, a is in the top indifference class of more than half of the votes, so by respect for cohesive majorities we should have $f(P) \subseteq \{a,b,c\}$. The scores are $S(a) = 4q+4$, $S(b) > S(c) \geqslant 3q + (2q-1)s(\tau_{211})_1 > 5q-1 > S(a)$ and $S(d) = 4qs(\tau_{211})_1$. Note that we are only lower bounding $S(c)$ and $S(b)$ as we do not know if $s(\tau_{211})_2 = 0$. Since $q > 1/(s(\tau_{211})_1 - 1)$, $S(d) > S(a)$. Thus, a is eliminated first. The new scores are $S(b) > S(c) = 5q - 1$ and $S(d) = 5q + 4$. Thus, c is eliminated next, and d wins the majority vote against b, which contradicts respect for cohesive majorities.

Let us now prove that $s(\tau_{211})_1 \geqslant 1$ with independence of clones. Assume that $s(\tau_{211})_1 < 1$. Take $q \in \mathbb{N}$ such that $q > 1/(1 - s(\tau_{211})_1)$ and consider the profile P:

$$\begin{aligned} 3 :&\{a\} \succ \{c\} \succ \{c'\} \succ \{b\} & q+2 :&\{c,c',a\} \succ \{b\} \\ 4 :&\{b\} \succ \{c\} \succ \{c'\} \succ \{a\} & q+2 :&\{c,c',b\} \succ \{a\} \\ q :&\{a,b\} \succ \{c\} \succ \{c'\} & & \end{aligned}$$

In this profile, $S(b) > S(a) = qs(\tau_{211})_1 + q + 5$ and $S(c) \geqslant S(c') \geqslant 2q+4$. Since $q > 1/(1 - s(\tau_{211})_1)$, we have $S(c) > S(a)$. Thus, a is eliminated first. The new scores are $S(b) = q+4$, $S(c) = q+5$ and $S(c') = q+2$ so c' is eliminated next, then c wins the majority vote against b. Now observe that c and c' are clones in P, so by independence of clones, c should also win in the profile P' without c'. However, in P', the scores are $S(b) > S(a) = 2q+5$ and $S(c) = 2q+4$, so c is eliminated first. This contradicts independence of clones, and proves that $s(\tau_{211})_1 = 1$. ☐

We now prove the result for τ_{12}, τ_{22} and τ_{13}, completing the proof for all dichotomous order types on $m \leqslant 4$ candidates.

Lemma 4.8
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau_{12}) = s(\tau_{22}) = s(\tau_{13}) = (1, 0)$.

Proof. We know from Lemma 4.6 that $s(\tau_{21}) = s(\tau_{31}) = (1, 0)$, and from Lemma 4.5 that the scoring vector of a linear order type is $(1, 0, \ldots, 0)$. We also know from Lemma 4.7 that $s(\tau_{211})_1 = 1$.

We first show that $s(\tau_{12})_1 = 1$. First, we show that $s(\tau_{12})_1 \geqslant 1$ using respect for cohesive majorities. Take $q \in \mathbb{N}$ such that $q > 1/(1 - s(\tau_{12})_1)$ and consider the profile P:

$$q : \{a\} \succ \{b, c\} \qquad\qquad q-1 : \{b, c\} \succ \{a\}$$

In this profile, a is ranked first in more than half of the votes, but the scores are $S(a) = qs(\tau_{12})_1$ and $S(b) = S(c) = q$. By hypothesis on q, $S(b) = S(c) > S(a)$ and a is eliminated first, a contradiction.

Assume now that $s(\tau_{12})_1 > 1$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{12})_1 - 1)$ and consider the following profile P:

$$\begin{aligned} q+2 &: \{a, b\} \succ \{c\} & q &: \{c\} \succ \{a, b\} \\ 4 &: \{b, c\} \succ \{a\} & 3 &: \{a\} \succ \{c\} \succ \{b\} \end{aligned}$$

In this profile, a is in the top indifference class of more than half of the votes, thus by respect for cohesive majorities $f(P) \subseteq \{a, b\}$. The scores are $S(a) = q+5$, $S(b) = q+6$ and $S(c) = 4 + qs(\tau_{12})_1$. Since $q > 1/(s(\tau_{12})_1 - 1)$, $S(c) > S(a)$, thus a is eliminated first, and c wins the majority vote against b. This contradicts respect for cohesive majorities. Therefore, we necessarily have $s(\tau_{12}) = (1, 0)$.

We now show that $s(\tau_{22})_1 = 1$. Assume that $s(\tau_{22})_1 < 1$. Take $q \in \mathbb{N}$ such that $q > 1/(1 - s(\tau_{22})_1)$ and consider the following profile P:

$$\begin{aligned} 3 &: \{a\} \succ \{b\} \succ \{b'\} \succ \{c\} & q+2 &: \{b, b'\} \succ \{c\} \succ \{a\} \\ 4 &: \{c\} \succ \{b\} \succ \{b'\} \succ \{a\} & q &: \{a, c\} \succ \{b, b'\} \end{aligned}$$

Recall that $s(\tau_{211})_1 = 1$ (Lemma 4.7). In this profile, $S(c) > S(a) = qs(\tau_{22})_1 + 3$ and $S(b) = S(b') = q+2$. Since $q > 1/(1 - s(\tau_{22})_1)$, $S(b) > S(a)$ and a is eliminated first, then b' is eliminated, and b wins the majority vote against c. Since b and b' are clones, this means that in the profile P' in which we remove b', b should still win. If we remove b', the scores are $S(c) > S(a) = q+3$ and $S(b) = q+2$, so b is eliminated first. This contradicts independence of clones.

Assume now that $s(\tau_{22})_1 > 1$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{22})_1 - 1)$ and consider the following profile P:

$$3 : \{a\} \succ \{b\} \succ \{b'\} \succ \{c\} \qquad q : \{b, b'\} \succ \{a, c\}$$
$$4 : \{c\} \succ \{b\} \succ \{b'\} \succ \{a\} \qquad q-2 : \{a, c\} \succ \{b\} \succ \{b'\}$$

Again, by Lemma 4.7, $s(\tau_{211})_1 = 1$. In this profile, $S(c) > S(a) = q+1$ and $S(b) = S(b') \geqslant qs(\tau_{22})_1$. Since $q > 1/(s(\tau_{22})_1 - 1)$, $S(b) > S(a)$ and a is eliminated, then b' is eliminated and b wins the majority vote against c. Since b and b' are clones, by independence of clones, b should also win in the profile P' in which we remove b'. If we remove b', the scores are $S(c) > S(a) = q+1$ and $S(b) = q$, so b is eliminated first. This contradicts independence of clones. Thus, we necessarily have $s(\tau_{22}) = (1, 0)$.

We finally prove that $s(\tau_{13}) = (1, 0)$. Assume first that $s(\tau_{13})_1 < 1$. Let $q > 1/(1 - s(\tau_{13})_1)$ and P be the profile:

$$q : \{a\} \succ \{b, c, d\} \qquad q-1 : \{b, c, d\} \succ \{a\}$$

By respect for cohesive majorities, a should be the winner, but in this case a is eliminated first (as $s(\tau_{31}) = (1, 0)$). This proves that $s(\tau_{13})_1 \geqslant 1$. Now assume that $s(\tau_{13})_1 > 1$. Let $q \in \mathbb{N}$ be such that $q > 1/(s(\tau_{13})_1 - 1)$ and consider the following profile P:

$$q+2 : \{a, b, c\} \succ \{d\} \qquad q : \{d\} \succ \{b, c, d\} \qquad 4 : \{a\} \succ \{d\} \succ \{b\} \succ \{c\}$$
$$6 : \{b, c, d\} \succ \{a\} \qquad 1 : \{a, b\} \succ \{c, d\}$$

In this profile, a is in the top indifference class of more than half of the votes, so respect for cohesive majorities implies that $f(P) \subseteq \{a, b, c\}$. The scores are $S(a) = q+7$, $S(b) = q+9$, $S(c) = q+8$ and $S(d) = qs(\tau_{13})_1 + 6$. Because of hypothesis on q, $S(d) > S(a)$ so a is eliminated first. The scores are now $S(d) = q+4$, $S(b) = q+3$ and $S(c) = q+2$. c is eliminated and d wins the majority vote against b. This contradicts respect for cohesive majorities. Therefore, $s(\tau_{13}) = (1, 0)$. □

We now prove the result for τ_{211}.

Lemma 4.9
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau_{211}) = (1, 0, 0)$.

Proof. We know from Lemmas 4.6 and 4.8 that the scoring vector of any order type of length $|\tau| = 2$ for $m \leqslant 4$ candidates is $(1, 0)$, and from Lemma 4.5 that the scoring vector of a linear order type is $(1, 0, \ldots, 0)$. We also know from Lemma 4.7 that $s(\tau_{211})_1 = 1$.

We now show that $s(\tau_{211})_2 = 0$. Assume for a contradiction that $s(\tau_{211})_2 > 0$. Take $q \in \mathbb{N}$ such that $q > 1/s(\tau_{211})_2$ and consider the profile P:

$$q : \{a, b\} \succ \{c\} \succ \{d\} \qquad 2q-2 : \{c, d\} \succ \{a, b\}$$
$$q : \{a, b\} \succ \{d\} \succ \{c\} \qquad 5 : \{b, c, d\} \succ \{a\}$$
$$4 : \{a\} \succ \{c\} \succ \{d\} \succ \{b\}$$

In this profile, a is in the top indifference class of more than half of the vote, so by respect for cohesive majorities we should have $f(P) \subseteq \{a, b\}$. The scores are $S(a) = 2q+4$, $S(b) = 2q+5$, $S(c) = S(d) = 2q+3+qs(\tau_{211})_2$. Since $q > \frac{1}{7}s(\tau_{211})_2$, $S(d) > S(a)$. Thus a is eliminated first.

The new scores are $S(b) = 2q$, $S(c) = 2q + 2$ and $S(d) = 2q - 2$ so d is eliminated next, and c wins the majority vote against b. This contradicts respect for cohesive majorities. We conclude that $s(\tau_{211})_2 = 0$. By combining this with Lemma 4.7, we obtain that $s(\tau_{211}) = (1, 0, 0)$. □

From the previous steps, we obtained that τ_{21}, τ_{31}, τ_{12}, τ_{22}, τ_{13} and τ_{211} are associated to approval score vectors. We continue to focus on specific order types before the induction step. In this step, we prove the result for τ_{112}, τ_{212} and τ_{121}, and thus we would have covered all order types for $m \leqslant 4$ candidates.

Lemma 4.10
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau_{112}) = s(\tau_{212}) = s(\tau_{121}) = (1, 0, 0)$.

Proof. We first show that $s(\tau_{112}) = (1, 0, 0)$. To see that $s(\tau_{112})_1 \geqslant 1$, we can use the proof from Lemma 4.8 for order type τ_{13}, but replacing the order types τ_{13} in the profile by order types τ_{112}.

We now prove that $s(\tau_{112})_1 \leqslant 1$. For this, assume by contradiction that $s(\tau_{112})_1 > 1$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{112}) - 1)$ and consider the profile P:

$$
\begin{array}{rlrlrl}
q+2: & \{a,b,c\} \succ \{d\} & q: & \{d\} \succ \{c\} \succ \{a,b\} & 4: & \{a\} \succ \{d\} \succ \{b\} \succ \{c\} \\
6: & \{b,c,d\} \succ \{a\} & 1: & \{a,b\} \succ \{c\} \succ \{d\} & &
\end{array}
$$

Then, we can again use the same reasoning as in Lemma 4.8 but with τ_{112} instead of τ_{13} to obtain a contradiction.

Finally, we prove that $s(\tau_{112})_2 = 0$. Assume that $s(\tau_{112})_2 > 0$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{112})_2)$ and consider the profile P:

$$
\begin{array}{rlrl}
q: & \{a\} \succ \{b\} \succ \{c,d\} & q: & \{a\} \succ \{c\} \succ \{b,d\} \\
q: & \{a\} \succ \{d\} \succ \{c,b\} & 3q-1: & \{b,c,d\} \succ \{a\}
\end{array}
$$

In this profile a is in the top indifference class of more than half of the vote, thus it should win the election by respect for cohesive majorities. The scores are $S(a) = 3q$ and $S(b) = S(c) = S(d) = 3q - 1 + qs(\tau_{112})_2$. By our hypothesis on q, a is eliminated first, which contradicts respect for cohesive majorities. We conclude that $s(\tau_{112}) = (1, 0, 0)$.

We now show that $s(\tau_{212}) = (1, 0, 0)$.

We first prove that $s(\tau_{212})_1 \geqslant 1$. Assume that $s(\tau_{212})_1 < 1$. Take $q \in \mathbb{N}$ such that $q > 1/(1 - s(\tau_{212})_1)$ and consider the profile P:

$$
\begin{array}{rlrl}
q: & \{a,b\} \succ \{d\} \succ \{c,c'\} & 4: & \{a\} \succ \{c\} \succ \{c'\} \succ \{b\} \succ \{d\} \\
q+3: & \{d,c,c'\} \succ \{a,b\} & 5: & \{b\} \succ \{c\} \succ \{c'\} \succ \{a\} \succ \{d\}
\end{array}
$$

In this profile, the scores are $S(b) > S(a) = qs(\tau_{212})_1 + 4$, $S(d) \geqslant S(c) = S(c') = q + 3$. Since $q > 1/(1 - s(\tau_{212})_1)$, a is eliminated first, then c' and d are eliminated, and c wins the majority vote against b. Note that c and c' are clones in this profile, so c should also win in the profile P' in which we remove c'. In P', we have $S(b) > S(a) = q + 4$ and $S(c) = S(d) = q + 3$, so c and d are eliminated first. This contradicts independence of clones. Therefore, $s(\tau_{212})_1 \geqslant 1$.

We now prove that $s(\tau_{212})_1 \leqslant 1$. Assume that $s(\tau_{212})_1 > 1$. Take $q \in \mathbb{N}$ such that $q >$

$1/(s(\tau_{212})_1 - 1)$ and consider the profile P:

$$q+2: \{d\} \succ \{c\} \succ \{c'\} \succ \{a\} \succ \{b\} \qquad q+1: \{a\} \succ \{c\} \succ \{c'\} \succ \{b\} \succ \{d\}$$
$$q+2: \{b\} \succ \{c\} \succ \{c'\} \succ \{a\} \succ \{d\} \qquad q: \{c, c'\} \succ \{d\} \succ \{a, b\}$$

In this profile, the scores are $S(d) \geqslant S(b) > S(a) = q+1$ and $S(c) = S(c') = qs(\tau_{212})_1$. Because of the hypothesis on q, $S(c) > S(a)$, so a is eliminated first. Then, c', b and d are eliminated and c is the winner. By independence of clones, c should also be the winner if we remove its clone c'. However, in this case, because $s(\tau_{112}) = (1, 0, 0)$, c is eliminated first with score $S(c) = q$. This contradicts independence of clones. Therefore, $s(\tau_{212})_1 = 1$.

We finally prove that $s(\tau_{212})_2 = 0$. Assume that $s(\tau_{212})_2 > 0$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau_{212})_2)$ and consider the profile P:

$$q-2: \{d\} \succ \{a\} \succ \{c\} \succ \{c'\} \succ \{b\} \qquad q-1: \{a\} \succ \{c\} \succ \{c'\} \succ \{b\} \succ \{d\}$$
$$q: \{b\} \succ \{a\} \succ \{c\} \succ \{c'\} \succ \{d\} \qquad q: \{c, c'\} \succ \{d\} \succ \{a, b\}$$

In this profile, the scores are $S(c) = S(c') = S(b) = q$, $S(a) = q-1$, and $S(d) = q-2+qs(\tau_{212})_2$. By our hypothesis on q, we have $S(d) > S(a)$ and a is eliminated first. By independence of clones, this implies that if we remove the clone c' of c, a should not be the winner. However, in this new profile P' without c', the scores are $S(c) = S(b) = q$, $S(a) = q-1$ and $S(d) = q-2$, so d is eliminated first. Then $S(c) = S(b) = q$ and $S(a) = 2q-3$, so c or b is eliminated next. In both cases, a wins the majority vote, which contradicts independence of clones. We can conclude that $s(\tau_{212}) = (1, 0, 0)$.

We now focus on $\tau_{121} = (1, 2, 1)$. To see that $s(\tau_{121})_1 \geqslant 1$, we can use the proof from Lemma 4.8 for order type τ_{13}, but replacing the order types τ_{13} in the profile by order types τ_{112}.

Assume now that $s(\tau_{121})_1 > 1$. Let $q \in \mathbb{N}$ with $q > 1/(s(\tau_{121})_1 - 1)$ and consider the profile P.

$$q+1: \{d\} \succ \{b\} \succ \{a, a'\} \qquad q: \{b\} \succ \{a, a'\} \succ \{d\}$$
$$q+2: \{a, a'\} \succ \{b\} \succ \{d\}$$

In this profile, the scores are $S(a) = S(a') > S(d) = q+1$ and $S(b) = qs(\tau_{121})_1$. By hypothesis on q, we have $S(b) > S(d)$, and d is eliminated first. Then a' (or a) is eliminated and b wins the majority vote against a (or a'). Note that a and a' are clones in P, so b should also be a winner in the profile P' without a'. In P', the scores are $S(a) > S(d) = q+1$ and $S(b) = q$, so b is eliminated first. This contradicts independence of clones.

We now prove that $s(\tau_{121})_2 = 0$. Let $q \in \mathbb{N}$ with $q > 1/s(\tau_{121})_2$ and consider the following profile P.

$$q: \{a\} \succ \{b, c\} \succ \{d\} \qquad q: \{a\} \succ \{b, d\} \succ \{c\} \qquad 2q-1: \{c, d, b\} \succ \{a\}$$

In this profile, a is in the top indifference class of more than half of the voters, so it should be the sole winner by respect for cohesive majorities. The scores are $S(b) > S(d) = S(c) = 2q-1+qs(\tau_{121})_2$ and $S(a) = 2q$. By hypothesis on q, a is eliminated first. This contradicts respect for cohesive majorities and proves that $s(\tau_{121}) = (1, 0, 0)$. □

We now proceed to induction steps. First, we focus on dichotomous orders and show that for all dichotomous orders $\tau = (k, k')$, we have $s(\tau) = (1, 0)$.

Lemma 4.11
Let f be a runoff scoring rule satisfying independence of clones and respect for cohesive majorities. Then, $s(\tau) = (1, 0)$ for all dichotomous orders $\tau = (k, k')$.

Proof. We prove it by induction on $m = k + k'$. We know this is true for τ_{21}, τ_{12}, τ_{13}, τ_{31}, and τ_{22} (Lemmas 4.6 and 4.8), so this is true for $m = 3$ and $m = 4$. More generally, we know that $s(\tau) = (1, 0, \dots, 0)$ for all order types τ on $m \leqslant 4$ candidates, and for τ_{212}.

Assume by induction that it is true up to some $m \geqslant 4$. Let $\tau = (k, k')$ with $k + k' = m + 1 \geqslant 5$. This means that either $k \geqslant 3$ or $k' \geqslant 3$.

Assume first that $k \geqslant 3$, we will show that $s(\tau)_1 = 1$. Assume for a contradiction that $s(\tau)_1 < 1$. Take $q \in \mathbb{N}$ such that $q > 1/(1 - s(\tau)_1)$ and consider the profile P:

$$\begin{aligned}
& & q &: \{c_1, \dots, c_{k-2}, a, a'\} \succ \{b_1, \dots, b_{k'}\} \\
&\forall j \in [1, k'] & q - 1 &: \{b_j\} \succ \{c_1\} \succ \dots \succ \{a\} \succ \{a'\} \\
&\forall i \in [2, k-2] & 1 &: \{c_i\} \succ \{c_1\} \succ \dots \succ \{a\} \succ \{a'\} \\
& & 1 &: \{a\} \succ \{a'\} \succ \{c_1\} \succ \dots \\
& & 1 &: \{a'\} \succ \{a\} \succ \{c_1\} \succ \dots
\end{aligned}$$

In this profile, the scores are $S(a) = S(a') = qs(\tau)_1 + 1$, $S(c_1) = qs(\tau)_1$, $S(c_i) = qs(\tau)_1 + 1$ for $i \in [2, k-2]$ and $S(b_j) = q - 1$ for $j \in [1, k']$. Since $q > 1/(1 - s(\tau)_1)$, we have $q - 1 > qs(\tau)_1$ and c_1 is eliminated first. Thus, c_1 is not the winner. Note that in this profile, a and a' are clones, so in the profile P' without a', c_1 should not be a winner. In P', by induction hypothesis all order types have scoring vector $(1, 0, \dots, 0)$. Thus, the scores are $S(a) = q + 1$, $S(c_1) = q$, $S(c_i) = q + 1$ for $i \in [2, k-2]$ and $S(b_j) = q - 1$ for $j \in [1, k']$. Therefore, b_j are successively eliminated. After this, the score of c_1 is $S(c_1) = q + (q-1)k'$ and the score of all other candidates is $q + 1$. Since $k' > 0$, they are successively eliminated and c_1 is the winner. This contradicts independence of clones.

Assume now for contradiction that $s(\tau)_1 > 1$. Take $q \in \mathbb{N}$ such that $q > 1/(s(\tau)_1 - 1)$ and consider the profile P described above, but with $q + 1$ of $\{b_j\} \succ \{c_1\} \succ \dots \succ \{a\} \succ \{a'\}$ for each j, instead of $q - 1$. In this profile, the scores are $S(a) = S(a') = qs(\tau)_1 + 1$, $S(c_1) = qs(\tau)_1$, $S(c_i) = qs(\tau)_1 + 1$ for $i \in [2, k-2]$ and $S(b_j) = q + 1$ for $j \in [1, k']$. Since $q > 1/(s(\tau)_1 - 1)$, $S(c_1) > S(b_j)$ for all b_j, and some b_j is eliminated first. From this point, all scoring vectors are $(1, 0, \dots, 0)$ by induction hypothesis. The score of c_1 is now $S(c_1) = 2q + 1$ and the score of any other candidate is $q + 1$. As long as c_1 is not eliminated, no candidate can have a score higher than $q + 2$, so c_1 survives until the end, and is the winner of the election. By independence of clones, it should also be the winner in P' in which we remove the clone a' of a. In this profile, all scoring vectors are $(1, 0, \dots, 0)$ by induction hypothesis. The scores are $S(c_1) = q$ and $q + 1$ for all other candidates. Thus, c_1 is eliminated first, which contradicts independence of clones. This implies that $s(\tau)_1 = 1$.

If $k \leqslant 2$ but $k' \geqslant 3$, we use a similar reasoning. Let $q \in \mathbb{N}$, and consider the following profile P:

$$\begin{aligned}
& & q :& \{c_1, \dots, c_k\} \succ \{b_1, \dots, b_{k'-2}, a, a'\} \\
\forall j \in [1, k'-2] & & 2q \pm 1 :& \{b_j\} \succ \{c_1\} \succ \dots \succ \{a\} \succ \{a'\} \\
\text{if } k = 2 & & q+1 :& \{c_2\} \succ \{c_1\} \succ \dots \succ \{a\} \succ \{a'\} \\
& & q :& \{c_1\} \succ \{a\} \succ \{a'\} \succ \dots \\
& & 2q+2 :& \{a\} \succ \{a'\} \succ \{c_1\} \succ \dots \\
& & 2q+2 :& \{a'\} \succ \{a\} \succ \{c_1\} \succ \dots
\end{aligned}$$

If we assume $s(\tau)_1 < 1$, we take $q > 1/(1-s(\tau)_1)$ and the profile in which each order ranking some b_j first appears $2q-1$ times. In this profile, the scores are $S(c_1) = q + qs(\tau)_1$, $S(b_j) = 2q-1$ for all $j \in [1, k'-2]$, $S(a) = S(a') = 2q+2$ and if $k = 2$, $S(c_2) > S(c_1)$. Because of hypothesis on q, $S(b_j) > S(c_1)$ for all b_j and c_1 will be eliminated first. In this profile, a and a' are clones, so in the profile P' in which we remove a', c_1 should not be a winner. In P', the scores are $S(c_1) = 2q$, $S(a) = 2q+2$, $S(b_j) = 2q-1$ for all $j \in [1, k'-2]$ and if $k = 2$, $S(c_2) = 2q+1$. Thus, the candidates b_j are eliminated, then c_2 if $k = 2$. At least two candidates are eliminated since $m \geqslant 4$, and at least one of them is a b_j since $k' \geqslant 3$, so the score of c_1 is now $S(c_1) \geqslant 2q + (q+1) + (2q-1) = 5q$, and the score of a is $S(a) = 4q+4$. c_1 is the winner, which contradicts independence of clones.

If we now assume that $s(\tau)_1 > 1$, we take $q > 1/(s(\tau)_1 - 1)$ and the profile P in which each order ranking some b_j first appears $2q+1$ times. This time, some b_j is eliminated first, giving $2q+1$ additional points to c_1, and all the other candidates b_j and c_2 (if $k = 2$) are eliminated successively, then a' is eliminated, and the scores are $S(a) = 4q+4$ and $S(c_1) \geqslant 5q+1$ for the same reasons as above, thus c_1 is the winner. However, if we remove the clone a' of a, c_1 is eliminated first with the lowest score of $S(c_1) = 2q$. This contradicts independence of clones. We conclude that $s(\tau) = (1, 0)$ for all dichotomous order types τ. □

We now proceed to the ultimate step. Note that for all remaining order types τ where $k = |\tau|$ is the length of the order, one of the following is true: (1) $\tau(1) \geqslant 3$, (2) $\tau(k) \geqslant 3$, (3) there is some $j \notin \{1, k\}$ such that $\tau(j) \geqslant 2$ or (4) there is some $j \notin \{1, k-1\}$ such that $\tau(j) = \tau(j+1) = 1$. Note that any order type of size $k = |\tau| \geqslant 4$ satisfies either (3) or (4). Moreover, we already prove the results for dichotomous order types (Lemma 4.11). The only order types of size $k = 3$ which do not satisfy any of the above conditions are $\tau_{111}, \tau_{211}, \tau_{112}$ and τ_{212}, and we already prove the results for all of these (Lemmas 4.9 and 4.10). Moreover, we also proved the result for τ_{121} and thus for all order types on $m \leqslant 4$ candidates.

We prove the result for the remaining order types by induction on the number of candidates m. Let us assume that the result is true up to some $m \geqslant 4$, let us prove it is true for $m+1$. Let τ be an order type for $m+1$ candidates. We will use a similar idea as in the previous steps to prove that $s(\tau) = (1, 0, \dots, 0)$. Let $q \in \mathbb{N}$ and P be the profile on $C = \{a, a', b, d, c_1, \dots c_{m-3}\}$ containing the following orders:

- q orders of the type τ. We have b in the top indifference class ($b \in C_1$) and d in the last indifference class ($d \in C_k$). a and a' are in a position to be clones, which is possible because τ satisfies one of the 4 conditions detailed above. We either put them both in indifference class $j \in [1, k]$ (if τ satisfies one of the first three conditions) or we put a alone in indifference class $j \notin \{1, k-1\}$ and a' alone in indifference class $j+1$ (if τ satisfies the fourth condition).

- q linear orders $\{b\} \succ \{d\} \succ \cdots \succ \{a\} \succ \{a'\}$.
- $2q \pm 1$ linear orders $\{d\} \succ \{b\} \succ \cdots \succ \{a\} \succ \{a'\}$.
- For all c_j that are in the top indifference class of orders of type τ, $q+2$ linear orders $\{c_j\} \succ \{b\} \succ \cdots \succ \{a\} \succ \{a'\}$.
- For all c_j that are not in the top indifference class of the orders of type τ, $2q+2$ linear orders $\{c_j\} \succ \{b\} \succ \cdots \succ \{a\} \succ \{a'\}$.
- If a and a' are in the top indifference class of the orders of type τ, 2 linear orders $\{a\} \succ \{a'\} \succ \{b\} \succ \ldots$, otherwise $2q+2$ such linear orders.
- If a and a' are in the top indifference class of the orders of type τ, 2 linear orders $\{a'\} \succ \{a\} \succ \{b\} \succ \ldots$, otherwise $2q+2$ such linear orders.

We assume first that $s(\tau)_1 > 1$. Then, we take the profile P with $2q+1$ orders $\{d\} \succ \{b\} \succ \cdots \succ \{a\} \succ \{a'\}$. Moreover, we take $q > 1/(s(\tau)_1 - 1)$. The scores are $S(a) \geqslant S(a') \geqslant 2q+2$, $S(b) = q + qs(\tau)_1$, $S(d) = 2q+1$ and for all c_j, $S(c_j) \geqslant 2q+2$. Because of hypothesis on q, $S(b) > S(d)$. Moreover, $S(a) > S(d)$ and $S(c_j) > S(d)$ for all $j \in [1, m-3]$. Thus, d is eliminated first. From this point all order types have scoring vector $(1, 0, \ldots, 0)$ by induction hypothesis. The new score of b is $S(b) = 4q+1$. As long as b is not eliminated, the score of any other candidate is at most $2q+2$, except for a which get a score $4q+4$ once its clone a' is eliminated. In the end, only a and b remain, with $S(b) \geqslant q+q+2q+1+(m-3)(q+2) \geqslant 5q+3$ (since $m \geqslant 4$) and $S(a) = 4q+4$. Thus, b wins the majority vote. By independence of clones, b should also win in the profile P' without the clone a' of a. By induction hypothesis, all orders have scoring vector $(1, 0, \ldots, 0)$ in this profile. The scores are $S(b) = 2q$, $S(d) = 2q+1$, $S(c_j) = 2q+2$ for all $j \in [1, m-3]$ and $S(a) = 4q+4$. Thus, b is eliminated first, which contradicts independence of clones.

We now assume that $s(\tau)_1 < 1$. Then, we take the profile P with $2q-1$ orders $\{d\} \succ \{b\} \succ \cdots \succ \{a\} \succ \{a'\}$. Moreover, we take $q > 1/(1 - s(\tau)_1)$. In this profile, the scores are $S(a) \geqslant S(a') \geqslant 2q+2$, $S(b) = qs(\tau)_1 + q$, $S(d) = 2q-1$ and $S(c_i) \geqslant qs(\tau)_1 + q + 2 > S(b)$. Because of hypothesis on q, we have that $S(d) > S(b)$. Thus, b is eliminated first. By independence of clones, this implies that in the profile P' without the clone a' of a, b should not be a winner. By induction hypothesis, all orders have scoring vector $(1, 0, \ldots, 0)$ in this profile, and thus the first candidate eliminated is d with score $S(d) = 2q-1$. The score of b is then $S(b) = 4q-1$, the score of a is $S(a) = 4q+4$ and all other candidates have score at most $2q+2$ until b is eliminated. Thus, the c_j are successively eliminated until only a and b remain. The scores are $S(a) = 4q+4$ and $S(b) \geqslant q + q + (2q-1) + (m-3)(q+2) \geqslant 5q+1$ (since $m \geqslant 4$). b wins the majority vote, which contradicts independence of clones. Therefore $s(\tau)_1 = 1$.

We finally show that $s(\tau)_2 = 0$. Assume that $s(\tau)_2 > 0$. Let $q > 1/s(\tau)_2$. If $\tau(1) = 1$, consider the profile P with $C = \{a, c_1, \ldots, c_m\}$ containing:

- For each $i \in [1, m]$, q orders of the type τ with $C_1 = \{a\}$ as the top indifference class and $C_2 = \{c_i, \ldots, c_{i+\tau(2)-1}\}$ as the second indifference class (subscripts should be considered modulo m).
- $mq - 1$ dichotomous orders $\{c_1, \ldots, c_m\} \succ \{a\}$.

In this profile, a appears at the top of more than half of the votes, so by respect for cohesive majorities it should be the sole winner. The scores are $S(a) = mq$ and $S(c_i) \geqslant mq - 1 + qs(\tau)_2 > mq$ by hypothesis on q. Thus, a is eliminated first, which contradicts respect for cohesive majorities.

Now, if $\tau(1) \geqslant 2$, consider the following profile P on $C = \{a, a', d, b, c_1, \ldots, c_{m-3}\}$:

- q orders of type τ with a and a' in the top indifference class C_1, b in the second indifference class C_2 and d in the last indifference class C_k (which is not the second one because τ is not dichotomous).
- $q-2$ linear orders $\{b\} \succ \{d\} \succ \{a\} \succ \{a'\} \succ \dots$
- $q-1$ linear orders $\{d\} \succ \{b\} \succ \{a\} \succ \{a'\} \succ \dots$
- For all c_i that are not in the top indifference class C_1 of the orders with type τ, q linear order $\{c_i\} \succ \{d\} \succ \{b\} \succ \{a\} \succ \{a'\} \succ \dots$

In this profile, the scores are $S(a) = S(a') = q$, $S(b) = q - 2 + qs(\tau)_2$, $S(d) = q - 1$ and $S(c_i) \geqslant q$ for $i \in [1, m-3]$. By our hypothesis on q, $S(b) > S(d)$ and d is eliminated first. By our induction hypothesis, all scoring vectors are now $(1, 0, \dots, 0)$. The score of b is now $S(b) = 2q - 3$ and the score of any other candidate is at most q until b is eliminated. Thus, all candidates are successively eliminated until only b remains and wins. By independence of clones, b should also win in the profile P' in which we remove the clone a' of a. In this profile, b has the lowest score $S(b) = q-2$, so it is eliminated first, which contradicts independence of clones. Therefore, $s(\tau)_2 = 0$.

We proved that $s(\tau)_1 = 1$ and $s(\tau)_2 = 0$, and by definition $s(\tau)_1 \geqslant s(\tau)_2 \geqslant s(\tau)_3 \geqslant \dots \geqslant s(\tau)_k \geqslant 0$, so $s(\tau) = (1, 0, \dots, 0)$. The induction concludes that for all order types τ, we have $s(\tau) = (1, 0, \dots, 0)$, which means that f is Approval-IRV. □

Each step of the proof depends on constructing families of profiles that witness violations of one of the two axioms in the characterization. We found it helpful to use linear programs to obtain such counterexample profiles for particular non-approval scoring systems, and then generalize these examples. To do this, we guess (by iterating) the elimination order of the profile (for independence of clones, of the two profiles) that might lead to a violation of an axiom, and then build an LP that has a continuous variable for each possible weak order, indicating the fraction of the profile(s) made up by voters with that weak order. Since the scoring system and the elimination order are fixed, we can encode the behavior of the runoff scoring rule as linear constraints.

Note that the axioms in the characterization are independent. A runoff scoring rule that respects cohesive majorities but fails independence of clones is the one that is like Approval-IRV except that $s(\tau_{21}) = (1/2, 0)$ (i.e., it flips to Split-IRV when it gets down to $m = 3$).[4] A runoff scoring rule that satisfies independence of clones but fails to respect cohesive majorities is the one that is like Approval-IRV, but with scoring vector $s(\tau) = (1/2, 0, \dots, 0)$ for order types τ such that $\tau(1) \geqslant 2$ or such that $|\tau| = 2$. Note that this rule also generalizes IRV on linear orders and approval voting on dichotomous orders, as Approval-IRV. To see why this rule might satisfy independence of clones, observe that adding a clone does not change the scores of the candidates, except when the candidate that is cloned was alone in the top indifference class. In that case, this candidate or its clone might get eliminated earlier, without changing the relative elimination order of the other candidates. We leave for future work the question of whether there exists a natural rule satisfying both axioms (but not being a runoff scoring rule).

IRV with monotonicity

Our second characterization of Approval-IRV has a somewhat different flavor: we are going to show that it is the unique monotonic way to extend IRV to weak orders. To make this claim precise, we need to be careful since IRV fails most monotonicity properties (which is in fact true for all runoff

[4]For this rule, the proof of Proposition 4.3 can be adapted by adding a special case when $m = 3$.

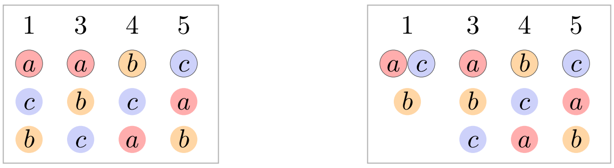


Figure 4.12: Split-IRV violates indifference monotonicity because c wins in the first profile but b is the sole winner in the second profile in which we applied a c-hover.

scoring rules, as shown by Smith (1973)). However, we identify a natural notion of "indifference monotonicity" that is satisfied by Approval-IRV.

Indifference Monotonicity

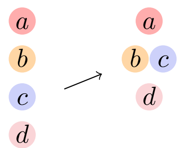


Figure 4.11: Indifference monotonicity: if c is the winner and a voter makes this change, then c stays winning.

Suppose $c \in f(P)$ is a winner given some profile P. Monotonicity requires that if we change the profile P to make c look stronger, then c should still be the winner (see Section 2.4.5). However, if we swap c with a candidate d above it (thereby making c stronger), then under a runoff scoring rule we might now have caused an earlier elimination of d which might ultimately lead to c losing. In a sense, the reason for the failure is that not only did we make c stronger, but we also made d weaker. Our notion of indifference monotonicity considers a very restricted class of changes that make c stronger without negatively affecting any other candidate. Formally, we say that a *c-hover* is the following transformation from one weak order to another:

$$
\begin{aligned}
&C_1 \succ \cdots \succ C_j \succ \{c\} \succ C_{j+2} \succ \cdots \succ C_k \\
\longmapsto \quad &C_1 \succ \cdots \succ C_j \cup \{c\} \succ C_{j+2} \succ \cdots \succ C_k
\end{aligned}
$$

A c-hover starts from a weak order in which c is in a singleton indifference class, and ends in the weak order where c has joined the indifference class just above it (see Figure 4.11). Note that a c-hover cannot be applied to a weak order where c is indifferent with another candidate.

Our indifference monotonicity axiom applies only to changes corresponding to c-hovers.

Indifference monotonicity
A voting rule f is *indifference monotonic* if for every profile P and every $c \in f(P)$, whenever $\hat{P}$ is obtained from P by applying c-hovers to some votes in P, we have $c \in f(\hat{P})$.

Figure 4.12 shows an example in which Split-IRV fails indifference monotonicity. In the first profile P, the scores are 4 for a and b and 5 for c. Thus, b can be eliminated, and c wins the majority vote against a. If we apply a c-hover on the first voter, we get the second profile $\hat{P}$. In this profile, the scores are 5.5 for c, 4 for b and 3.5 for a. Thus, a is eliminated first and b wins the majority vote against c. This contradicts indifference monotonicity. We now show that Approval-IRV satisfies indifference monotonicity.

Proposition 4.12
Approval-IRV is indifference monotonic, but Split-IRV is not.

Proof. Let f be Approval-IRV, let $w \in f(P)$, and let $\hat{P}$ be obtained from P by applying some w-hovers. Suppose that in P, Approval-IRV eliminates candidates in the order $c_1, \dots, c_{m-1}, w$.

We will show that this is also a valid elimination order in $\hat{P}$, which implies that $w \in f(\hat{P})$, as required.

Suppose for a contradiction that this was not the case, and let round $1 \leqslant t \leqslant m-1$ be the first time when in $\hat{P}$, Approval-IRV cannot eliminate candidate c_t (which can be eliminated in round t under P). Let us compare the scores of candidates at this point in P and $\hat{P}$ after the elimination of candidates $c_1, \dots, c_{t-1}$. Note that, by definition of w-hover, every voter has the same current top indifference class under P and $\hat{P}$, except that some voters may additionally have w in their top indifference class under $\hat{P}$.

Thus, all candidates have the same scores under P and $\hat{P}$, except that w may have a higher score under $\hat{P}$. But under P, the score of c_t was lowest (and thus weakly lower than the score of w), and so the same is true in $\hat{P}$, contradicting our assumption that c_t could not be eliminated at this time by Approval-IRV. □

Characterization

We now characterize Approval-IRV using indifference monotonicity. This time, we do not characterize it among all runoff scoring rules, but among the ones that generalize IRV. To this aim, we define the following axiom.

Consistency with IRV
We say that a voting rule f is *consistent with IRV* if for every profile P of linear orders, $f(P)$ is the set of IRV winners.

We can now state our second characterization of Approval-IRV.

Theorem 4.13
Approval-IRV is the unique runoff scoring rule that is consistent with IRV on profiles of linear orders and that satisfies indifference monotonicity.

Consistency with IRV implies that linear orders must have score vector $(1, 0, \dots, 0)$. Observe that any order type τ can be obtained from a linear order by successively applying candidate-hovers, for example $\tau_{111111} \to \tau_{21111} \to \tau_{3111} \to \tau_{312}$. Using indifference monotonicity, this allows us to inductively deduce that every order type has score vector $(1, 0, \dots, 0)$ by constructing counterexample profiles ruling out all other score vectors. The formal proof is provided below.

Proof. Proposition 4.12 already shows that this rule satisfies indifference monotonicity. The consistency with IRV is clear. Let us now show that no other runoff scoring rule satisfies these two axioms. For this, we prove that for all order types $\tau = (\tau(1), \dots, \tau(k))$, the associated scoring vector is $s(\tau) = (1, 0, \dots, 0)$.

First, we use consistency with IRV to show that it is the case for all linear orders. Because there is only one possible linear order type $\tau = (1, \dots, 1)$ for each number of candidates m, we define $s(m) = s(\tau)$ for $|\tau| = m$. We know that $s(m)_1 > 0$ otherwise $s(m) = (0, 0, \dots, 0, 0)$ and a is not the only winner in the profile with one ranking in which a is ranked first. Thus, we can assume without loss of generality that for all m, $s(m)_1 = 1$ (as there is only one linear order type for each number of candidates). We now prove that for all m, $s(m)_2 = 0$.

We prove it by induction on m. It is clearly true for $m = 2$ as $\tau = (1, 1)$ so $s(2) = (1, 0)$. Assume it is true for $m \geqslant 2$, we prove it for $m + 1$.

Assume for contradiction that $s(m+1)_2 > 0$. Let $q \in \mathbb{N}$ with $q > 1/s(m+1)_2$ and consider the profile P on $C = \{a, c_1, \dots, c_m\}$:

- For each $i \in [1, m]$, q linear orders $\{c_i\} \succ \{a\} \succ \{c_{i+1}\} \succ \cdots \succ \{c_{i+m-1}\}$, where the index of the c_i have to be taken modulo m.
- $q-1$ linear orders $\{a\} \succ \{c_1\} \succ \cdots \succ \{c_m\}$.

In this profile, observe that each c_i appears at least q times *last* in a linear order (in the ones in which c_{i+1} is ranked first). With IRV, a is eliminated in the first round, as it has score $S(a) = q-1$ while all other candidates have score $s(c_i) = q$. However, here the scores are $S(a) = (q-1)+mqs(m+1)_2$, and $S(c_i) \leqslant q+(m-1)qs(m+1)_2+q\cdot 0$. We only upper bound because we know that $s(m+1)_i \leqslant s(m+1)_2$ for all $i \geqslant 3$. However, because we assumed $q > 1/s(m+1)_2$, we have $S(a) > S(c_i)$ for all c_i. Thus, one c_i is eliminated, assume c_1 without loss of generality. By induction hypothesis, all scoring vectors are of the form $(1, 0, \ldots, 0)$ from this point. The new score of a is $S(a) = 2q-1$ and the score of all other candidates is upper bounded by q until a is eliminated. Therefore, a is never eliminated, and wins the election, a contradiction. This shows that $s(m+1)_2 = 0$. Since $s(m+1)_i \leqslant s(m+1)_2$ for all $i \geqslant 3$, then the scoring vector of the linear order on $m+1$ candidates is $s(m+1) = (1, 0, \ldots, 0)$.

We can now show that this is true for all order types. The proof is done by induction on the number of candidates m. It is clearly true for $m = 2$ as the only possible order is linear. Assume that it is true up to $m-1$ and let us show it for $m \geqslant 3$.

To show it is the case for all order types on some number of candidates $m \geqslant 3$, we will do another induction, this time on the number of candidates that are not alone in their indifference class (i.e., on $p = \sum_{i:\tau(i)>1} \tau(i)$). If $p = 0$, this means that the order is linear, so we already know that its scoring vector is $(1, 0, \ldots, 0)$. It is not possible to have $p = 1$, as if there is one candidate that is not alone in their indifference class, there is at least another one. Assume it is true up to some $p \geqslant 1$, and let us prove this is also true for $p+1$.

Assume by contradiction that there is an order type $\tau = (\tau(1), \ldots, \tau(k))$ with $\sum_{i:\tau(i)>1} \tau(i) = p+1$ but the scoring vector is such that $s(\tau)_1 \neq 1$ or $s(\tau)_2 > 0$. Let j be the minimal index such that $\tau(j) > 1$. Define the order type $\tau' = (\tau(1), \ldots, \tau(j)-1, 1, \tau(j+1), \ldots, \tau(k))$. Equivalently, we have $\tau'(i) = \tau(i)$ if $i < j$, $\tau'(j) = \tau(j)-1$, $\tau'(j+1) = 1$ and $\tau'(i) = \tau(i-1)$ for $i > j+1$. We have $\sum_{i:\tau'(i)>1} \tau'(i) \leqslant \sum_{i:\tau(i)>1} \tau(i) - 1 \leqslant p$, so by induction on p its associated scoring vector is $s(\tau') = (1, 0, \ldots, 0)$.

Let us assume first that $s(\tau)_1 < 1$. Take $q \in \mathbb{N}$ such that $q > 3/(1-s(\tau)_1)$, and consider the following profile P' with candidate set $C = \{a, b, d, c_1, \ldots, c_{m-3}\}$.

- q orders with order type τ' such that $a \in C_{j+1}$ (alone in its indifference class), $b \in C_1$, and $d \notin C_1$ (this is possible since τ has at least two indifference classes, so τ' has at least three).
- $q+2$ linear orders $\{b\} \succ \{d\} \succ \{a\} \succ \ldots$.
- $2q$ linear orders $\{a\} \succ \{d\} \succ \ldots$.
- $2q-1$ linear orders $\{d\} \succ \{a\} \succ \ldots$.
- For all $j \in [1, m-3]$, $2q$ linear orders $\{c_j\} \succ \{d\} \succ \{a\} \succ \ldots$.

Since τ' satisfies the inductive hypothesis on p and the other orders are linear orders, all order types in this profile are associated with scoring vectors of the form $(1, 0, \ldots, 0)$. Then, the score of every c_j is $S(c_j) \geqslant 2q$. The score of a is $S(a) = 2q$, the score of b is $S(b) = 2q+2$ and the score of d is $S(d) = 2q-1$. Therefore, d is eliminated first. By induction on m, for all the following steps all scoring vectors are also of the form $(1, 0, \ldots, 0)$. Thus, the new score of a is $S(a) = 4q-1$, the one of b is $S(b) = 2q+2$ and the score of all other candidates is $S(c_i) \leqslant 3q$. After each elimination, the score of a increases and the one of all other candidates is always upper bounded by $3q$ as long as a is not eliminated. Thus, a is the winner of this election.

Now, consider the profile P in which we applied a-hover transformation to every order of type τ', and thus obtained orders of type τ (by merging a with the indifference class above it). By indifference monotonicity, a is still a winner in P. In P, the score of b is $S(b) = q + 2 + qs(\tau)_1$, the score of d is $S(d) \geqslant 2q - 1$, the score of a is $S(a) \geqslant 2q$ and the score of all other candidates c_j is $S(c_j) \geqslant 2q$. Because $q > 3/(1 - s(\tau)_1)$, we have $S(d) > S(b)$, so b is eliminated first. Therefore, d get ranked first in $q+2$ additional orders. By induction on m, for all the following steps all scoring vectors are of the form $(1, 0, \ldots, 0)$. The new score of d is $S(d) \geqslant 3q + 1$, while the score of all other candidates is at most $3q$ as long as d is not eliminated. Thus, d is the sole winner instead of a. This shows by contradiction that $s(\tau)_1 \geqslant 1$.

Assume now that $s(\tau)_1 > 1$. Take $q \in \mathbb{N}$ such that $q > (2s(\tau)_1 - 1)/(s(\tau)_1 - 1) > 1/(s(\tau)_1 - 1)$ and denote by P' the following profile.

- $q - 2$ orders with order type τ' with $a \in C_{j+1}$ (alone in its indifference class), $b \in C_1$ and d at the bottom of the order: If $j < k$, then $d \in C_{k+1}$ and if $j = k$, $d \in C_k$ (in the second case, $C_{k+1} = \{a\}$).
- q linear orders $\{b\} \succ \{a\} \succ \{d\} \succ \ldots$.
- $2q$ linear orders $\{a\} \succ \{b\} \succ \ldots$.
- $2q - 1$ linear orders $\{d\} \succ \{b\} \succ \{a\} > \ldots$.
- For all $j \in [1, m-3]$, $2q$ linear rankings $\{c_j\} \succ \{b\} \succ \{a\} \succ \ldots$.

In this profile, all scoring vectors are $(1, 0, \ldots, 0)$. The score of a is $S(a) = 2q$, the score of b is $S(b) = 2q - 2$, the score of d is $S(d) = 2q - 1$ and the score of all c_i is $S(c_i) \geqslant 2q$. Thus, b is eliminated first. Now, by induction on m all scoring vectors are $(1, 0, \ldots, 0)$. The score of a is $S(a) \geqslant 3q$, the score of d is $S(d) \leqslant 3q - 3$ and the score of all other c_j is $S(c_i) \leqslant 3q$. Moreover, since d is in the last possible position in the first $q - 2$ orders (originally with order type τ'), the score of d is necessarily smaller than the one of all c_j. Thus, d is eliminated next, and the score of a is now at least $S(a) \geqslant 5q - 1$. The score of all other candidates c_j is upper bounded $3q$ as long as a is not eliminated. The candidates c_j get successively eliminated and a is the winner.

Consider the profile P obtained from P' by applying a-hover transformations to all orders with type τ' (thus giving an order with order type τ). By indifference monotonicity, a is also a winner in this profile. Note that in the rankings with order type τ, d is always in the last indifference class, therefore getting score 0 from the voters with this order type. In P, the score of b is $S(b) = (q-2)s(\tau)_1 + q$, the score of d is $S(d) = 2q - 1$, the score of a is $S(a) \geqslant 2q$ and the score of all other candidates c_j is $S(c_j) \geqslant 2q$. Since $(2s(\tau)_1 - 1)/(s(\tau)_1 - 1)$, $S(b) > S(d)$, thus d is eliminated first. By induction on m, we can now assume that all order types are associated with scoring vector $(1, 0, \ldots, 0)$. Therefore, b has score $S(b) = 4q - 3$, and a and all c_j all have score upper bounded by $3q - 2$ as long as b is not eliminated. Moreover, the elimination of a or any c_j increases the score of b. Therefore, b wins instead of a in P. This shows by contradiction that $s(\tau)_1 = 1$.

We now show that $s(\tau)_2 = 0$. Assume by contradiction that $s(\tau)_2 > 0$. Take $q \in \mathbb{N}$ such that $q > 1/s(\tau)_2 + 2$ and consider the following profile P'.

- $q-2$ orders with type τ' with $a \in C_{j+1}$ (alone in its indifference class), $b \in C_1$ and d in the second indifference class (a excluded): If $j \neq 1$, then $d \in C_2$ and otherwise $d \in C_3$ (because $C_2 = \{a\}$).
- $q+1$ linear orders $\{b\} \succ \{d\} \succ \{a\} \succ \dots$
- $2q$ linear orders $\{a\} \succ \{d\} \succ \dots$
- $2q-2$ linear orders $\{d\} \succ \{a\} \succ \dots$
- For all $j \in [1, m-3]$, $2q$ linear orders $\{c_j\} \succ \{d\} \succ \{a\} \succ \dots$.

In this profile, all order types are associated with scoring vectors of the form $(1, 0, \dots, 0)$. The score of a is $S(a) = 2q$, the score of b is $S(b) = 2q-1$, the score of d is $S(d) = 2q-2$ and the score of all c_j is $S(c_j) = 2q$. Thus, d is eliminated first. By induction on m, all order types are now associated with scoring vectors of the form $(1, 0, \dots, 0)$. The score of a is now $S(a) = 4q-2$, the score of b is $S(b) = 2q-1$, and all other candidates have score lower than $3q-2$, but greater than $2q$. b is eliminated next, then the candidates c_j. After each elimination (of b or some c_j), the score of a increases, and the score of all other candidates remains bounded by $3q-2$ as long as a is not eliminated. Therefore, a is the winner in P'.

If we apply a-hover transformations (by moving a in the indifference class above) for all orders with type τ', we obtain orders with type τ, and a new profile P. By indifference monotonicity, a should be a winner in P. Observe that d is always in the second indifference class in orders with order type τ. In P, the score of b is $S(b) = 2q-1$, the score of d is $S(d) = (q-2)s(\tau)_2 + 2q - 2$, the score of a is $S(a) \geqslant 2q$ and the score of other candidates c_j is $S(c_j) \geqslant 2q$. Thus, since $q > 1/s(\tau)_2 + 2$, $S(d) > S(b)$ and b is eliminated first. Now, by induction on m, all order types are associated with scoring vectors of the form $(1, 0, \dots, 0)$. The score of d is $S(d) \geqslant 3q-1$. The score of all other candidates (including a) is upper bounded by $3q-2$ as long as d is not eliminated. Therefore, a and the candidates c_j will be successively eliminated and there eliminations can only make the score of d increase, as the other scores will still be upper bounded by $3q-2$. d wins the election in P, which contradicts indifference monotonicity. This proves that $s(\tau)_2 = 0$.

Since by definition of scoring vectors we have $s(\tau)_2 \geqslant s(\tau)_j$ for all $j \geqslant 2$, this implies that the order type τ is associated to the scoring vector $s(\tau) = (1, 0, \dots, 0)$. Induction on p concludes that this is true for all order types on m candidates. Induction on m concludes that it is true for all order types. □

The properties of this characterization result are logically independent. Split-IRV is a runoff scoring rule that is consistent with IRV but fails indifference monotonicity. A runoff scoring rule that satisfies indifference monotonicity but is not consistent with IRV can be constructed by extending the Baldwin rule (defined in Example 2.2). For this, take the runoff scoring rule that associates to each order type τ the Borda-style scoring vector

$$s(\tau) = (m-1, m-1-\tau(1), m-1-\tau(1)-\tau(2), \dots, m-1-\sum_{i=1}^{k-1}\tau(i)).$$

One can use the same proof as in Proposition 4.12 to show that this rule is indifference monotonic. Finally, there are rules that satisfy both properties of the characterization but that are not runoff scoring rules, such as the rule that returns the result of IRV on profiles of linear orders, and the whole set of candidates C when the profile is not linear.

	Approval-IRV	Split-IRV
Independence of Clones	✓[1]	✗
Respect of Cohesive Majorities	✓[1]	✗
Consistency with IRV	✓[2]	✓
Indifference Monotonicity	✓[2]	✗

Table 4.2: Summary of the axiomatic properties of the rules Approval-IRV and Split-IRV. The superscripts [1] and [2] indicate the characterization results.

Summary of the Results

Table 4.2 summarizes the results from the axiomatic analysis. We showed that Approval-IRV can be characterized among runoff scoring rules in two ways: as the only rule that satisfies the natural generalization of the main axioms satisfied by IRV in the ranking setting, namely *independence of clones* and *respect of cohesive majorities*, and as the only rule that generalizes IRV and satisfies *indifference monotonicity*. Split-IRV, in contrast, fails these three axioms.

4.4 Experimental Analysis

Let us now experimentally compare the generalizations of IRV to weak orders using synthetic and real data. We first present the datasets we used, and then detail our experimental results.

Datasets

We tested our rules on a variety of datasets, including synthetic data and real data. In both cases, we used this data to sample profiles of linear orders (rankings). In a second step, we introduced indifferences in these orders to obtain weak orders.

We used three types of models for synthetic datasets. First, we used the impartial culture model (rankings are drawn i.i.d. among all possible rankings). Second, we used a mixture of $k = 4$ Mallows with $\phi = 0.5$ (as a reminder, in a Mallows model, rankings tend to be similar to one "central" ranking). Third, we used d-dimensional Euclidean models for $d \in \{1, 2\}$, where voters and candidates are associated to ideal positions in a metric space and preferences are based on the distances between the voters and the candidates. We sampled their positions uniformly at random in $[0, 1]^d$. We refer to Section 2.5.1 for a more detailed description of these models.

We also used datasets of real preferences. First, we used the 2017 and 2022 *online* datasets of the *Voter Autrement* project, collected during French presidential elections, and in which participants could vote using truncated rankings (Bouveret et al., 2018; Delemazure and Bouveret, 2024). We refer to Section 2.5.4 for more details on *Voter Autrement*. Second, we used datasets of elections held in Dublin, Ireland, in 2002, and for which the IRV method with truncated ballots was used. These datasets are available on PrefLib (Mattei and Walsh, 2013).

In both cases, voters were allowed to only give truncated rankings, for instance by giving only their top-4 candidates, and implicitly reporting indifference between all the others. In our experiments, we only kept the voters who gave full rankings. We also assigned weights to voters in the *Voter Autrement* datasets to reduce selection bias (see Section 2.5.4) Table 4.3 gives the resulting number of voters n and candidates m for each election.

In our experiments, we randomly sample rankings from these datasets, with replacement (based on voters' weights if they exist, otherwise uniformly at random).

	France 2017	France 2022	Dublin Meath	Dublin West
n	5 126	404	3 165	4 810
m	11	12	14	9

Table 4.3: Number of voters n and candidates m for dataset of real preferences.

Introducing Indifferences

All these datasets concern profiles of rankings, so we need a method to turn them into weak orders. We used two such methods.

The *"coin-flip"* method with parameter $p \in [0, 1]$ works as follows: in a ranking $\succ$, for each pair of consecutively ranked candidates a and b, we add a tie between them (and thus put them in the same indifference class) with probability p. For example, for the linear order $a \succ b \succ c \succ d$ we flip 3 independent coins, one for each occurrence of the "$\succ$" symbol, and replace a strict preference by an indifference when the coin comes up heads (which happens with probability p). If the coins come up tails, heads, tails, the resulting weak order is $\{a\} \succ \{b, c\} \succ \{d\}$.

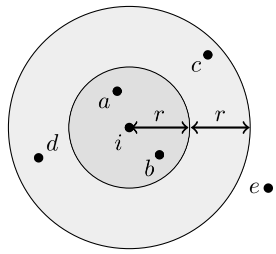


Figure 4.13: A voter $i \in V$ with $\{a, b\} \succ_i \{c, d\} \succ_i \{e\}$.

The *"radius"* method is specific to Euclidean models, in which voters $i \in V$ and candidates $c \in C$ are placed in random locations $p(i), p(c) \in [0, 1]^d$ in the Euclidean space. The method is parameterized by a radius $r \geqslant 0$, which from the perspective of voter $i \in V$ divides the candidates into sets $C_j = \{(j-1)r \leqslant \|p(i) - p(c)\| < jr\}$. This produces the weak order $C_1 \succ_i C_2 \succ_i \cdots \succ_i C_k$ for voter i. Figure 4.13 illustrates this model with an example.

Results

To compare Split-IRV and Approval-IRV, we randomly sampled 10 000 instances from each of our datasets for each value of the parameters p and r. For the coin-flip method, we tested values of p between 0 (introducing no indifferences) and 0.9; we exclude $p = 1$ since this leads to complete indifference. For the radius method, we tested values of r between 0 (no indifferences) and 0.5. Each instance has $n = 500$ voters and $m = 10$ candidates, except for real data where we keep the original number of candidates m (between 9 and 14, see Table 4.3).

Similarities with Ranking-based Rules

First, we looked at the frequency of agreement between the two rules on our datasets. In Figure 4.14, we computed the proportion of profiles in which Approval-IRV and Split-IRV return the same candidate. This proportion is the highest when the parameter of the indifference model p or r is small (as profiles are almost linear), or large. The rules are the most different for intermediate values of p or r, when profiles are not linear, but also not full of indifferences. Finally, the rules agree more frequently on the Dublin dataset as well as for the Impartial culture and the Mallows model. They disagree the most in the Euclidean model.

Then, we looked at the properties of the winners selected by the two rules, and how they could be related to winners of classical voting rules on profiles of rankings. In particular, we computed the Borda scores of the winners, and checked whether they were the Condorcet winner or the IRV winner in the original linear order profile (before introducing indifferences).

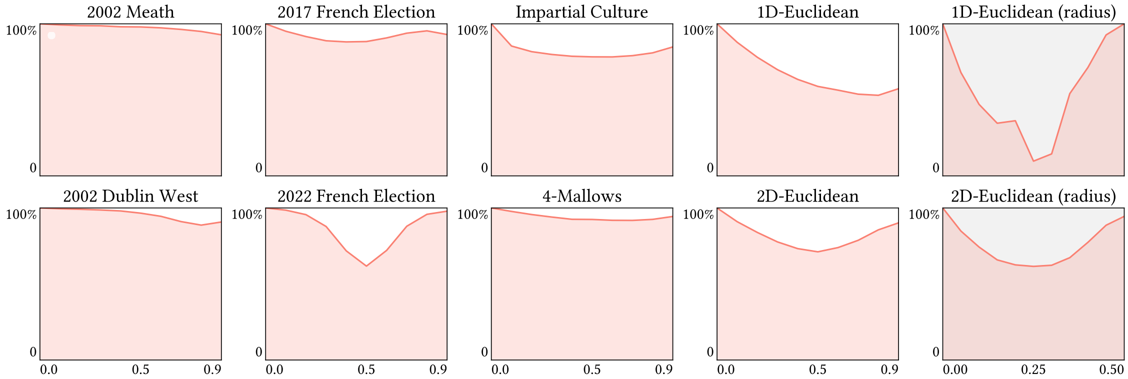


Figure 4.14: Frequency of agreement between Approval-IRV and Split-IRV on our datasets.

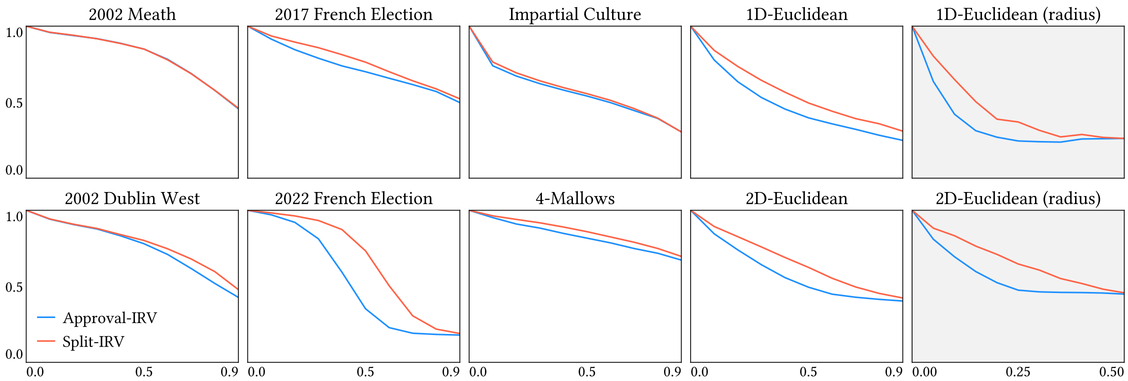


Figure 4.15: Frequency of agreement between the rules and linear-order IRV on our datasets.

In Figure 4.15, we show the proportion of profiles in which Approval-IRV and Split-IRV return the candidate who is the IRV winner in the corresponding profile of rankings. This value decreases when the parameter p or r of the indifference model increases. This is expected: by creating more indifferences, we are going further away from the original linear order profile. In particular, when $p = 0$ or $r = 0$, the frequency of agreement with IRV is obviously 1. For all of the datasets we studied, and for all values of the parameters p and r, Split-IRV agrees more frequently than Approval-IRV with linear-order IRV. Intuitively, this can be related to the alternative definition of Split-IRV that we described in the introduction, which says that we first replace all weak orders in the profile by all the possible linear orders that are compatible with these weak orders, and then run the IRV rule on this profile of linear orders.

In Figure 4.16, we compute the proportion of profiles in which each rule selects the Condorcet winner of the corresponding linear-order profile (before indifferences are introduced). The figure also shows the proportion of profiles in which a Condorcet winner exists. The evolution of this frequency of selecting the Condorcet winner for each rule heavily depends on the dataset: in some it decreases (e.g. the Dublin datasets), in some it increases (e.g. the Euclidean datasets with the coin-flip model for indifferences), and in others it is unclear (e.g. the Euclidean datasets with the radius model for indifferences, or the 2017 French election). However, for all of the datasets, the frequency of Condorcet winners is consistently higher for Approval-IRV than for Split-IRV.

In Figure 4.17, we compute the average Borda score of the winner where the Borda score is computed with respect to the original profile of linear orders. Here, we normalized the Borda

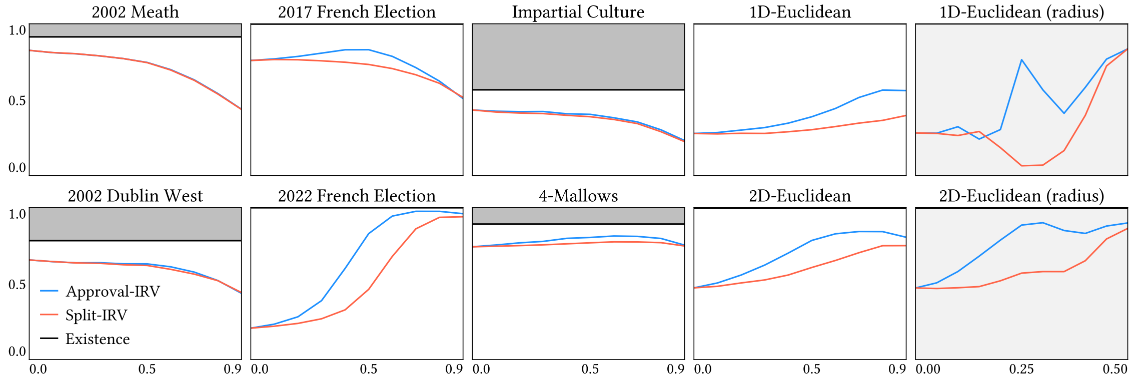


Figure 4.16: Frequency of finding the Condorcet winner, and frequency of such candidate existing on our datasets.

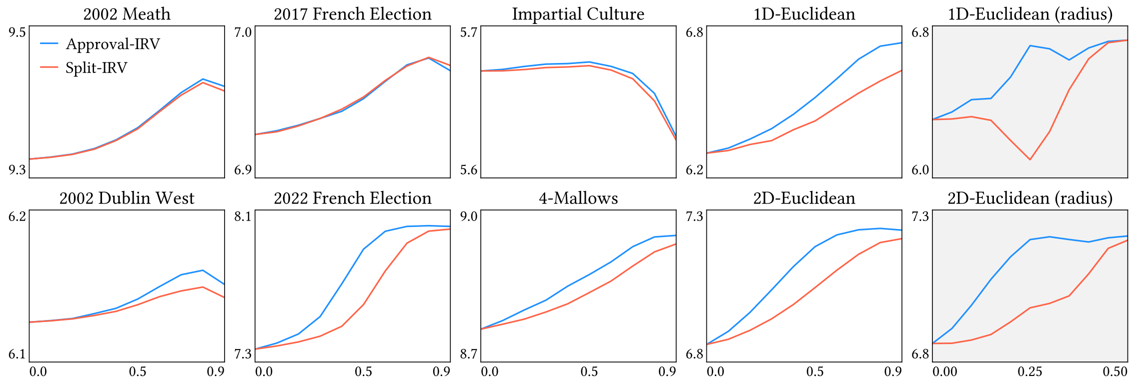


Figure 4.17: Average Borda score of the winner (normalized by dividing by the number of voters n) on our datasets.

score by dividing by the number of voters n in the profile. (Thus, what is actually shown is the average number of Borda points given by the voters). While one might expect that introducing more indifferences (higher p or r) would lead our rules to select less interesting candidates, and thus candidates with lower Borda score, in fact the opposite is the case. In almost all datasets, the Borda score of the candidate selected by linear-order IRV in the original profile ($p = 0$ or $r = 0$) is lower on average than the Borda score of the candidate selected by Approval-IRV and Split-IRV (The main exception is impartial culture, for which the Borda score of the winner is going down when p increases). We believe that the reason for this is that linear-order IRV depends mostly on plurality scores, whereas the weak orders allow the rules to identify candidates that are frequently ranked in high positions, but not necessarily first. Figure 4.17 also shows that if the difference between Approval-IRV and Split-IRV is very small, the candidate selected by Approval-IRV has on average higher Borda score than the one selected by Split-IRV, consistently across datasets.

We checked the robustness of this observation on various distributions over profiles of rankings, using the *map of elections* framework (Boehmer et al., 2022b, Figure 1(a), 10×50 isomorphic swap). Figure 4.18 shows a dot for each profile of the map, which is colored according to the difference in Borda score between the winner of Approval-IRV and Split-IRV, where we used the coin-flip method to transform the linear-order profile into a weak order one. For each dot, we averaged over random weak order profiles sampled with the coin-flip method for each value of

$p \in \{0.1, 0.2, \ldots, 0.9\}$ (giving 450 profiles per dot). In order to highlight the differences, we set the color scale to range from -100 to 100, though some datasets have a difference higher than 2 000.

We observe that the Borda score of the Approval-IRV winner is generally higher than that of the Split-IRV winner (blue dots), especially for structured preferences like single-peaked ones. For profiles close to those drawn from impartial culture, the difference is less pronounced (similar to what we see in Figure 4.17), and some datasets show a small advantage for Split-IRV. Profiles with little difference between voters (those close to ID) show almost no difference between the two rules, as there is often a clear winner that both rules select.

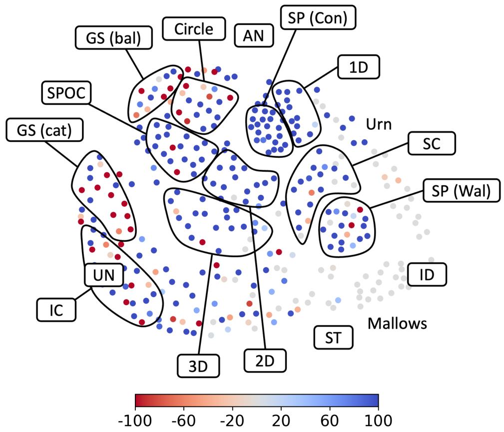


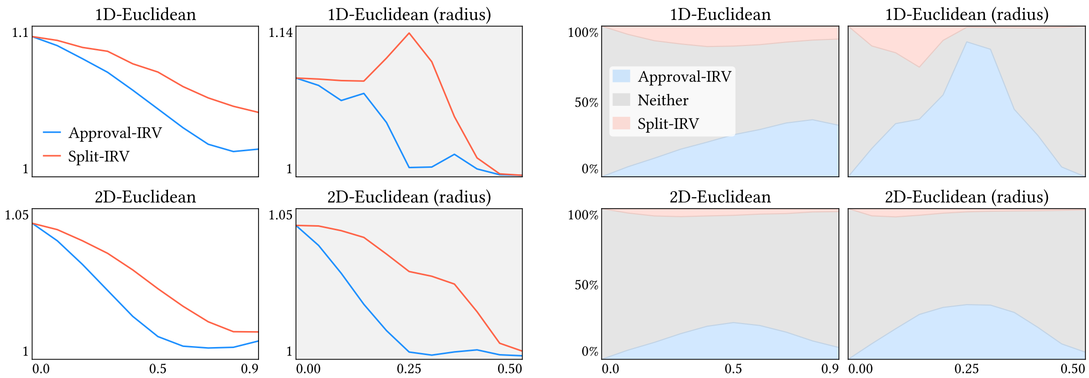
Figure 4.18: Map of elections, showing the difference in Borda score between the Approval-IRV and Split-IRV winners in the coin-flip model, with blue dots indicating that Approval-IRV selected on average a winner with higher Borda score.

Note that the frequency of agreement with IRV is not particularly a measure that a rule is well-behaved: when we are adding indifferences to the profile, this changes the preferences of the voters. In other words, this does not mean that we lost information. We might actually gain more information, since some preferences can only be expressed using indifferences. Since we changed the preferences of the voters, it is not surprising to obtain a different result. Similarly, the Borda score and the frequency of Condorcet winners are not measures that a rule is well-behaved either. However, this shows that while Split-IRV tends to behave more like IRV and to follow the *majoritarian* principle, Approval-IRV, as a mix of approval voting and IRV, tends to select more *consensual* winners, like Borda and Condorcet extensions. A typical example of this phenomenon is the *Voter Autrement* dataset of the 2022 election, for which IRV and Borda disagree on the winner (Borda selects the ecologist *Yannick Jadot* in 82% of the sampled profiles, while IRV selects *Emmanuel Macron* in 68% of them). Both Approval-IRV and Split-IRV select the IRV winner when there are no indifferences ($p = 0$) and the Borda winner when there are a lot of indifferences ($p = 0.9$), and we can see for which value of p each rule is shifting from one to the other. For Approval-IRV, this occurs for $p \approx 0.4$, while for Split-IRV, this occurs for $p \approx 0.6$. In a sense, Split-IRV "sticks" to classical IRV a bit longer than Approval-IRV.

Average Distortion

For Euclidean datasets, we also computed the average *metric distortion* (Anshelevich et al., 2018) induced by the winner selected by the rules, using the method detailed in Section 2.5.2. As a reminder, the distortion of a rule f on a profile P based on an Euclidean model is defined as the cost of the winner (i.e., the sum of the distances from this candidate to the voters) divided by the cost of the cost-optimal candidate. The distortion of any candidate w is lower bounded by 1, and the lower it is, the closer it is to optimal. Based on this definition, we compute the *average distortion* over the 10 000 profiles for the Euclidean datasets.

Figure 4.19 (a) shows the average distortion of Approval-IRV and Split-IRV winners for the Euclidean datasets with the coin-flip method and the radius method of adding indifferences. In both cases, the distortion is decreasing when the parameters p and r are increasing, implying that the average welfare of the winner gets higher the more we add indifferences in the profile, and the



(a) Average distortion caused by the winner.

(b) Rule with lowest distortion.

Figure 4.19: The average distortion caused by the winner (Figure a) and the frequency of returning the candidate with the lowest distortion for each rule (Figure b).

average distortion gets very close to the optimal distortion 1 for large p or r. It is particularly true for the datasets using the radius method, in which indifferences have meaning in terms of Euclidean distance: a voter is indifferent between two candidates if they are very close to each other. In datasets using the coin-flip method, indifferences are not related to anything from the metric space and appears randomly, but even in this case adding indifferences significantly decreases the average distortion induced by the winner. Moreover, Approval-IRV consistently has better average distortion than Split-IRV: it needs less indifferences than Split-IRV to identify "good" candidates.

Figure 4.19 (b) completes the picture and shows for the same datasets the proportion of sampled profiles for which each rule returns a candidate with a better distortion, and the proportion of profiles in which both rules agree on the winner (in gray). Overall, Approval-IRV selects more often a "better" candidate in terms of distortion, especially in the 2-dimensional Euclidean model.

These results on distortion show that under a utilitarian view, Approval-IRV selects better winners. This conclusion is supported by the fact that it more often returns Condorcet winners and select candidates with higher Borda scores.

4.5 Discussion

In this chapter, we have studied generalizations of IRV to weak orders and have given formal arguments why the "Approval" generalization behaves better than the "Split" generalization.

Which Rule to Choose?

We showed in Section 4.3 that Approval-IRV satisfies the generalizations to weak orders of the main axioms satisfied by IRV, while Split-IRV does not (and in fact, no other runoff scoring rule does). We also showed that it is the only runoff scoring rule extending IRV that satisfies a weak notion of monotonicity. In Section 4.4, we experimentally compared the two rules on various datasets, and showed that Approval-IRV selects better candidates in terms of Borda scores and distortion, while Split-IRV returns candidates that more often correspond to the IRV winner in the original linear order profile. Finally, another argument in favor of Approval-IRV is that the two most successful strands of the electoral reform movement in America are advocates for IRV such as FairVote[5], and

[5] https://fairvote.org/

advocates for approval voting such as the Center for Election Science[6]. Approval-IRV is a rule that combines properties of both rules, and may serve as a compromise position.

Further Work

However, there are additional avenues for comparing Approval-IRV and Split-IRV. For example, we have not looked into strategic aspects, and the two rules may well differ in how often voters can obtain a better outcome by misrepresenting their preferences. Another direction would be to study the (worst-case) utilitarian welfare provided by these rules, for example in the metric distortion model, where the worst-case performance of linear-order IRV has been studied (Anshelevich et al., 2018; Anagnostides et al., 2022). It is also worth mentioning that in the original paper (Delemazure and Peters, 2024), we also define generalizations of the multi-winner rule STV to weak orders, and we show that Approval-STV satisfies generalized PSC, a proportionality axiom introduced by Aziz and Lee (2020), while Split-STV fails it.

As to whether it would be desirable to move from linear-order IRV to Approval-IRV, future work could address this question from several angles. User studies could explore whether voters understand how weak order ballots work, and whether they understand Approval-IRV. Generally, approval-based voting methods suffer from an intuitive (mis)impression that some voters get "more votes" than others by ranking several candidates first, and it would be interesting to determine whether this objection can be answered in an intuitively persuasive way. Models of political ideology (such as strategic candidacy models) could be used to understand the impact on the political landscape of a move to Approval-IRV. Finally, it would be interesting to provide a theoretical foundation for our experimental finding that Approval-IRV selects higher-quality candidates (with respect to Borda scores or distortion) than linear-order IRV.

Implementing Approval-IRV

Assuming there are no ties, computing Approval-IRV can clearly be done in polynomial time, though just like for linear-order IRV the task cannot be efficiently parallelized (Csar et al., 2017), because counting needs to proceed in rounds. Since some jurisdictions (notably Australia) count IRV elections by hand using paper ballots, it makes sense to optimize the computation in terms of the number of times each ballot needs to be handled, which Ayadi et al. (2019) analyze using a query complexity model. It may be worth performing such an analysis for Approval-IRV. It seems that this rule can still be easily counted by hand, as follows: For each candidate, make a stack with all ballot papers placing this candidate in the first indifference class. If a ballot puts $t \geqslant 2$ candidates in the first indifference class, assign the ballot to any one of the t stacks, and additionally take $t-1$ *tokens* (e.g., specially colored pieces of paper) and add one to each of the other $t-1$ stacks. Determine the candidate c with the smallest stack and eliminate it, by throwing away all the tokens in the stack and reassigning the ballot papers in the stack. To reassign, if a ballot paper is indifferent between c and some non-eliminated candidate d, then add the ballot paper to d's stack in exchange for one of the tokens in that stack. Otherwise, we pass to the next highest indifference class of the ballot, assign the ballot paper to one candidate in that indifference class, and add tokens to other stacks as appropriate. Note that Split-IRV does not admit a similar protocol, since we need to update the scores of all top-ranked candidates after each elimination. It would also be interesting to adapt the work of Jelvani and Marian (2022), who proposed methods to identify (possible) IRV winners when not all ballots are known.

[6] https://electionscience.org/

Chapter 5

Rankings in Parliamentary Elections with Threshold

5.1 Introduction

In the previous chapters, we have tackled the single-winning voting problem, focusing on voting systems that are implemented in practice for large-scale elections, and proposing ways to improve these systems by allowing voters to give more expressive preferences when voting. In this chapter, we turn to another voting system, that is also widely used in practice, but for a different type of election: parliamentary elections. In these elections, voters are electing several representatives, who form the parliament. As for single-winner voting, many different systems are being used in practice for parliamentary elections. In France, the country is divided into 577 districts, and in each district the representative is elected using a form of plurality with runoff. In the United Kingdom, the country is similarly divided, but the local representatives are elected using plurality. In many other countries, parliaments are elected using proportional representation.

Proportional representation is typically implemented using lists of candidates, and each voter votes for one of the lists. As these lists are often associated with a party, we will use in this chapter the word *party* to refer to the candidates of the election. Then, the seats in the parliament are allocated to the parties in proportion to the number of votes they received (this corresponds to the apportionment problem, briefly discussed in Section 2.3.2). Proportional systems of this kind are widely used in European and South American countries (but also in some African and Asian countries) to elect the national parliament, and it is the norm for the election of the EU parliament. The specificities of the different systems differ from one country to another, but they overall all agree on the proportional representation objective. On the other hand, systems like the ones in France and the United Kingdom ignore this proportionality aspect, and focus on the local representation and the governability of the parliament.

Many countries that use proportional representation impose an *electoral threshold*, a minimum percentage of votes (usually between 0 and 6%) that is necessary for a party to enter the parliament (Farrell, 2011; Pukelsheim, 2014). Lists that do not gather the required votes get no seats, and the votes for those lists are "lost", or "wasted", and not used to distribute seats in parliament. In this chapter, we explore ways to prevent this phenomenon of wasted votes.

The Threshold and the Lost Votes

Not all jurisdictions that use proportional representation impose a threshold. While no votes are lost in such a system, it comes with the risk of having a fragmented parliament with many parties, which makes forming and maintaining a governing coalition difficult. Infamously, the Weimar Republic (1918–1933) did not use a threshold and saw the number of parties present in the *Reichstag* steadily grow to up to 15 in 1930. This led to political chaos: over 13 years there were 16 governments (only five of which had a majority) and 8 elections. The lack of a threshold and the resulting popular dissatisfaction with the political system are widely seen as one contributing factor to the rise of National Socialism (Falter, 2020), though the magnitude of its influence is disputed (Antoni, 1980). Citing this experience, post-war Germany instituted a 5% threshold in 1953. Limiting the fragmentation of the parliament is also the argument given by the French Constitutional Council to justify the 5% threshold used in the election of the French representatives to the EU parliament.[1]

While thresholds limit the number of parties in the parliament and improve governability, they also have drawbacks. In particular, they can lead to a significant number of wasted votes, which violates the principles of proportionality and equality of votes. Benken (2023) has cataloged the fraction of votes that were lost in German elections since 1970 and finds a steady upward trend (Figure 5.1). His dataset includes the 2022 election in the state of Saarland, where a record of 22% of votes were lost. The 2025 German federal election also illustrates the issue: even though 'only' 13.7% of votes were lost, two parties were just below the 5% threshold, with respectively 4.98% and 4.33% of the votes, and thus received no seats. Another example is the 2019 election of the French representatives to the EU parliament, where the 5% threshold led to 19.8% of wasted votes. The most extreme case of wasted votes happened in the 2002 Turkish general election, in which more than 46% of the votes were lost due to a 10% threshold. This led to the first success of the AK party of Recep Tayyip Erdoğan, which did not leave power since then.

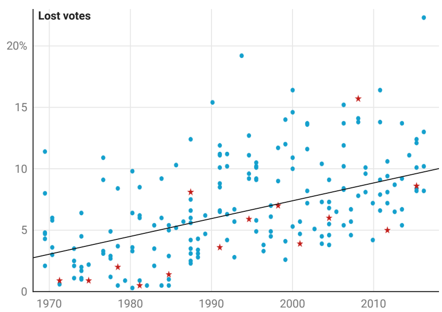


Figure 5.1: The fraction of lost votes in German federal and state elections (shown as red stars and blue circles, respectively) between 1970 and 2022. Reproduced from Benken (2023).

Besides wasting votes and thereby harming representativeness, thresholds also discourage the formation of new parties, hinder the growth of small parties, and require voters to vote strategically (Decker, 2016). In particular, voters need to decide whether to vote for a party that might or might not reach the threshold, or whether to instead vote for a safer party. There is also strategizing on the part of the parties, who sometimes encourage their supporters to vote for allied parties in order to help them to clear the threshold. This is for instance what we observed in the 2024 European election in France: the ecologist party was dangerously close to the threshold according to the polls, and some politicians from other parties encouraged their supporters to vote for it.

Strategic voting behavior in elections using proportional representation systems with or without thresholds have been observed in several empirical studies, including in a 2003 Israel election (Blais et al., 2006), in the 2010 election in Sweden (Fredén, 2014), and in the 2014 elections in Belgium (Verthé and Beyens, 2018), as well as in laboratory studies (Lebon et al., 2018; Fredén, 2016).

[1]Conseil Constitutionnel, Décision n° 2019-811 QPC du 25 octobre 2019, paragraph 10, https://www.conseil-constitutionnel.fr/decision/2019/2019811QPC.htm

Note that once again, the several drawbacks caused by thresholds can be linked to the *independence of clones* axiom, which was central to the discussion in the previous chapters. Indeed, if a party gets a little bit more supporters than required by the threshold, introducing a clone of this party in the election would probably cause some supporters of the original party to vote for the clone instead, increasing the chance none of them reach the threshold.

Second-chance Vote

Thus, naïvely, there appears to be a tradeoff between the problems of a threshold and the risk of political fragmentation. However, there are promising proposals that could alleviate the problems of the threshold without taking away its advantages. In particular, we could elicit additional information from voters regarding their preferences over parties. For example, we could ask voters for a second choice of party. If their first choice of party misses the threshold, their vote is instead counted for the second choice. More generally, we could allow voters to provide a (truncated) *ranking* of the parties, and keep redistributing the vote until we reach a party that met the threshold. Thus, voters would not have to worry that their vote will be wasted, as long as they rank at least one party that meets the threshold.

This idea has been extensively discussed in Germany under the name "replacement vote" (*Ersatzstimme*, sometimes translated into English as "spare vote"). It appears in the election program of one party for the 2025 German parliament election,[2] and laws implementing it have been proposed (but not adopted) in three German states between 2013 and 2015. It is also the main subject of a recent academic edited volume in German language (Benken and Trennheuser, 2023). Elsewhere, the idea has been discussed by the Independent Electoral Review (2023, numbers 4.34 and 4.58) in New Zealand, a committee established by the Justice Minister, which noted the strong support that the proposal received during their consultation, though they recommended to instead lower the threshold to not complicate the voting process.

Party Selection Rules

In the social choice literature, to the best of our knowledge, second chance voting has only been studied from a computational complexity perspective with respect to bribery and control (Laußmann et al., 2024). As we will see, there are many interesting voting-theoretic questions not answered by the high-level description of how to process the voters' second choices or rankings.

To study these questions, we introduce a new framework of *party selection rules*, which take as input a profile of (possibly truncated) rankings over parties and a threshold $\tau \in \mathbb{N}$ (an absolute number of votes), and output a subset of parties: those that will be included in the parliament. For a given selection of parties, a voter is *represented* by their most-preferred party c among the ones in the selection. If the voter did not rank any party in the selection, they are considered *unrepresented*. We require that a selection should be *feasible*, in the sense that each selected party represents at least τ voters. We view such a feasible selection as fully specifying the make-up of a parliament, though in practice we will need to apply an apportionment rule (such as D'Hondt) to determine exactly how many seats each party obtains, as a function of the number of voters it represents. One can conceptualize our formal model as ranking-based multi-winner voting, with a variable number of winners in the output committee, though the literature generally studies this topic with complete rankings (not truncated), and without the threshold condition. We refer

[2] Programm Volt Deutschland zur Bundestagswahl 2025, p. 17 (https://voltdeutschland.org/storage/assets-btw25/volt-programm-bundestagswahl-2025.pdf)

to Section 2.3.2 for a discussion on this particular model. Note that because our model allows truncated rankings, it in particular allows for applications where voters can rank at most two parties, which is the most commonly discussed variant in the threshold context.

As has been recognized in the discussion in Germany, there are at least two possible party selection rules (Benken and Trennheuser, 2023, pp. 52–62). The simplest is what we call the *direct winners only* (DO) rule, which selects exactly those parties who are ranked in first place by at least τ voters, and assigns voters who do not rank any of those parties in first place to their most-preferred party among this selected subset. Thus, under DO, there is just a single round of reassignments. Another option reassigns votes in multiple rounds. We call the resulting rule the *single transferable vote* (STV) due to its close similarity to the multi-winner rule of the same name used in systems that let voters rank candidates instead of parties (see Section 2.3.2). In our model, STV works by repeatedly identifying the party with the fewest first-place votes, and eliminates it from the profile. It repeats this until the set of remaining parties is feasible. We add another party selection rule to the collection: the *greedy plurality* (GP) rule sorts parties by the number of voters placing it in first position, then iteratively adds parties to the selection starting from those with largest score, as long as the addition keeps the selection feasible, and skipping the parties for which it is not the case. Finally, we consider two optimization-based rules, that respectively select the subset of parties (among feasible subsets) that maximizes the number of voters represented, and the subset that maximizes the number of voters represented by the party they rank first.

Other Applications of the Problem

Although our main motivation is to design better parliamentary election systems, our formal model of party selection rules applies more generally to any multi-item selection context where each item is required to have enough voter support, where 'voter support' means being the voter's preferred item within the selection. This applies whenever we have to find a clustering of voters, each cluster being associated with some item. For example, consider a university program that needs to select which optional courses to open in a particular year, given that each student will choose their preferred course from the selection, and that a course should be opened only if it is taken by a minimal number of students. Other examples are selecting a set of activities to be organized for a group on a given day, such that every participant will choose one, or ordering a set of dishes for a company lunch, where everyone will eat a portion of their preferred dish out of the selection, given that the caterer will not prepare a dish in small quantities.

In that sense, our model becomes a special case of the group activity selection problem with group size constraints studied by Darmann et al. (2012, 2022), who focus on stability and efficiency notions, their mutual compatibility, and the computational difficulty of finding solutions. The model can also be seen as a type of facility location problem where facilities may only be opened if they serve a minimum number of users. Algorithmic questions about such problems have been studied (Svitkina, 2010; Ahmadian and Swamy, 2012; Li, 2019). However, *upper* bounds on the number of users of a facility are much more common in the literature (see Aziz et al. (2020)). The Monroe multi-winner voting rule (Monroe, 1995), designed for proportional representation, also shares some similarities to our model, since it involves assigning voters to candidates. However, this rule operates with a fixed number of winners, and does not necessarily assign voters to their most-preferred committee member. Two major differences between parliamentary elections and clustering-based settings are that the latter do not involve an apportionment step, and that voters' preferences bear only on the item they are ultimately assigned to, while in parliamentary elections they usually depend on the complete composition of the parliament.

Outline of the Chapter

In the remainder of this chapter, we first formally define in Section 5.2 the model and the voting rules. In Section 5.3, we analyze these rules using the axiomatic method, by defining a variety of properties appropriate for the party selection model with thresholds. For example, we show that DO is the only rule among the ones we study satisfying monotonicity, and we give an axiomatic characterization of DO using a reinforcement-like consistency condition. For STV, we show that it satisfies clone-proofness and represents solid coalitions, and we characterize STV using an independence condition that is quite strong but formalizes a common normative intuition advanced in favor of STV. For GP, we note that it satisfies set-maximality, which is a kind of efficiency axiom. Since avoiding strategic behavior due to the threshold is one of the main motivations of this model, we also introduce a sequence of strategyproofness axioms. We find that STV fails those axioms in the worst case, but are able to establish that DO and GP are strategyproof in a number of politically plausible situations. In Section 5.4, we experimentally compare the rules on a dataset of real preferences that we collected during the 2024 European election that was held in France, motivated by the 2019 election in which a high percentage of votes were wasted. We find that the rules we propose significantly reduce the number of wasted votes without fragmenting the parliament. We also study the strategic behavior of the voters, and find that between 5% and 29% of voters strategically vote for a larger party in the actual election, while choosing a smaller party as their top-ranked party when given the possibility to provide a ranking. Finally, in Section 5.5, we conclude and discuss future work.

5.2 Party Selection Rules

First note that in this chapter, we will use the word *parties* instead of *candidates* when referring to the elements of C, as our main motivation is parliamentary elections. We assume that we have a profile $P = (\succ_1, \dots, \succ_n)$ of *truncated* rankings, where each voter i ranks a subset of parties (see Section 2.2.4). In particular, we write $a \succ_i \emptyset$ to indicate that a party is ranked by voter i. Moreover, for sets of parties $S, T \subseteq C$, we write $S \succ T$ if for all $a \in S$ and $b \in T$, we have $a \succ b$.

We say that a profile is a *full* profile if every voter ranks all parties, and that it is a *uninominal* profile if every voter ranks exactly one party (as in the current system).

Feasible Outcomes

In this chapter, an *outcome* is a (possibly empty) subset of parties $S \subseteq C$. Together, an outcome and a profile jointly define a unique mapping $\mathsf{best}_{S,P} : V \to S \cup \{\emptyset\}$ that assigns every voter i to their most-preferred party $\mathsf{best}_{S,P}(i)$ among those in S, and to the empty set if they do not rank any party in S. We say that $c = \mathsf{best}_{S,P}(i)$ is the *representative* of voter i in S, and that i is *unrepresented* in S if $\mathsf{best}_{S,P}(i) = \emptyset$. We sometimes write that the votes of unrepresented voters are *wasted* or *lost*. For a party $c \in S$, we define the *supporters* of c as the set of voters of whom it is the representative in S, $\mathsf{supp}_{S,P}(c) = \{i \in V : \mathsf{best}_{S,P}(i) = c\} \subseteq V$, and the *score* of c is $\mathsf{score}_{S,P}(c) = |\mathsf{supp}_{S,P}(c)|$. Finally, we define as $\mathsf{share}_{S,P}(c) = \mathsf{score}_{S,P}(c)/(\sum_{x \in S} \mathsf{score}_{S,P}(x))$ the share of representation of each party in the outcome. Thus, any outcome comes with a *share allocation*, that we assume might be used in a later step to apportion the parliament seats among the parties in the outcome. In particular, we assume that the higher the share of representation of a party, the more seats it will get in the parliament.[3]

[3]This assumption will matter in Section 5.3, where we define strategyproofness notions.

Given a profile P and a *threshold* $\tau \in \mathbb{N}$ with $0 \leqslant \tau \leqslant |V|$, an outcome S is *feasible* if every party $c \in S$ has at least τ supporters, that is, $\mathsf{score}_{S,P}(c) \geqslant \tau$ for all $c \in S$. Note that in our model, the threshold is an absolute number of voters rather than a fraction. This choice makes the notation clearer. Clearly, the empty set is feasible, and a subset of a feasible set is feasible. If P is a full profile, then every singleton set $S = \{c\}$ is feasible, because we will then have $\mathsf{score}_{S,P}(c) = n \geqslant \tau$. When $\tau = 0$, every subset of C is feasible. When $\tau = 1$ and we have a full profile, then S is feasible if and only if it does not contain two parties c and c' such that c Pareto-dominates c'. When $\tau = n$ (or more generally $\tau > n/2$) and we have a full profile, then S is feasible if and only if it is a singleton or the empty set. Hence, single-winner voting on full profiles is a special case of our model with $\tau = n$ if we additionally force rules to return a non-empty outcome.

Once the party selection S has been determined, the parliament is made up using an apportionment method, each party $c \in S$ being represented proportionally to $\mathsf{score}_{S,P}(c)$. Our setup abstracts away issues arising due to apportionment: while in practice an outcome will need to be reduced to a fixed number of available parliament seats, we will not study this "second step", trusting that it won't affect our conclusions in interesting ways.

Party Selection Rules

A *party selection rule* is a function f that takes as input a profile P and a threshold τ, and returns a feasible outcome $f(P, \tau)$. Note that in this chapter, we assume that the rules are *resolute*, that is, they always return one outcome only. This assumption makes it easier to analyze the rules in this case. However, all the results we present in this chapter can be extended to irresolute rules with parallel-universe tiebreaking (Conitzer et al., 2009; Freeman et al., 2015), where all possible ways to break ties are considered to obtain the set of outcomes (this is for instance what we did in Chapter 4). To obtain resoluteness in rules, we assume that there exists some fixed tie-breaking order on the parties to decide how to proceed when a rule encounters a tie. We now define the rules that we study in this chapter.

Procedural Rules

This first rule we consider is arguably the most natural one and easy to explain to voters. It selects in the outcome all parties that are the first choice of at least τ voters. In other words, the parties that have a plurality score of at least τ. We call this rule *Direct Winners Only* (DO).

Direct Winners Only
This rule selects the outcome whose support consists of all parties who are ranked in top position by at least τ voters:

$$\mathrm{DO}(P, \tau) = \{c \in C : |\{i \in V : c \succ_i C \setminus \{c\}\}| \geqslant \tau\} = \{c \in C : \mathsf{score}_{C,P}(c) \geqslant \tau\}.$$

A nice way to implement this rule is to do two rounds of election, such that only the parties reaching the threshold in the first round are allowed to participate in the second round. We will discuss this implementation in more detail in Section 5.5.

Note that the uninominal rule that selects the parties above the threshold in actual parliamentary elections with thresholds selects exactly those parties whose plurality score exceeds the threshold. Therefore, as a party selection rule, it coincides with DO. However, because it forces every voter to cast a *uninominal* ballot instead of a ranking, this rule cannot be defined directly within our model (one can also think of it as first transforming the profile by removing everything below the first party ranked by each voter, and then running DO on the resulting uninominal profile). Thus, the uninominal system *ignores* all votes that did not rank one of the selected parties on top, wherea DO reassigns such ballots to the highest-ranked selected party.

This can lead to very different distributions of scores, and thus distributions of seats in the parliament, as a large number of votes may be wasted. For example, in the profile

$$100\colon\ a \qquad\qquad 100\colon\ b \qquad\qquad 99\colon\ c \succ b$$

with threshold $\tau = 100$, the selected parties are $\{a, b\}$. Under the uninominal system, both parties have 100 supporters, while under DO, party b receives the votes of c's supporters, and thus has almost twice as many supporters as a.

One drawback of this rule is that it does not always select some parties that would deserve to be in the outcome because they have enough support. Consider the profile:

$$10\colon\ a \qquad\qquad 10\colon\ b \qquad\qquad 9\colon\ c \qquad\qquad 6\colon\ d \succ c$$

with $\tau = 10$. Here, DO selects $\{a, b\}$ but it could be argued that c should be selected, since it has 15 supporters if we take into account the votes of d's supporters.

We propose two ways to extend the set of selected parties from direct winners: either by successively *eliminating* parties until we reach a feasible outcome, or by successively *adding* parties to the outcome as long as it remains feasible. The first method is directly inspired by the *Single Transferable Vote* (STV) rule, which is used in multi-winner elections, and successively eliminates candidates until an acceptable outcome is reached (see Section 2.3.2).

Single Transferable Vote
This rule starts with the set $S_0 = C$. Then, at each step $k \geqslant 0$, if S_k is feasible, it returns this set. Otherwise, the rule identifies the party $c \in S_k$ which minimizes $\mathsf{score}_{S_k,P}(c)$ (i.e., it is ranked first the least among parties of S_k), and sets $S_{k+1} = S_k \setminus \{c\}$.

The second method proceeds by adding parties to the outcome as long as it remains feasible. We call this rule *Greedy Plurality* (GP).

Greedy Plurality
This rule starts with the empty set and goes over each party in decreasing order of plurality score $\mathsf{score}_{C,P}(c)$, adding it to the outcome set if the outcome remains feasible; otherwise it is skipped.

On uninominal profiles, it is clear that DO, STV and GP all coincide with the uninominal system. However, in the general case, these rules can give different outcomes, as in the following example.[4]

Example 5.1
Let P be the following profile:

$$4\colon\ a \succ b \succ c \qquad 3\colon\ b \succ c \qquad 2\colon\ c \succ b \succ a \qquad 2\colon\ d \qquad 4\colon\ d \succ b$$

with the threshold $\tau = 5$. In this profile, the only party with a plurality score of at least 5 is d, thus DO returns $\{d\}$. If we run the STV rule, we eliminate c first, as it has the lowest plurality

[4]Note that this example can easily be adapted to show that the rules are different even if we allow only two ranked parties per voter.

score, and then a because the c voters are now supporting b, and we obtain the outcome $\{d, b\}$. Finally, if we run the GP rule, we add d first, then we add a since it is the party with the second highest plurality score and $\{d, a\}$ is feasible. However, we do not add b, since $\{d, a, b\}$ is not feasible (a has only 4 supporters), and same for c. Thus, the outcome of GP is $\{d, a\}$.

DO, STV, and GP are polynomial-time computable, as well as easy to understand and implement for parliamentary elections. However, other interesting rules can be defined. In particular, one can consider rules that optimize some objective function over the set of feasible outcomes.

Optimization-based Rules

We now define two rules that optimize some objective function over the set of feasible outcomes. More formally, let ϕ be a function that assigns a real value $\phi(P, S) \in \mathbb{R}$ to each profile P and subset of parties $S \subseteq C$. The optimization rules are of the form $f(P) = \arg\max_{S \subseteq C, S \text{ feasible}} \phi(P, S)$ for some function ϕ. We will consider two rules of this kind: the first one maximizes the number of voters who rank a party of the outcome in first position, the second one maximizes the number of voters who rank any party of the outcome (and thus are represented). We call these rules respectively *Maximal Plurality* (MaxP) and *Maximal Representation* (MaxR).

Maximal Plurality
This rule returns the feasible set S of parties that maximizes the number of voters who rank a party of S in first position. Formally, it is based on:

$$\phi(P, S) = |\{i \in V : \text{for all } c \in C, \mathsf{best}_{S,P}(i) \succcurlyeq c\}| = \sum_{c \in S} \mathsf{score}_{C,P}(c)$$

Maximal Representation
This rule returns the set S of parties that is feasible and that maximizes the number of voters who include at least one party from the set in their ranking. Formally, it is based on:

$$\phi(P, S) = |\{i \in V : \mathsf{best}_{S,P}(i) \succ \emptyset\}| = \sum_{c \in S} \mathsf{score}_{S,P}(c)$$

In case multiple outcomes maximize the objective function, we assume that ties are broken in favor of the set that is lexicographically maximal according to the fixed tie-breaking order on the parties. Note that we could also break ties by maximizing some other objective function, such as the number of voters who rank at least one party of the set in the first two positions, and so on.

Example 5.2
Consider the profile introduced in Example 5.3, still with a threshold $\tau = 5$. The MaxP rule returns $\{d, a\}$, with 10 voters having their top choice selected, and the MaxR rule either selects $\{d, b\}$ or $\{d, c\}$ depending on the tie-breaking, with all voters represented.

We can show that in contrast to DO, STV and GP, the problem of computing the outcome of MaxP and MaxR is NP-hard.

Theorem 5.1
The problem of computing the outcome of MaxP and MaxR is NP-hard.

Proof. We prove it by reduction from the independent set problem. Given a 3-regular graph G (in which every vertex has degree 3), the problem is to find a set of vertices of maximal size such that no two vertices are connected.

Let $G = (U, E)$ be a 3-regular graph. Construct a profile P with party set $C = U$ and such that for each $(u, v) \in E$, we add one voter with preference $u \succ v$ and one voter with preference $v \succ u$. We set the threshold $\tau = 6$. An outcome is feasible if and only if no two vertices of it are connected by an edge in G (as there are exactly 6 voters ranking each vertex, they *all* need to be represented by this vertex). Thus, since every vertex appears in 6 rankings, and 3 times in first position, then the outcome of maximal size is necessarily the outcome of MaxP and MaxR.

Now, assume that there exists an independent set of size k in G, and denote it $S \subseteq U$. Then, since vertices in S are independent, no voter in P ranks two vertices of S. Thus, every vertex represents 6 voters, and the outcome S is feasible. Conversely, if the outcome is feasible, this means that no voter ranks two vertices of the outcome, thus there are no edges between the vertices of the outcome, and the outcome is an independent set of G.

Therefore, there is an independent set of size k in G if and only if there is a feasible outcome of size k in P, and the problem of computing the outcome of MaxP and MaxR is NP-hard.[5] □

In addition to being hard to compute, MaxP and MaxR are probably not appropriate to be used in practice for parliamentary elections, as the way the outcome is selected can hardly be explained to the voters. However, they can be useful in clustering contexts, where it can be more important for voters to be satisfied with the outcome than to understand how it was selected. In particular, for those contexts, it makes sense to maximize the number of voters who are assigned to an element in the outcome.

5.3 Axiomatic Analysis

In this section, we define a set of axioms that we believe are desirable for a party selection rule in the context of proportional representation with thresholds. We then analyze the different rules with respect to these axioms. The proofs that are entirely due to my co-authors are omitted (more specifically Theorems 5.9 and 5.13), and can be found in the original paper (Delemazure et al., 2025b).

Efficiency Axioms

The definition of Pareto efficiency in our model is not completely clear, as this requires reasoning about voters' preferences over outcomes rather than parties. We can define an uncontroversial axiom of *weak efficiency* that forbids rules from selecting the empty outcome unless they are forced to.

Weak Efficiency
A party selection rule satisfies *weak efficiency* if $f(P, \tau) \neq \emptyset$ whenever there exists a non-empty feasible outcome.

A stronger axiom can be defined by interpreting voters to prefer outcomes which contain a party they like more. In this perspective, note that if we have two outcomes S and S' such that $S' \supseteq S$, then $\mathsf{best}_{S',P}(i) \succcurlyeq_i \mathsf{best}_{S,P}(i)$ for every $i \in V$. This motivates the definition of the following axiom.

Set-maximality
A party selection rule satisfies *set-maximality* if $f(P, \tau)$ is inclusion-maximal among feasible

[5] The reduction used in this proof was proposed by Deepseek, prompted by Dominik Peters.

outcomes for every profile P and threshold τ.

Note that set-maximality implies weak efficiency. Note also that while set-maximality leads to selecting representatives that might be more preferred, it may also lead to selecting more parties, which as we discussed may be undesirable: in the context of parliamentary elections, we are searching for a good trade-off between representation and dispersion. Set-maximality ensures in general a better representation, but at the price of a higher dispersion (which thresholds aim at limiting in the first place). GP, MaxP and MaxR satisfy both efficiency axioms, but the other rules do not.

Proposition 5.2
GP, MaxP and MaxR satisfy set-maximality. DO and STV fail weak efficiency.

Proof. For GP, let S be its outcome on some profile, and assume for a contradiction that there is a party $c \notin S$ such that $S \cup \{c\}$ is feasible. Let $S' \subseteq S$ be the set of parties that GP has added to the outcome by the time it considered c. Then $S' \cup \{c\}$ is feasible since $S' \cup \{c\} \subseteq S \cup \{c\}$ and feasibility is preserved under taking subsets. This contradicts that GP did not select c.

For MaxP and MaxR, the result is clear since we are maximizing some form of representation among voters. So if there exists a superset of the outcome that is feasible, that means that we can increase the representation of some voters, which is a contradiction.

For DO and STV, take the profile $P = \{2 : b \succ c, 1 : c\}$ with $\tau = 3$. Both rules return $\emptyset$, even though $\{c\}$ is feasible. □

We can always complete a non-maximally feasible outcome to make it maximally feasible using some completion algorithm. This is for instance what GP is doing, starting with the outcome of DO, but we could do the same by starting with the outcome of STV.

Representation Axioms

We now turn to representation axioms, which guarantee in different ways that groups of voters of size at least τ are represented.

Inclusion of Direct Winners

The most basic axiom requires that all parties that receive enough first-place votes must be winners. This axiom seems essential for political applications, but it also makes sense in other contexts.

Inclusion of Direct Winners
A party selection rule satisfies *inclusion of direct winners* if for every profile P and threshold τ, whenever c is a party such that at least τ voters rank c in top position, then $c \in f(P, \tau)$.

Since DO returns *only* the direct winners, a party selection rule f satisfies this axiom if and only if $\text{DO}(P, \tau) \subseteq f(P, \tau)$ for every profile P and threshold τ. It is easy to see that DO, STV, and GP satisfy this axiom. Indeed, we always have $\text{DO}(P, \tau) \subseteq \text{STV}(P, \tau)$ and $\text{DO}(P, \tau) \subseteq \text{GP}(P, \tau)$. Thus, there are at least as many unrepresented voters under DO as under STV or GP, by the argument seen before introducing set-maximality.

Perhaps surprisingly, MaxP and MaxR fail inclusion of direct winners. Consider for instance the profile $P = \{2 : a \succ b, 2 : a \succ c, 3 : c, 3 : b\}$ with $\tau = 4$. The only direct winner is a, however both MaxP and MaxR return $\{c, b\}$.

Proposition 5.3
DO, STV and GP satisfy inclusion of direct winners, but MaxP and MaxR do not.

Representation of Solid Coalitions

One can strengthen inclusion of direct winners to apply to cases where enough voters support a *set* of parties. For example, suppose that there are three "green" parties and that more than τ voters rank them in the top three ranks, though they may disagree on their relative ordering. The following axiom requires that at least one of the green parties is included in the outcome. It is inspired by the "proportionality for solid coalitions" (PSC) axiom in multi-winner voting (Dummett, 1984), which we discussed in Section 2.4.2.

Representation of Solid Coalitions
A party selection rule satisfies *representation of solid coalitions* if for every profile P and threshold τ, if $T \subseteq C$ is a set of parties such that $|\{i \in V : T \succ_i C \setminus T\}| \geqslant \tau$, then $T \cap f(P, \tau) \neq \emptyset$.

Note that representation of solid coalitions implies inclusion of direct winners (consider singleton T). Thus, it is failed by MaxP and MaxR. It is also failed by DO, since it can happen that none of the parties supported by a solid coalition has enough first-place votes. Actually, among the five rules we are studying, only STV satisfies this axiom.

Proposition 5.4
STV satisfies representation of solid coalitions, but DO, GP, MaxP and MaxR do not.

Proof. For STV, assume that there is a profile P where the axiom is violated, i.e. there exists a set $T \subseteq C$ of parties with $|\{i \in V : T \succ_i C \setminus T\}| \geqslant \tau$, but $T \cap \mathrm{STV}(P, \tau) = \emptyset$. Consider the first step in the execution of STV at which all but one party of T are eliminated. At that point, this party $c \in T$ is ranked first by at least τ voters such that $T \succ_i C \setminus T$. Thus, c cannot be eliminated in the subsequent steps and must therefore be included in the outcome, which is a contradiction.

DO and GP fail the property on the full profile $P = \{4 : a \succ b \succ c, 3 : b \succ c \succ a, 2 : c \succ b \succ a\}$ with $\tau = 5$. The last 5 voters form a solid coalition for $\{b, c\}$. However, we have $\mathrm{DO}(P, \tau) = \emptyset$ since no party is ranked first by at least 5 voters, and $\mathrm{GP}(P, \tau) = \{a\}$ since a is the party who is ranked first by the most voters and any set containing more than one party is not feasible, as $\tau > n/2$.

For MaxP and MaxR, simply observe that the representation of solid coalitions is a stronger property than inclusion of direct winners, which they fail. □

Local Stability

One could strengthen this axiom further by forbidding that there is a party c outside the outcome S for which τ voters prefer c to all parties in S. This is a version of the local stability axiom studied in multi-winner voting (Aziz et al., 2017b; Jiang et al., 2020).

Local Stability
A party selection rule satisfies *local stability* if for every profile P and threshold τ, for all parties $c \notin S = f(P, \tau)$, we have $|\{i \in V : c \succ_i S\}| < \tau$.

Note that this axiom implies representation of solid coalitions, and thus inclusion of direct winners. Unfortunately, this axiom cannot be satisfied for any $\tau \notin \{1, n\}$.

Proposition 5.5
No party selection rule satisfies local stability when $\tau \notin \{1, n\}$.

Proof. For $n \geqslant 3$ and $\tau = n - 1$, we can take a simple profile of $m = n$ parties $c_1, \dots, c_n$ with a Condorcet cycle. This corresponds to the profile $P = (\succ_1, \dots, \succ_n)$ in which $\succ_i = c_{i+1} \succ \dots \succ c_n \succ c_1 \succ \dots \succ c_{i-1}$ for all i. Because the threshold is greater than $n/2$, at most one party can be part of the outcome. However, in a Condorcet cycle every c_i is better ranked than c_{i+1} in $n-1$ rankings (and c_n is better ranked than c_1 in $n - 1$ rankings), thus for any feasible outcome $S = \{c_i\}$, we have $|\{i \in V : c_{i+1} \succ_i S\}| = n - 1 \geqslant \tau$, and the rule fails the axiom. If $S = \emptyset$, the result is even clearer.

For all other $\tau \notin \{1, n\}$, we can take a profile P with $m = \tau + 2$ parties $c_1, \dots, c_{\tau+1}, d$, such that P contains the Condorcet cycle of the example above with $m' = \tau + 1$ parties and $n' = \tau + 1$ voters (so with parties $c_1, \dots, c_{\tau+1}$), and add $n - n'$ dummy voters that only rank d. In this profile, the outcome must contain at most one of the c_i, as only $\tau + 1 < 2\tau$ voters ranked them. However, for any $c_i \in S$, we have $|\{i \in V : c_{i+1} \succ_i S\}| = n' - 1 = \tau$, which breaks local stability. If no c_i is part of the outcome, again the result is even easier to see. □

Note that for $\tau = n$, the rule that selects a party with full support and which is not Pareto-dominated (if such party exists) satisfies the axiom. Similarly, for $\tau = 1$, the rule that selects all parties that are ranked first by at least one voter satisfies the axiom.

Representation of Unrepresented Voters

Proposition 5.5 seems to show that it is too demanding to require that no party outside the outcome should be better for more than τ voters. However, we can weaken the local stability axiom by restricting the voters that can ask for a better representative to the set of *unrepresented* voters. In other words, no party outside the outcome should be included in the (truncated) rankings of τ unrepresented voters.

Representation of Unrepresented Voters
A party selection rule satisfies *representation of unrepresented voters* if for every profile P and threshold τ, for all party $c \notin S = f(P, \tau)$, we have $|\{i \in V : \mathsf{best}_{S,P}(i) = \emptyset \text{ and } c \succ_i S\}| < \tau$.

The counterexample from the proof of Proposition 5.5 would not work anymore, since all voters that are part of the Condorcet cycle are represented in the outcome. However, we can show that this axiom is not satisfied either by any of the rules we introduced.

Proposition 5.6
DO, GP, STV, MaxP and MaxR do not satisfy representation of unrepresented voters.

Proof. To show that DO and STV fail this axiom, consider the following profile $P = \{3 : a, 2 : b \succ c, 1 : c\}$ with $\tau = 3$. In this profile, $\text{DO}(P, \tau) = \text{STV}(P, \tau) = \{a\}$. But the last three voters all ranked c and are unrepresented, thus breaking the axiom.

For GP, consider the profile $P = \{3 : a, 2 : b \succ a, 2 : c \succ b, 2 : d \succ b\}$ with $\tau = 4$. The first party added to the outcome is a, since it is ranked first by the most voters. Then, c and d cannot be added because they are ranked by only two voters each, and b cannot be added because it will reduce the number of supporters of a to only three, which is below the threshold, making the set $\{a, b\}$ unfeasible. Thus, $GP(P, \tau) = \{a\}$, but the last $\tau = 4$ voters all ranked b and are unrepresented, breaking the axiom.

For MaxP and MaxR, consider the profile $P = \{5 : a, 1 : a \succ b, 1 : a \succ c, 4 : b, 4 : c\}$ with $\tau = 5$. The outcome of MaxP and MaxR is $\{b, c\}$, but the first $\tau = 5$ voters all ranked a and are unrepresented, breaking the axiom. □

Still, it is possible to satisfy this axiom, and all our rules can be transformed into rules satisfying it through a local search procedure. The idea is simply to add to the outcome any party that is ranked by at least τ unrepresented voters (potentially removing other parties), as long as such a party exists. Below, we provide an algorithm that implements this idea. We left some freedom in the algorithm, that we will discuss later.

1. Start with any feasible set of parties S_0 (e.g., the outcome of DO, STV or GP).
2. Repeat until there is no party c such that $|\{i \in V : \mathsf{best}_{S_k,P}(i) = \emptyset \text{ and } c \succ_i \emptyset\}| \geqslant \tau$:
   (a) Let c be a party such that $|\{i \in V : \mathsf{best}_{S_k,P}(i) = \emptyset \text{ and } c \succ_i \emptyset\}| \geqslant \tau$.
   (b) Identify all parties c' which do not get enough support anymore if we add c. More formally, this corresponds to the set S_- such that $c' \in S_-$ if $|\{i \in V : \mathsf{best}_{S_k,P}(i) = c' \text{ and } c' \succ c\}| < \tau$.
   (c) Add c to the outcome, and remove all parties from S_-: $S_{k+1} = S_k \cup \{c\} \setminus S_-$.

Example 5.3
Consider the following profile with $\tau = 7$:

$$7\colon a \qquad 6\colon b \succ f \succ a \qquad 6\colon c \succ f \succ a \qquad 1\colon e \succ d \succ c \qquad 1\colon d \succ b \qquad 4\colon e \succ d \qquad 3\colon d$$

The outcome of DO is $\{a\}$, and the outcome of STV and GP is $\{a, b, c\}$. However, the last 7 voters are unrepresented and all ranked d. If we run the algorithm described above starting with $\{a, b, c\}$, we first add d to the outcome. This removes b and c from the outcome, as they now represent only 6 voters each. We obtain $\{a, d\}$, and the algorithm terminates, as all voters are now represented. However, observe that the new outcome is not maximal, as we could add f and still obtain a feasible outcome.

We can show that this rule terminates. If it terminates, this means we cannot find any party satisfying the condition of the axiom in step (2), and thus the representation of unrepresented voters axiom is satisfied. As we mentioned, we left some freedom in the algorithm, as we can choose the initial set of parties S_0 and the way we break ties in the selection of the party to add. For a rule f, we denote f^+ the rule obtained by starting the algorithm with the set $f(P, \tau)$, and breaking ties by selecting the party that is supported by the most unrepresented voters $c = \arg\max_{c' \notin S_k} |\{i \in V : \mathsf{best}_{S_k,P}(i) = \emptyset \text{ and } c' \succ_i \emptyset\}|$.

Proposition 5.7
The algorithm described above terminates.

Proof. To see why it terminates, observe that if it does not, this means that it cycles. Thus, if we denote S_k the outcome obtained at step k, then there exists $k' > k$ such that $S_{k'} = S_k$. For a subset $S \subseteq C$, denote $r(S)$ the number of voters who ranked at least one alternative from S:

$$r(S) = |\{i \in V \mid \mathsf{best}_{S,P}(i) \neq \emptyset\} = \sum_{c \in S} \mathsf{score}_{S,P}(c)$$

We have $r(S_{k'}) = r(S_k)$. Note that at each step we add a party to the outcome such that there exists at least τ unrepresented voters who have ranked this party. If this causes some other parties to be removed from the outcome, this means that these parties only had strictly less than τ supporters putting them above all other parties from the outcome. Thus, by removing a party, we add at most $\tau - 1$ unrepresented voters. Therefore, we have the following for all j:

$$r(S_{j+1}) \geqslant r(S_j) + \tau - (|S_j| + 1 - |S_{j+1}|) \cdot (\tau - 1)$$
$$r(S_{j+1}) - r(S_j) \geqslant 1 - (|S_j| - |S_{j+1}|) \cdot (\tau - 1)$$

Thus, we have

$$\begin{aligned} r(S_{k'}) - r(S_k) &= \sum_{j=k}^{k'-1} r(S_{j+1}) - r(S_j) \\ &\geqslant \sum_{j=k}^{k'-1} 1 - (|S_j| - |S_{j+1}|) \cdot (\tau - 1) \\ &= (k' - 1 - k) - |S_k| + |S_{k'}| = (k' - 1 - k) \end{aligned}$$

since $S_k = S_{k'}$. Moreover, we have $k' > k + 1$, otherwise this means that $S_{k+1} = S_k$, which is impossible since we add a new party at each step. Thus, $r(S_{k'}) - r(S_k) \geqslant 1$, which contradicts the fact that $r(S_{k'}) = r(S_k)$. This proves that there is no cycle and the algorithm terminates. □

Therefore, any rule of the type f^+ satisfies representation of unrepresented voters. However, this comes at the price of failing other axioms that are satisfied by the original rules. For instance, GP$^+$ fails set-maximality while GP satisfies it (see for instance Example 5.3).

Varying the Threshold

We now discuss what should happen to the outcome when the threshold changes. The first axiom says that a losing party should stay losing if the threshold increases, and conversely a winning party should stay winning if the threshold decreases. This is a natural requirement, as a higher threshold should make it harder for a party to be selected.

Threshold Monotonicity
A party selection rule satisfies *threshold monotonicity* if for all profiles P and all thresholds $\tau \leqslant \tau'$, we have $f(P, \tau) \supseteq f(P, \tau')$.

It is easy to see that DO and STV satisfy this axiom, but not the other rules.

Proposition 5.8
DO and STV satisfy threshold monotonicity, but GP, MaxP and MaxR do not.

Proof. The result for DO is clear: if a party is ranked first by τ' voters, it is ranked first by $\tau \leqslant \tau'$ voters. For STV, the order of elimination for the parties is the same, but the rule might stop earlier for a lower threshold; meaning fewer parties are eliminated, resulting in a superset outcome.

For GP, consider the profile $P = \{3 : a \succ b, 2 : b\}$. With $\tau = 3$, the outcome is $\{a\}$ and with $\tau' = 4$, the outcome is $\{b\}$. For MaxP and MaxR, consider the profile $P = \{5 : a, 1 : a \succ b, 1 : a \succ c, 4 : b, 4 : c\}$. With $\tau = 5$, the outcome is $\{b, c\}$ and with $\tau' = 7$, the outcome is $\{a\}$. □

Proponents of STV often argue in its favor using the following principle of procedural fairness: once we have decided that some parties are losing, we should continue the procedure as if that party hadn't run in the first place (Meek, 1969). We can formalize this principle using the following independence axiom, which says that once some parties are losing at some threshold, then for all larger thresholds, the rule should behave as if none of the losing parties had been available.

Independence of Definitely Losing Parties
A party selection rule satisfies *independence of definitely losing parties* if for every profile P and thresholds $\tau \leqslant \tau'$, we have $f(P, \tau') = f(P|_S, \tau')$, where $S = f(P, \tau)$.

Here, P_S denotes the profile obtained from P by deleting all parties outside of S. Independence of definitely losing parties implies threshold monotonicity, because it requires that $f(P, \tau') = f(P_S, \tau') \subseteq S = f(P, \tau)$. Thus, GP, MaxP and MaxR necessarily fail it. The axiom is related to the "independence at the bottom" axiom of Freeman et al. (2014b), and it encodes a key intuition behind the functioning of the STV rule. In fact, in combination with inclusion of direct winners, this axiom characterizes STV. To state the result formally, we need to restrict ourselves to profiles on which we never have to use the tie-breaking order. Thus, we say that a profile P is *generic* if for every $S \subseteq C$, in the profile P_S there is a unique party $c \in S$ with the lowest plurality score $\mathsf{score}_{S,P}(c)$. On generic profiles, the STV rule never encounters a tie.

Theorem 5.9
Let f be a party selection rule satisfying inclusion of direct winners and independence of definitely losing parties. Then f equals STV on all generic profiles.

This characterization crucially depends on a variable-threshold setup, and can thus not be adapted to characterize STV in other frameworks.

Corollary 5.10
DO, GP, MaxR and MaxP do not satisfy independence of definitely losing parties.

Independence of Clones

Another recurrent argument in favor of the STV rule in many models is that it satisfies the independence of clones (Tideman, 1987). In this model too, the STV rule satisfies an independence of clones axiom. As a reminder, we say that two parties $c, c' \in C$ are *clones* if for all $i \in V$ and all $x \in C \setminus \{c, c'\}$, $c \succ_i x \Leftrightarrow c' \succ_i x$ and $x \succ_i c \Leftrightarrow x \succ_i c'$. This implies that every voter ranks c and c' consecutively. Note that we will consider here only *pairs* of clones, but our results could easily be extended to *sets* of clones. The axiom says that when we have clones in a profile, the outcome should not change if we remove one clone. For simplicity, we define independence of clones only on generic profiles, on which STV never encounters a tie.

Independence of Clones
A party selection rule satisfies *independence of clones* if for every generic profile P and threshold τ, if c and c' are clones in P, writing $S = f(P, \tau)$ and $S' = f(P_{C \setminus \{c'\}}, \tau)$ then

1. $\{c, c'\} \cap S \neq \emptyset$ if and only if $c \in S'$, and
2. for all $x \in C \setminus \{c, c'\}$, $x \in S$ if and only if $x \in S'$.

Note that this axiom is equivalent to independence of clones for single-winner irresolute voting rules. It is satisfied by STV and MaxR, but not by the other rules.

Proposition 5.11
STV and MaxR satisfy independence of clones, but DO, GP and MaxP do not.

Proof. Let us first show that STV satisfies independence of clones. Let P be a generic profile and P' another profile equivalent to P but in which some party $c \in C$ has been cloned into another party $c' \notin C$. In both profiles, we will eliminate parties one by one.

As long as neither c nor c' is the plurality loser in P', the order of elimination will be the same in P and P', since cloning c does not affect the relative order of all other parties in the rankings. Note that STV stops when it finds a feasible outcome, which happens if and only if all remaining parties are ranked first (among the remaining parties) by at least τ voters .

If after some eliminations (the same in both profiles), STV finds a feasible outcome in P' before eliminating either c or c', then this means every remaining party is ranked first among the set of remaining party by at least τ voters, and thus this is also the case in P after the same elimination steps, and $\text{STV}(P', \tau) \subseteq \text{STV}(P, \tau) \cup \{c'\}$. Moreover, STV could not have found a feasible outcome in P before this step, as the scores of parties other than c and c' are the same in both profiles. Thus $\text{STV}(P', \tau) = \text{STV}(P, \tau) \cup \{c'\}$, and the conditions of the axiom are satisfied.

Now, assume that we reach a step such that c or c' is the plurality loser in P' before finding a feasible outcome. Without loss of generality, assume that c' is the plurality loser, otherwise we exchange the names of c and c'. Then, we eliminate c' in P' and do nothing in P (because P is generic, we can assume that c' is eliminated in P'). We now reached a step such that the two profiles are perfectly equivalent as the set of remaining parties are identical, and thus after this point the outcomes of STV on P and P' will necessarily be the same (remember that P is generic, so we will never encounter any tie). This concludes the proof that STV satisfies independence of clones.

For MaxR, simply observe that the number of voters represented by a set $S \subseteq C$ in P' is the same as in P, and the number of voters represented by $S \cup \{c'\}$ in P' is the same as in P if $c \in S$, and is equal to that of $S \cup \{c\}$ otherwise. Thus, MaxR satisfies the conditions of the axiom.[6]

To show that DO, GP and MaxP do not satisfy independence of clones, consider the following generic profile $P = \{6 : a, 4 : c' \succ c \succ a, 3 : c \succ c' \succ a\}$ with $\tau = 7$. In this profile, $\text{DO}(P, \tau) = \emptyset$ and $\text{GP}(P, \tau) = \text{MaxP}(P, \tau) = \{a\}$. However, c and c' are clones. Denote $P' = \{6 : a, 7 : c \succ a\}$ the profile without the clone c'. In this profile, $\text{DO}(P', \tau) = \text{GP}(P', \tau) = \text{MaxP}(P', \tau) = \{c\}$, contradicting independence of clones. Note that for GP and MaxP, we could also deduce it from the fact that with $\tau = n$ and a full profile they are equivalent to the plurality rule in the single-winner setting, which is not independent of clones. □

When using parallel-universe tie-breaking, STV satisfies independence of clones even without restricting the axiom to generic profiles.

Finally, we prove that if a rule f satisfies the independence of clones axiom, then its variant f^+ also does. Note that the proof relies on the weak condition that the clones c and c' are also clones in the tie-breaking order.

Proposition 5.12
If a party selection rule f satisfies independence of clones, then its variant f^+ also satisfies it.

[6] Note that the proof for MaxR needs the additional condition that c and c' are next to each other in the tie-breaking order.

Proof. Let τ be a threshold, f be a party selection rule satisfying independence of clones, P a profile and P' a profile identical to P with a clone c' of $c \in C$. Let $S = f(P, \tau)$ and $S' = f(P', \tau)$. Since f is independent of clones, S and S' satisfy the conditions of the axiom. We will show that the completion algorithm performs (almost) the same steps in both profiles (with possibly skipping steps in P when a clone is removed or added in P'), and thus the sets of parties obtained in both profiles satisfy the condition of the axioms after $k \geqslant 0$ steps.

We know that it is true for $k = 0$. Now assume that it is true after $k \geqslant 0$ steps. If there are no groups of τ unrepresented voters who rank the same party a, then by induction hypothesis, the conditions of independence of clones are satisfied. Otherwise, assume that there exists a group of τ unrepresented voters in P', and that a is the party which maximizes the number of unrepresented voters ranking it (ties are broken the same way in both profiles). Since we assumed by induction hypothesis that the sets of parties at this point are either the same on both profiles, or that the one for P additionally contains c', then the set of represented voters is the same in both profiles (since every voter who ranks c' also ranks c). Thus, if $a \notin \{c, c'\}$, we add a to the set of parties for both profiles, and if $a \in \{c, c'\}$, we add a to the set of parties for P', and c to the set of parties for P (they have the same number of supporters among unrepresented voters and are next to each other in the tie-breaking order). Adding a might cause the elimination of some other parties. If these parties are neither c nor c', then they will clearly be eliminated in both cases. If c is eliminated in P, then c and c' are eliminated in P' (if they were part of the current set of parties) and conversely, if c and c' are not part of the current set of parties for P' because one or both of them are eliminated, then c must be eliminated in P. Finally, it is possible that only c or c' is eliminated in P', while both were part of the set of parties. In this case, since one is remaining, this means it has enough support, and thus c will also have enough support in P and not be eliminated. At the end of the step, the conditions of the axiom are still verified.

Since we know that the algorithm terminates, then $f^+(P)$ and $f^+(P')$ will satisfy the conditions of the axiom, therefore f^+ satisfies independence of clones. □

Reinforcement for Winning Parties

The next axiom connects the outcomes of a party selection rule on profiles defined on different sets of voters, and imposes a consistency condition. The axiom is inspired by the reinforcement axiom introduced by Young (1974) that we discussed in Section 2.4.3. Our axiom says that if a party is winning in a profile P_1 with a threshold τ_1 and in a profile P_2 with a threshold τ_2, then it should also be winning in the profile $P_1 + P_2$ with threshold $\tau_1 + \tau_2$, where $P_1 + P_2$ is the profile obtained by "concatenating" P_1 and P_2.

Reinforcement for Winning Parties
A party selection rule satisfies *reinforcement for winning parties* if for all profiles P_1 and P_2 and all thresholds τ_1 and τ_2, we have $f(P_1, \tau_1) \cap f(P_2, \tau_2) \subseteq f(P_1 + P_2, \tau_1 + \tau_2)$.

DO satisfies this. In fact, DO is the only party selection rule satisfying inclusion of direct winners and reinforcement for winning parties, thus providing an axiomatic characterization of this rule.

Theorem 5.13
DO is the only party selection rule that satisfies inclusion of direct winners and reinforcement for winning parties.

A direct corollary is that STV and GP fail reinforcement for winning parties. We can show that MaxP and MaxR also fail it, with the profiles $P_1 = \{\tau_1 - 1 : b \succ a, 1 : a\}$ and $P_2 = \{\tau_2 : a, \tau_2 + 2 : b\}$. The outcomes of MaxP and MaxR are $\{a\}$ in P_1 and $\{a, b\}$ in P_2, but in $P_1 + P_2$ the outcome is $\{b\}$ for both rules.

Proposition 5.14
STV, GP, MaxP and MaxR do not satisfy reinforcement for winning parties.

Monotonicity

The next axiom is the classical monotonicity axiom which says that if a party is selected in the outcome, then it should remain selected if some voters place this party in a better position in their vote, leaving unchanged the relative ranking of the other parties. It is based on the same notion of c-improvement that we defined in Section 2.4.5, adapted to the case of truncated rankings.

Monotonicity
A party selection rule satisfies *monotonicity* if for every profile P and threshold τ, if $c \in f(P, \tau)$ and P' is a c-improvement of P, then $c \in f(P', \tau)$.

Note that this axiom corresponds to monotonicity for single-winner irresolute voting rules, which is known to be failed by IRV, the single-winner version of STV.

Proposition 5.15
DO satisfies monotonicity, but STV, GP, MaxP and MaxR do not.

Proof. For DO, consider a profile P and a threshold τ, and suppose that $c \in \mathrm{DO}(P, \tau)$. Now, consider a profile P' obtained from P by increasing the rank of c in one ranking without changing the relative ranking of the other parties. Then, the voters who ranked c in top position in P still rank c in top position in P', and thus c still has at least τ supporters in P', so $c \in \mathrm{DO}(P', \tau)$.

For STV, consider the following profiles P and P' with $\tau = 13$:

P:	$5 : a \succ c$	$6 : c$	$13 : d$	$4 : b \succ a$	$2 : b \succ c$
P':	$5 : a \succ c$	$6 : c$	$13 : d$	$4 : b \succ a$	$2 : \underline{c} \succ \underline{b}$

In P, a is eliminated first, then b, and the outcome is $S = \{c, d\}$. Now, observe that P' is a c-improvement of P. In P', b is now eliminated first, then c, then a and the outcome is $S' = \{d\}$. Thus, $c \notin \mathrm{STV}(P', \tau)$, contradicting monotonicity.

For GP, consider the following profiles P and P' with $\tau = 7$:

P:	$5 : a \succ c$	$2 : a \succ b \succ c$	$6 : c \succ b$	$2 : b$
P':	$5 : a \succ c$	$2 : \underline{b} \succ \underline{a} \succ c$	$6 : c \succ b$	$2 : b$

In P, a is added to the committee first. Then c is considered, but $\{a, c\}$ is not feasible. Then b is added to the committee, as $\{a, b\}$ is feasible. Now, observe that P' is a b-improvement of P. In P', c is added first to the committee, then a is considered, but $\{c, a\}$ is not feasible. Then b is considered, but $\{c, b\}$ is not feasible. Thus, $b \notin \mathrm{GP}(P', \tau)$, contradicting monotonicity.

For MaxP and MaxR, consider the following profiles P and P' with $\tau = 9$:

P:	$5 : b$	$1 : b \succ c$	$5 : c$	$1 : c \succ b$	$3 : a \succ b$	$3 : a \succ c$	$4 : a$
P':	$5 : b$	$1 : b \succ c$	$5 : c$	$1 : \underline{b} \succ \underline{c}$	$3 : a \succ b$	$3 : a \succ c$	$4 : a$

In P, if a is in the outcome, then neither b nor c can be, because they will represent at most 8 voters each. Thus, the two possible outcomes of maximal size are $\{a\}$ and $\{b, c\}$, and $\{b, c\}$ is the one maximizing both the number of voters represented and the number of voters represented by their first choice. Now, observe that P' is a b-improvement of P. In this profile, a is still incompatible with b and c, but now b and c are also incompatible together, as c will only represent 8 voters. Thus, the only possible outcomes of maximal size are singletons, and $\{a\}$ is the one that maximizes the objectives of MaxP and MaxR. This contradicts monotonicity. □

Monotonicity can be seen as a weak version of strategyproofness: If it is possible that a party goes from selected to not selected by increasing its rank in one ranking, then it is possible for a voter to manipulate the outcome and change the status of this party from not selected to selected by decreasing its rank. In the next subsection, we consider other notions of strategyproofness.

Incentive Issues

A major drawback of uninominal voting is that it incentivizes voters to strategically misreport their preferences. In particular, in standard uninominal elections, voters whose favorite party will not reach the threshold may instead vote for a large party (and thereby increase its share of the parliament) or vote for a party near the threshold (and thereby potentially move it above the threshold). This type of strategic voting is similar to what we observe in single-winner elections with uninominal preferences, in which voters are incentivized to vote for a candidate who has a chance to win. We refer to this type of strategic voting as *tactical voting*. We will study the extent to which tactical voting can be avoided when using party selection rules.

A second distinct type of manipulation has been called *coalition insurance voting* (Gschwend, 2007; Fredén, 2017; Susumu Shikano and Thurner, 2009). Here, a voter whose preferred party is guaranteed to reach the threshold decides to instead vote for a less-preferred party that is in danger of missing the threshold. If that smaller party reaches the threshold, it may form a governing coalition with the voter's preferred party. Thus, while the voter is now contributing their support to a worse representative, the voter will be more satisfied with the parliament as a whole. This manipulation is specific to parliamentary elections with thresholds and has no analogue in single-winner voting. We will consider this second type of manipulation separately.

Tactical Voting

Consider the common tactical vote in a uninominal election: a voter who supports a sub-threshold party instead votes for a larger party in order to have their vote counted towards the parliament's final composition. By doing this, the voter increases the share of representation of the party they vote for without any cost to their true favorite party (which anyway was not going to meet the threshold). In the extreme case, a tactical vote can even cause a new party to enter the winning set. In our model, we define tactical voting in two natural ways, depending on how we measure the satisfaction of the voter with an outcome S:

1. The satisfaction corresponds *solely* to the highest position of a party in S in the voter's truthful ranking.[7]

2. The satisfaction corresponds to the highest position of a party in S in the voter's truthful ranking, *and* to the share of representation $\mathsf{share}_{S,P}(c) = \mathsf{score}_{S,P}(c)/(\sum_{x\in S} \mathsf{score}_{S,P}(x))$ of that party.

Note that the second notion implicitly assumes that in practice, the allocation of seats would be done proportionally to the share of representation. It also assumes that the higher the share of representation, the more seats a party gets.

Of course, other notions of satisfaction are possible and could in principle depend on the entire vector of shares of representation and the voter's truthful truncated ranking. In particular, these two notions are not sufficient to model the *coalition insurance voting* type of manipulation. However, they will be sufficient to capture typical manipulations while being permissive enough to allow for positive results. Note that obtaining full strategyproofness with respect to either of these notions is hopeless. Assume that $\tau = n$ and that all voters submit complete rankings. Then, if we additionally enforce efficiency, this exactly corresponds to single-winner voting, and we know from the Gibbard–Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) that any rule that is strategyproof is either a dictatorship (i.e., the outcome is always the favorite party of some voter $i \in V$) or imposing (i.e., some party is never selected). We will therefore consider strategyproofness in restricted cases, considering a sequence of politically plausible situations where some of our rules turn out to be immune to manipulations.

The notion of strategyproofness that we will first explore requires that no voter can improve their most-preferred party among those selected.

Representative-Strategyproofness
A party selection rule is *representative-strategyproof* for a profile P and a threshold τ, if for any voter i and any misreport $\succ'_i$, if $P' = (\succ_1, \dots, \succ'_i, \dots, \succ_n)$, $S = f(P, \tau)$, and $S' = f(P', \tau)$, we have $\mathsf{best}_{S',P'}(i) \not\succ_i \mathsf{best}_{S,P}(i)$.

In search of positive results, we will make statements that distinguish three types of parties *from the perspective of a voter i* on a particular profile P, assuming that all other votes are fixed.

- A party is *safe* if it is included in the outcome no matter how i votes.

- A party is *risky* if it might be included or not in the outcome depending on how i votes.

- A party is *out* if it is not included in the outcome no matter how i votes.

Note that this partition of the parties depends on the rule f that is used. However, for all rules that satisfy inclusion of direct winners (including DO, STV and GP), parties that are ranked first by more than $\tau + 1$ voters are always *safe*, and parties that are ranked by fewer than $\tau - 1$ voters among the voters who did not rank a safe party first are always *out*. The remaining parties can be either safe, risky or out. Intuitively, risky parties are the ones that are neither clear winners nor clear losers. In a real-world scenario, this corresponds to the parties that are dangerously close to the threshold according to the polls.

We begin by considering the case where there exists at most one risky party. This is often not an unrealistic assumption. For example, in the 2023 New Zealand general election, with a

[7] We note that this notion is particularly natural in a clustering-based context, where an agent prefers an outcome to another whenever they are assigned to a better representative.

threshold of 5%, only one party's vote share fell in the interval $(3.08\%, 8.64\%)$. The same is true in 2020 for the interval $(2.60\%, 7.86\%)$ and in 2017 for $(2.44\%, 7.20\%)$.

In the current uninominal system, even with only one risky party, a voter can manipulate the outcome by misreporting their preferences. For example, if a voter prefers a party that is *out* first, and the *risky* party second, then they can misreport their preferences by putting the risky party first, causing the risky party to be included in the outcome. Interestingly, for party selection rules, we can show the following result.

Proposition 5.16
If a party selection rule satisfies set-maximality, then it satisfies representative-strategyproofness when there is at most one risky party from the perspective of each voter.

Proof. Let f be a party selection rule that satisfies set-maximality, P a profile in which there is one risky party from the perspective of every voter, and $i \in V$ a voter that successfully manipulates by misreporting, giving another profile P'. Let $S = f(P, \tau)$ and $S' = f(P', \tau)$. We assume $\mathsf{best}_{S',P'}(i) \succ_i \mathsf{best}_{S,P}(i)$, and thus $\mathsf{best}_{S',P'}(i) \notin S$. Thus, $c' = \mathsf{best}_{S',P'}(i)$ is a risky party from the perspective of i. However, since c' is better ranked than $\mathsf{best}_{S,P}(i)$ in the ranking of voter $\succ_i$ and S' is feasible in P', it is also feasible in P. Thus, by set-maximality, $S' \neq S \cup \{c'\}$, but this means that $S \setminus S' \neq \emptyset$, and thus there exists a risky party in S different than c', a contradiction. □

A direct corollary is that GP, MaxP and MaxR satisfy representative-strategyproofness when there is at most one risky party from the perspective of each voter. We can show that it is not the case for DO and STV.

Proposition 5.17
GP, MaxP and MaxR satisfy representative-strategyproofness when there is at most one risky party from the perspective of each voter. DO and STV do not.

Proof. For GP, MaxP and MaxR, this is a direct consequence of Proposition 5.16.

For STV, consider the profile $P = \{1 : b \succ a, 1 : c \succ a\}$ with $\tau = 2$. Parties b and c are out, while a is risky. We have $\mathrm{STV}(P, \tau) = \emptyset$, but either voter could manipulate the outcome by placing a first in their vote, which would result in the outcome $\{a\}$ (assuming that ties are broken in favor of a). For DO, consider the profile $P = \{1 : b \succ a, 1 : a \succ c\}$ with $\tau = 2$. We have $\mathrm{DO}(P, \tau) = \emptyset$, but the first voter can manipulate by voting $a \succ b$, resulting in outcome $\{a\}$. □

Note that GP, MaxP and MaxR fail strategyproofness if we allow two risky parties. Consider for instance the profile $P = \{4 : a, 2 : b \succ c, 1 : c \succ b \succ a\}$ with $\tau = 3$ and GP. Party a is safe but b and c are risky from the perspective of the last voter. GP selects $\{a, b\}$ and is then unable to add c. However, if the last voter reports $c \succ a \succ b$, GP will first add a, then consider b but not add it (since $\{a, b\}$ is no longer feasible), and finally add c and output $\{a, c\}$, which is better for the manipulating voter.

We now turn to the second, stronger notion of strategyproofness, where the satisfaction of a voter depends on both the position of their favorite selected party c and its share of representation $\mathsf{share}_{S,P}(c) = \mathsf{score}_{S,P}(c)/(\sum_{x \in S} \mathsf{score}_{S,P}(x))$.

Share-strategyproofness
A party selection rule is *share-strategyproof* for a profile P and a threshold τ if for any voter i and any misreport $\succ'_i$, if $P' = (\succ_1, \dots, \succ'_i, \dots, \succ_n)$, $S = f(P, \tau)$, $S' = f(P', \tau)$, and $c = \mathsf{best}_{S,P}(i)$,

we have

$$(1)\ \mathsf{best}_{S',P'}(i) \not\succ_i c \qquad \text{and} \qquad (2)\ \mathsf{share}_{S',P'}(c) \leqslant \mathsf{share}_{S,P}(c)$$

First, we consider the case where every voter has a safe party among their top two preferences. Clearly, uninominal voting fails share-strategyproofness in this case, as a voter can elevate a safe party that is their second choice ahead of a party that is out but their true favorite, thus increasing the share of the vote allocated to the safe party. Notably, this strategy is ineffective under DO.

Proposition 5.18
DO satisfies share-strategyproofness whenever the most-preferred or second-most-preferred party of every voter is safe from the perspective of that voter. GP, STV, MaxP and MaxR do not.

Proof. For DO, consider a voter i with preference $c_j \succ_i c_k \succ_i \dots$. If c_j is selected then i has no incentive to misreport, as it is clear that no manipulation will increase the share of representation of c_j (in fact, they will all decrease it). If c_j is not included, then c_k is safe by assumption, and therefore selected. Moreover, it is the representative of voter i. Since any manipulation by i can only decrease the number of first-place votes received by c_j, it is impossible for c_j to be selected after the manipulation: the party is *out*. Similarly, any manipulation by i can only decrease the number of supporters of c_k without decreasing the support of any other parties. Therefore, no manipulation can result in an increase of share of representation for c_k. This proves the result for DO.

For MaxP, consider the profile $P = \{5 : b, 4 : a \succ b, 4 : c \succ a, 1 : b \succ c\}$ with $\tau = 5$. The feasible outcome that maximizes the plurality score is $S = \{b, a\}$ ($\{b, c\}$ is not possible because c would only be supported by 4 voters). Moreover, party b is safe for every voter, and a is safe for the voters with ranking $c \succ a$: individually, they cannot make a replaced by c. The share of representation of b is $\mathsf{share}_{S,P}(b) = 6/14$. Now, assume that the last voter misreports and votes $c \succ b$. Now the only feasible outcome that maximizes the plurality score is $S' = \{b, c\}$. The share of representation of b is now $\mathsf{share}_{S',P'}(b) = 9/14 > 6/14$. This violates share-strategyproofness. We use the same profile for MaxR. Again, the outcome is $S = \{b, a\}$, b is safe, and a is safe for the four voters with ranking $c \succ a$ for the same reasons. With the same manipulation, the outcome $S' = \{b, c\}$ becomes feasible and has maximal score, and assuming that ties are broken in favor of c, it is the outcome of MaxR.

For GP, we consider the profile $P = \{1 : a \succ b, 5 : a, 3 : b \succ c, 3 : c \succ a\}$ with $\tau = 4$. GP selects $S = \{a, c\}$ and we have $\mathsf{share}_{S,P}(a) = 6/12$. Note that party a is safe from the perspective of every voter since it is a direct winner even if it loses a vote. Furthermore, party c is safe from the perspective of the voters with preference $b \succ c$, since no unilateral deviation from any of them can cause b to be selected, and therefore c will be feasible to add to the set $\{a\}$ with at least five supporters (even if the deviator does not rank c). Therefore, every voter has a safe party in first or second position and this profile satisfies the conditions of the proposition. However, if the first voter changes their report to b then GP selects $S' = \{a, b\}$ and we have $\mathsf{share}_{S',P'}(a) = 8/12 > 6/12$. The identical example also demonstrates a violation of share-strategyproofness for STV, provided that ties between b and c are broken in favor of b being eliminated. □

Finally, we prove a result in which the set of possible misreports of a voter is restricted. In particular, we assume that voters will only misreport by promoting their most-preferred party that is selected under truthful voting to the first position in their misreport. This restriction can be thought of as giving voters perfect knowledge of which parties are out (and therefore not worth voting for), but not enough sophistication to perform arbitrarily "complex" manipulations.

Proposition 5.19
DO and GP are share-strategyproof under the restriction that $c = \mathsf{best}_{S,P}(i)$ is ranked first in $\succ'_i$, but STV, MaxP and MaxR fail even representative-strategyproofness under this restriction.

Proof. For DO, it is clear that if a voter elevates $\mathsf{best}_{S,P}(i)$ to first position then neither the set of direct winners nor their supporters undergo any change.

For GP, any misreport allowed by the proposition results in c receiving one additional plurality vote, while i's true first choice c_j receives one fewer. GP may now consider c earlier and c_j later than under truthful voting, but the relative order of consideration for all other parties remains unchanged. It is easy to see that if S' is the outcome after the manipulation, then $c \in S'$ and $c_j \notin S'$. For any other party c_k, assume by induction that all parties considered before c_k are selected in S' if and only if they are selected in S. If $c_k \in S$ then it must be feasible to add c_k to S', since the outcome set at the time that c_k is considered is a strict subset of S. Similarly, if $c_k \notin S$ then it must not be feasible to add c_k to S', since the outcome set at the time that c_k is considered in the misreported instance is a superset of the outcome set at the time that c_k is considered in the truthful instance (it might additionally contain c).

For STV, consider the following profiles P and P' with $\tau = 10$:

P:	$10 : b$	$4 : c \succ d$	$3 : d \succ c$	$2 : d \succ b$	$3 : a \succ b \succ d$	$1 : a \succ c \succ b \succ d$
P':	$10 : b$	$4 : c \succ d$	$3 : d \succ c$	$2 : d \succ b$	$3 : a \succ b \succ d$	$1 : \underline{b} \succ c \succ \underline{a} \succ d$

We assume that the tie-breaking order is $a > b > c > d$. In P, c is eliminated first, then a, then the outcome $S = \{b, d\}$ is feasible with $\mathsf{score}_{S,P}(b) = 11$ and $\mathsf{score}_{S,P}(d) = 12$. Now, in P' in which the last voter is manipulating, a is eliminated first, then d and the outcome is $S' = \{b, c\}$ with $\mathsf{score}_{S',P'}(b) = 13$ and $\mathsf{score}_{S',P'}(c) = 10$. Thus, the manipulating voter is more satisfied because $\mathsf{share}_{S',P'}(b) = 13/23 > 11/23 = \mathsf{share}_{S,P}(b)$, and also because this voter prefers c to $b = \mathsf{best}_{S,P}(i)$.

For MaxP, consider the profile $P = \{3 : b \succ d, 4 : c, 2 : d \succ b, 1 : a \succ b \succ c\}$ with $\tau = 5$. In this profile, c and b cannot be both part of the outcome, otherwise it is not feasible. Similarly, b and d cannot be both part of the outcome. Moreover, a is ranked by only one voter and thus cannot be part of the outcome. Thus, the feasible outcome with maximum plurality score is $S = \{c, d\}$. Now, if the last voter changes their vote to $c \succ a \succ b$, putting their representative $c = \mathsf{best}_{S,P}(i)$ first, then the outcome $\{c, b\}$ is now feasible and has a higher plurality score. However, the last voter prefers b to c, thus it is a successful manipulation. We can use the same profile for MaxR. The only feasible outcome that covers all voters in P is $\{c, d\}$, but $\{c, b\}$ becomes feasible after the manipulation. If we assume that ties are broken in favor of b, then we have a successful manipulation. □

It should be observed that Proposition 5.19 covers the typical manipulation that occurs with uninominal voting: voters place their favorite *safe* party first instead of their favorite party, in order to increase the vote count of this party (otherwise, their ballot would not be considered). Consider for instance the profile $P = \{1 : a \succ b, 3 : b, 3 : c\}$ with $\tau = 3$. With our rules, the first voter can vote sincerely: their vote will support b since a is out. Under uninominal voting, they have an incentive to vote for b.

Finally, note that STV is not covered by any of these positive results. However, in cases where STV is manipulable, the manipulating voter causes the whole elimination order to change, and thus some parties not to be added to the outcome anymore, possibly increasing the vote share of their representative. This is arguably a very unnatural manipulation for voters, who need almost full knowledge of the preferences of the other voters to be able to predict the correct manipulation. This is consistent with observations by Van der Straeten et al. (2010), who showed with lab experiments that voters tend to vote sincerely in elections using IRV (the single-winner equivalent of STV), as they are not able to predict the correct manipulation.

Coalition Insurance Voting

A separate type of manipulation is *coalition insurance voting* (Cox, 1997), where a supporter of a safe party c instead votes for a risky party d that they also like in order to push d over the threshold, thereby potentially allowing c and d to form a governing coalition (while in a parliament without d, there would be no majority for c). Indeed, while c loses one supporter, party d gains τ supporters by virtue of being included in the outcome. In many cases, the voter will be more satisfied with the new outcome as a whole. In many countries with proportional representation systems, parties announce intended coalitions in advance of the elections, and safe parties such as c might even encourage their supporters to vote for d instead. Coalition insurance voting has been observed in several countries including Germany and Sweden, and is well-studied using survey and lab experiments (e.g., Fredén, 2017; Fredén et al., 2024).

Can such voting behavior be avoided when using party selection rules? Unfortunately, a simple example suffices to show that no rule that satisfies the inclusion of direct winners axiom is immune to manipulations of this type. Consider the profile $P = \{3 : a, 3 : b, 4 : c, 2 : d, 1 : c \succ d\}$ with $\tau = 3$, and suppose that the last voter likes both c and d and is close to indifferent between them, but dislikes a and b. The outcome under truthful voting is $S = \{a, b, c\}$ by inclusion of direct winners, as all these parties have at least τ first-place votes, and including d would violate feasibility. The last voter only likes party c from S, which makes up $5/11 \approx 45\%$ of the parliament, and $\{a, b\}$ may form the governing coalition. Now, if they change their vote to $d \succ c$, then the outcome will be $S' = \{a, b, c, d\}$ by inclusion of direct winners, as all parties get at least τ first choices. After this manipulation, the last voter likes $7/13 \approx 54\%$ of the parliament, and $\{c, d\}$ may now be forming a governing coalition, thus including the most-preferred party of the manipulating voter.

While party selection rules cannot completely avoid this effect, one might expect that in practice there is less motivation for this kind of manipulation in the case that voters can submit a ranking than if they can only submit a uninominal ballot. In particular, if the smaller party d is not selected, then presumably many votes cast for d will transfer to c as a second or third choice. For instance, in the previous example, the d voters might have put c in second place if c and d were running on similar platforms or had announced an intention of forming a coalition. However, there is also a possibility that coalition insurance voting might *increase* under rankings, since this strategy is less risky in this situation. Indeed, if voters preferring c instead cast the ranking $d \succ c$, then either the manipulation is successful (and d is selected), or it is unsuccessful and the vote is transferred to c, which does not hurt the manipulator. In contrast, under uninominal voting, a vote for d carries a risk that the vote will be lost. Further experiments are needed to determine the actual impact of ranking ballots on this kind of manipulation.

Summary of the results

We conclude this section with a summary of the axiomatic results, displayed in Table 5.1, which shows which of the five rules satisfy which axioms. It also shows the characterization results for DO and STV. In addition to these results, we also described in this section an algorithmic method to transform any party selection rule into another rule satisfying representation of unrepresented voters.

	DO	STV	GP	MaxP	MaxR
Set-maximality	✗	✗	✓	✓	✓
Inclusion of direct winners	✓[1]	✓[2]	✓	✗	✗
Representation of solid coalitions	✗	✓	✗	✗	✗
Representation of unrepresented voters	✗	✗	✗	✗	✗
Threshold monotonicity	✓	✓	✗	✗	✗
Independence of definitely losing parties	✗	✓[2]	✗	✗	✗
Independence of clones	✗	✓	✗	✗	✓
Reinforcement for winning parties	✓[1]	✗	✗	✗	✗
Monotonicity	✓	✗	✗	✗	✗
Representative-strategyproofness (one risky party)	✗	✗	✓	✓	✓
Share-strategyproofness (safe first or second)	✓	✗	✗	✗	✗
Share-strategyproofness (representative ranked first)	✓	✗	✓	✗	✗

Table 5.1: Properties satisfied by the rules. The superscripts [1] and [2] indicate characterization results.

5.4 Experimental Analysis

In the classical uninominal system, voters either vote sincerely, at the risk of wasting their vote, or they choose to vote strategically. Inspired by our theoretical analysis, we want to check empirically if under ranking-based systems, voters would vote less strategically, for instance by putting less popular parties on top of their rankings. We also want to evaluate the extent to which these systems allow better representation by decreasing the number of unrepresented voters, without drastically increasing the number of parties included in the parliament (as the main argument for the threshold is to reduce the number of parties).

To the best of our knowledge, the only existing empirical study on party selection rules is the one by Graeb and Vetter (2018), who asked $n = 828$ participants in Germany in 2017 how they would vote if they could indicate a second-choice replacement vote. They found that participants vote more frequently for smaller parties, and that the number of lost votes decreases (they applied the DO rule). However, their study does not compare different party selection rules, and it introduces two different changes to the voting system simultaneously: addition of a replacement vote, but also a combination of the person-bound and party-bound votes (*Erststimme* and *Zweitstimme*) into a single vote, making it more difficult to estimate the impact of the replacement vote alone.

We base our study on a survey we ran in the context of the election of the French representatives to the EU Parliament held in June 2024. The French representatives are elected by a nationwide party-list proportional representation system, using the D'Hondt apportionment method with a 5% threshold. In the 2024 election, 38 lists took part in the election and 12.08% of votes were cast on lists that did not reach the 5% threshold. Seven lists reached it, two of which were just above the threshold (with 5.47% and 5.5% of vote share), so the proportion of wasted votes could have been much worse (which happened in 2019, when it was around 20%).

The Datasets

Our datasets (Delemazure et al., 2024a) were collected through a survey that invited participants to consider how they would vote in a system that allows them to rank either at most two lists, or an arbitrary number of lists. The survey was administered through a custom-built online platform in French language.[8] Participants were led through several steps:

1. Participants were briefly informed about the problem of wasted votes, with the help of data visualizations of the results of the 2019 election, where 19.8% of ballots had been lost due to the threshold. We then familiarized participants with the 38 lists participating in this election: we displayed official posters and provided access to the campaign manifestos. We sampled a random ordering of the lists for each voter, and we always displayed the lists in that order in the subsequent steps.

2. In the second step, participants were told that to avoid losing votes for lists that do not reach the threshold, they were allowed to specify a second choice if they wished so, which would be taken into account if their first choice fell below the threshold of 5%. We then asked them to indicate how they would vote under this system. Participants could rank 0, 1, or 2 lists.

3. In the third step, we explained further that voters could now vote for as many lists as desired. We explained that if the first choice were to receive less than 5% of the votes, the second-choice vote would be counted. If the second choice still receives less than 5%, the third-choice vote would be counted, and so on. We then asked participants to indicate how they would vote under this system. Participants could rank between 0 and 38 lists.

4. In the next step, participants were asked to say for which list they intended to vote on election day (for participants taking the survey prior to election day), or for which list they had voted (for those taking it after election day). Answering was optional.

5. Finally, participants were shown two questionnaires: one in which they could express their opinion about moving to more expressive voting, and a second one in which they could provide some socio-demographic data.

We ran this survey on two samples of participants:

- *Self-selected sample.* We recruited 3 046 participants through social media (mainly *Twitter*) and mailing lists. These participants tended to be interested in the political process and were very diligent in answering the survey. However, the resulting sample is clearly not representative of the French population: it is very left-leaning (66% intended to vote for one of the four largest left-oriented parties LFI, PCF, PS and the Ecologists, while they received less than 32% of the votes at the election), young (72% are between 18 and 39, while 30% of the French population is between 20 and 39), and highly educated (92% indicated a university education level, while it corresponds to 42% of the general population). Participants were not paid. This dataset only includes answers from those who indicated being registered to vote. It was gathered between May 30th and June 26th 2024, however for this analysis we only kept the 2 840 participants who completed the survey before the election day (June 6th).

- *Representative sample.* We recruited 1 000 participants through a survey research company. Participants were paid a fixed amount of money to participate in the study. This sample is more representative of the French population with respect to demographics and voting behavior. However, it is of lower quality as some participants filled out the form as quickly as possible. This dataset was collected between June 17th and 25th, after the election.

[8] The survey is available at www.lamsade.dauphine.fr/vote/.

	Self-selected		Representative	
	Two votes	Rankings	Two votes	Rankings
Inconsistent (c^* not ranked)	3.5%	0.7%	8.6%	10.4%
Sincere ($c = c^*$)	73.5%	70.2%	84.3%	80.4%
Strategic ($\mathsf{score}(c) < \mathsf{score}(c^*)$)	22.7%	28.6%	5.0%	6.2%
$\hookrightarrow$ out $\rightarrow$ safe	16.4%	21.0%	2.7%	3.2%
$\hookrightarrow$ out $\rightarrow$ risky	2.2%	3.0%	0.3%	0.7%
$\hookrightarrow$ risky $\rightarrow$ safe	2.6%	3.0%	0.8%	0.4%
$\hookrightarrow$ others	1.5%	1.6%	1.3%	2.0%
Strategic ($\mathsf{score}(c) > \mathsf{score}(c^*)$)	0.3%	0.6%	2.1%	3.0%
$\hookrightarrow$ safe $\rightarrow$ out	$< 0.1\%$	0.1%	0.7%	0.6%
$\hookrightarrow$ safe $\rightarrow$ risky	0.3%	0.2%	0.3%	1.1%
$\hookrightarrow$ safe $\rightarrow$ safe	$< 0.1\%$	0.2%	0.8%	0.7%
$\hookrightarrow$ risky $\rightarrow$ out	$< 0.1\%$	$< 0.1\%$	0.2%	0.3%
$\hookrightarrow$ others	$< 0.1\%$	$< 0.1\%$	$< 0.1\%$	0.2%

Table 5.2: Comparisons of voters' voting intention c^* and the list they ranked first c.

All the collected data were anonymized. Participants were informed before participating that the data would be used for research purposes only, and they had the possibility to skip any question they did not wish to answer. The study received ethics approval (Université Paris Dauphine - PSL, décision du comité d'éthique de la recherche n°2024-01 and University of Virginia IRB protocol number 6756). The dataset is available on Zenodo (Delemazure et al., 2024a).

For both samples, we used the voting intentions (or actual votes for participants who answered the survey after the election) to assign weights to voters in order to reduce the representation bias of our samples. We refer to Section 2.5.4 for more details on the weighting method. Participants who did not provide any voting intention were assigned weight zero, leaving $n = 2\,728$ participants with non-zero weight for the self-selected sample, and $n = 895$ for the representative sample. This gives us a total of four preference profiles: self-selected and representative samples, together with either two votes or rankings.

Strategic Voting

Using these datasets, we first investigate the strategic behavior of the voters in the actual election. To do so, we compare the party c they put in first position in their ranking to the party c^* they actually voted for, or intended to vote for at the election. Here, we are making the assumption that c is their true top choice: even though the ranking-based rules are still manipulable, voters had no real incentive to vote strategically, as there are no stakes to the survey. Moreover, participants were not familiar with ranking-based rules, and would have difficulty knowing how to strategize. Thus, we assume a voter voted sincerely in the actual election if c and c^* are equal, and strategically otherwise.

For this analysis, we divide the parties into three groups according to their score in the actual election. Intuitively, we want this partition to reflect the safe/risky/out categorization that we introduced in Section 5.3. We have:

- 5 *safe* parties that received more than 7% of the votes at the actual election, and were expected to reach the threshold,

- 2 *risky* parties that received between 5% and 6% of the votes and were in danger of not reaching the threshold according to the polls, and
- 31 *out* parties that received less than 3% of the votes and were expected to not reach the threshold.

Then, we divided the voters into four categories depending on their voting intention c^* and their presumed favorite party c. The percentages of voters in each category are given in Table 5.2.

1. The *inconsistent* voters are those who did not include c^* *at all* in their ranking. We call them inconsistent as there are no sensible reasons to select some party c^* when it is allowed to select only one, but not select it when it is allowed to select several.[9] There are much fewer inconsistent voters in the self-selected sample than in the representative one.
2. The presumably *sincere* voters who ranked c^* in first position (i.e., $c^* = c$), and who correspond to the large majority of participants. These voters represent around 70% of the participants in the self-selected sample and 80% in the representative sample.
3. The *strategic* voters who ranked in first position a party c with a lower score (that is, vote count) that of c^*. There are 22.7% and 28.6% (respectively, 5% and 6.2%) of such participants in the two datasets from the self-selected (respectively, representative) sample. For a majority of them, c is an out party while c^* is a larger party (either risky or safe); this is the canonical example of *tactical voting*.
4. The other *strategic* voters who ranked in first position a party c with a higher score that of c^*. This behavior is less common. For a majority of these voters, c is safe and c^* is either risky (which resembles patterns observed in *coalition insurance voting*) or also safe.

The fact that voters are more likely to vote for an *out* party when they can cast rankings than when they have to vote for only one party can be shown to be statistically significant (for the representative sample, we obtain $\chi^2(1, N = 895) = 9.8, p = 0.002$ for length-two rankings and $\chi^2(1, N = 895) = 16.2, p < 0.001$ for unlimited-length rankings). This corroborates the results from Graeb and Vetter (2018).

From the questionnaire at the end of the survey, we also know that the question *"Would you be more likely to vote for a small list if you could give additional choices?"* has been answered positively by 75% and 52% of the participants respectively in the self-selected and representative sample (and negatively by 19% and 29%), while *"Would you be likely to vote for a small list closer to your interests even if there is a chance for your vote to not be taken into account?"* was answered negatively by 60% and 37% of participants respectively in the self-selected and the representative sample (and positively by 33% and 47%). These answers confirm that many voters are indeed strategizing in uninominal elections and that the possibility to rank more parties limits the need to do so.

Representativity

We now compare the results of our different rules with that of the actual election, in particular the representativity of the outcomes. In this analysis, we will focus on the three procedural rules DO, STV and GP, as they are the most relevant in the context of parliamentary elections.

[9] For the participants who completed the survey *after* the election, they might have changed their mind about the list they voted for, but we assume that this is not the case for the majority of the participants.

	Self-selected		Representative	
	Two votes	Rankings	Two votes	Rankings
Actual	12.1%	12.1%	12.1%	12.1%
Uninominal	38.0%	34.6%	20.5%	21.4%
DO	11.7%	3.2%	10.8%	8.7%
STV	7.0%	2.3%	9.2%	7.2%
GP	7.0%	2.3%	9.2%	7.2%

Table 5.3: Percentages of unrepresented voters.

	Self-selected		Representative	
	Two votes	Rankings	Two votes	Rankings
Actual	7	7	7	7
DO	6	7^+	6	6
STV	7	8^+	7	7
GP	7	8^+	7	7

Table 5.4: Number of selected parties. The '+' indicates that a party not selected at the actual election is selected.

Unrepresented Voters

We first compare the share of voters who are unrepresented (i.e., that did not rank any party that is selected by the rule). In Table 5.3, we show these shares obtained from

(1) the actual election (12.1%),

(2) the uninominal rule which deletes everything below voters' first choice, selects all parties meeting the threshold, but leaves voters unrepresented if their first choice is not selected,

(3) the party selection rules DO, STV, and GP.

The discrepancy between (1) and (2) is due to voters' strategic behavior in the actual election. Indeed, as we just saw in Table 5.2, many voters, especially in the self-selected sample, ranked a party that is *out* first in their ranking, but voted for a *safe* party at the election. Thus, applying the uninominal rule on the "sincere" rankings (that we collected) would lead these voters to be unrepresented, while they were represented in the actual election. This partly explains why we observe a much higher percentage of wasted votes for the uninominal rule with our datasets than in the actual election. Another part of the explanation is that because participants voted less strategically, one risky party did not reach the threshold in first-rank votes anymore in each dataset, and their supporters became unrepresented. Arguably, the numbers in (2) are counterfactual: if the voters know they have to cast a uninominal vote, they will strategize and the outcomes will be those in (1).

In that sense, the party selection rules are reducing the share of unused ballots not from the 12.1% of the actual election, but from the values obtained in (2). The difference between (1) and (3) is therefore even more relevant, and the gain is all the more remarkable as voters have less incentives to strategize in our model. In other words, allowing voters to cast rankings decreases the amount of unrepresented voters *and* the need for them to strategize.

Recall that the outcome of DO is always a subset of the outcomes of STV and GP. For each of our datasets, it turns out that STV and GP return the same set of parties, and DO returns one fewer party (see Table 5.4), explaining why there are more unrepresented voters with DO than with STV and GP.

The representation gain is higher with ranking ballots than with 2-truncated ballots. This is not surprising: when participants can rank more than two parties, they are more likely to rank at least one safe party than when they can rank only two. In addition, for the ranking dataset of the self-selected sample, one additional party (namely the *Pirate party*) is selected in the outcome for each rule, further decreasing the number of unrepresented voters. However, this is almost entirely due to a selection bias: the survey was shared among the supporters and people familiar with this party, leading to more people ranking it first than in a representative sample.[10]

The representation gain is also higher for the self-selected sample than for the representative sample. This is partly explained by the fact that in the representative sample, a significant number of voters ranked only one party, thus limiting their chances to be represented. This corresponds to around 40% of the unrepresented voters in the representative sample, but to less than 10% of them in the self-selected sample.

Note that because more voters are represented, the score distribution at the end will differ from the one of the actual election, and if we were to distribute the seats according to the scores of the parties, the seat distribution would also be different. With our datasets, this difference is accentuated by the fact that the Pirate party is selected in the outcome of STV and GP for the self-selected sample, leading to completely different seat distributions.

Finally, recall that local stability is not satisfied by any of our rules; therefore, it makes sense to check if it is violated in our datasets. It turns out that it is indeed violated by DO in all datasets: for both samples there exists one list (the one that is selected in the actual election but not in our datasets because participants voted less strategically, see Table 5.4) which is preferred to all selected lists by at least 5% of the voters. It is not violated by STV and GP. On the other hand, we do not observe any violation of the representation of unrepresented voters axiom in our datasets. Still, it is close to being violated by DO for the self-selected sample, as 4.7% of the voters were unrepresented and ranked the same list (the same list that caused a violation of local stability).

Impact of the Ballot Size

Another interesting observation is that the representativity gain with ranked ballots is already quite high with short ballots. Indeed, a large majority of voters are represented by a party ranked very high in their ranking. For instance, Figure 5.2 (a) shows that for the ranking dataset of the self-selected sample, if we use STV or GP, around 70% of voters are represented by their first choice, and almost all voters have their representative in their top 3 choices. This is even clearer for the representative sample (see Figure 5.2 (b)) as voters in this sample ranked fewer parties on average (see Figure 5.3).

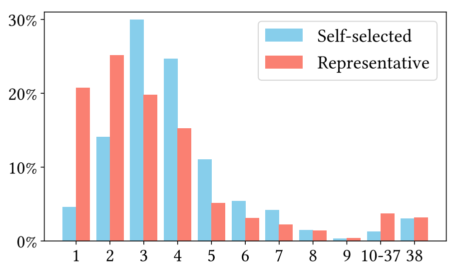
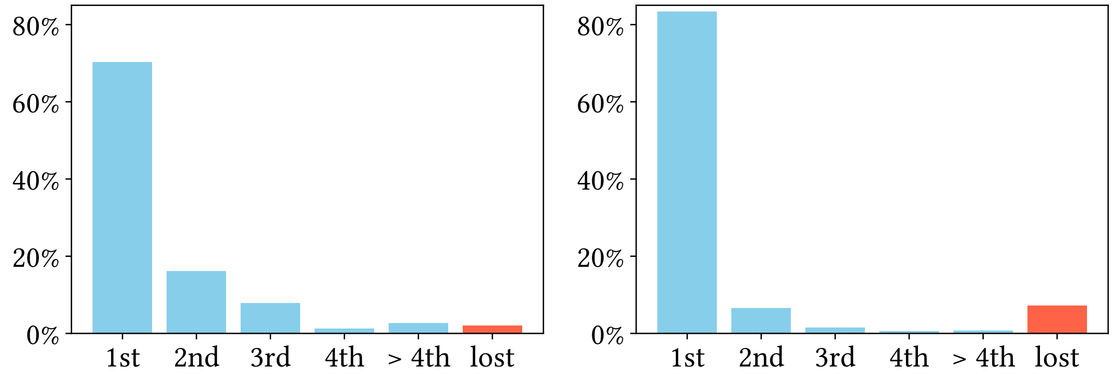


Figure 5.3: Distribution of the number of candidates ranked by the voters.

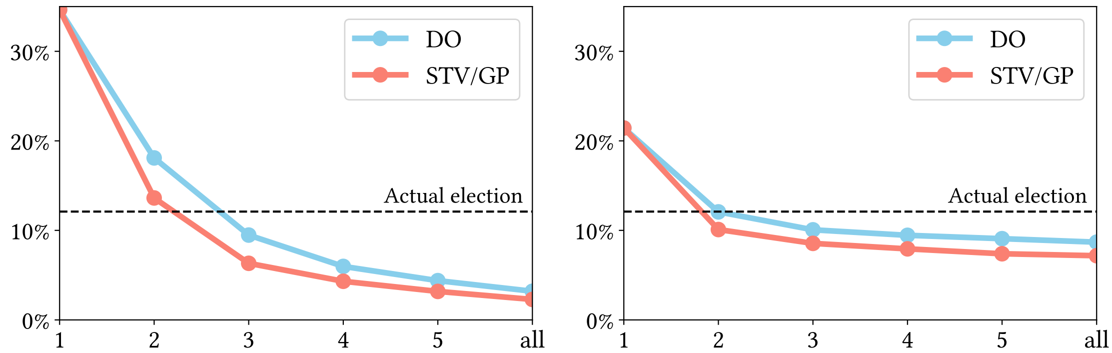
To complete this analysis, Figure 5.4 shows the fraction of lost votes if all rankings are truncated to rank k. For the representative sample, as soon as $k \geqslant 3$, increasing k has almost no impact on

[10]Note that this bias is not corrected enough by our weighting method, as many participants ranked this party first without indicating voting for it at the election, probably for strategic reasons.



(a) Self-selected sample. (b) Representative sample.

Figure 5.2: Distribution of the ranks of the representatives in the rankings of the voters for STV and GP with a 5% threshold, in the ranking datasets.



(a) Self-selected sample. (b) Representative sample.

Figure 5.4: Fraction of lost votes if all rankings are truncated to rank k.

representativity. For the self-selected sample, as voters ranked more parties (see Figure 5.3), we need $k \geqslant 5$ to reach an almost maximal representativity level. This is important for the practical implementation of such rules: limiting voters to rank at most three parties leads to an acceptable cognitive load for them (and is more likely to be adopted in real-world political settings), while still ensuring good representativity.

Varying the Threshold

The results above were obtained with a threshold of 5%, the one used in this election. As a robustness check, we ran the same analyses with other thresholds. In particular, we applied the rules to our datasets with thresholds varying between 1% and 10%, and computed the percentage of unrepresented voters, as well as the distribution of ranks of the representatives. The results are shown in Figure 5.5. They are compatible with our previous observations: STV and GP consistently give similar results and a better representation than DO, especially for high threshold values, for which DO ignores a significant part of the voters. Moreover, we observe that even for high thresholds, most voters are represented by one of their top 3 choices.

Finally, we conducted experiments with random noise added to the data. The goal here is to have enough noise to make the risky parties sometimes reach the threshold, and sometimes not.

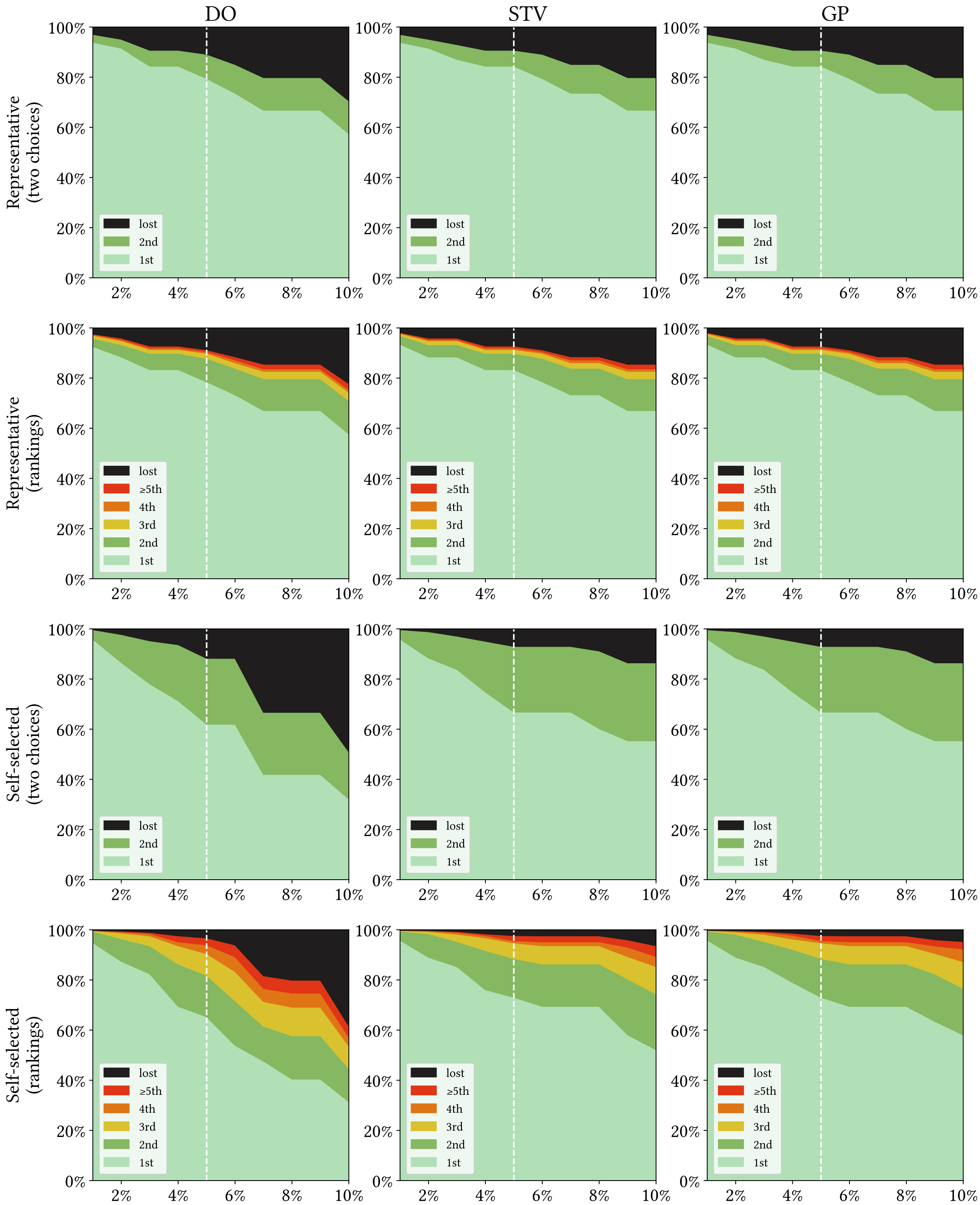


Figure 5.5: Distribution of the ranks of the representatives in the rankings of the voters, with different thresholds. The threshold used is indicated on the horizontal axis. For each threshold, the vertical slice above it shows how the voters are divided into unrepresented voters (black area) and represented voters (colored areas, colored according to the rank of the voter's representative).

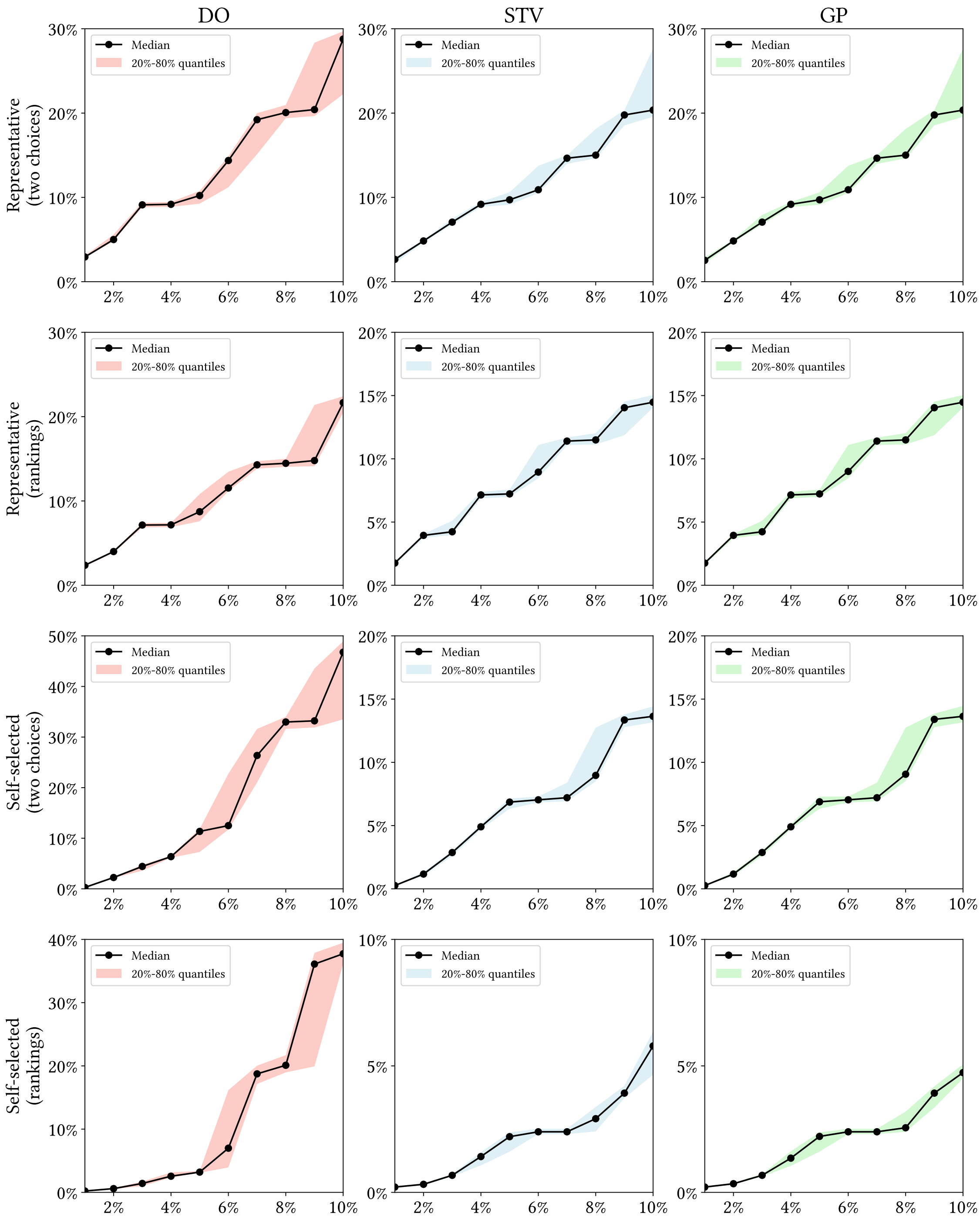


Figure 5.6: Median percentage of unrepresented voters (over 100 random profiles) for different thresholds.

We applied the noise to the weights of the voters, and we used the following random model to alter the weights of the voters: for every single simulation, we sampled a multiplier for each party c from a Gaussian distribution $\sigma_c \sim \mathcal{N}(1, 0.1)$, and one multiplier for each voter i from another Gaussian distribution $\sigma_i \sim \mathcal{N}(1, 0.1)$. We then multiplied the original weights w_c of the voters (which depend on their voting intention or actual vote $c \in C$ at the election) by these multipliers. More formally, the new weights are $w_i^* = w_c \times \sigma_c \times \sigma_i$. Thus, the global weights of the different parties can change, and the two risky parties sometimes reach the threshold, and sometimes not. We sampled 100 profiles for each dataset and each rule using this model, and computed the median percentage of unrepresented voters for different values of the threshold, as well as the 20 and 80 percentiles. The results are displayed in Figure 5.6. It appears that the observation that STV and GP give much better results than DO holds even when we add noise to the data. We also observe that STV and GP are more robust to noise, and that their outcomes are again very similar.

5.5 Discussion

In this chapter, we studied whether allowing voters to rank party lists instead of voting for a single list could help obtain more representativeness in parliamentary elections by reducing the amount of unrepresented voters. To this end, we defined five *party selection rules*, three of which seem applicable to parliamentary elections: DO, STV and GP. Both our theoretical and our empirical results suggest that rankings can indeed be helpful, with results varying by rule.

Which Rule to Use?

On the one hand, STV and GP allow more parties to be represented, and relatedly, leave fewer voters unrepresented than DO. On the other hand, DO is the easiest to implement. In particular, one simple way to implement it is to do two rounds of voting, both using uninominal preferences, such that only the parties receiving more than 5% of the votes in the first round are allowed to be in the second round. Such a system would make sense in some countries that are used to elections with a runoff, such as France, and it is actually quite close to the voting systems used in France for city councils and regional elections, in which lists of candidates that received more than 10% of the votes can go to the second round, and lists that received more than 5% are also allowed to go if they merge with a larger list. However, in these elections, the list that receives the most votes in the second round gets a bonus (either 25% or 50% of the seats). Note also that DO is the only party selection rule that can be implemented this way, ensuring a minimal communication complexity. Despite this, all three rules are simple and easy to understand, with DO being closest to the uninominal voting system, and STV being closely related to the ranking-based election systems that are widely used in practice, especially in English-speaking countries (such as Ireland and Australia).

Axiomatically, the three rules are incomparable: DO and GP enjoy stronger strategyproofness guarantees than STV, while STV satisfies independence of clones and representation of solid coalitions. Moreover, we characterized STV with an independence condition. We showed that DO is the only one of the three rules that satisfies monotonicity, and we characterized it using a reinforcement axiom. GP is the only one of these three rules that satisfies set-maximality. Note that MaxP and MaxR also satisfy set-maximality. However, they fail inclusion of direct winners, which is a crucial axiom in the context of a parliamentary election. Indeed, in such election it would be hard to defend that a party that reaches the threshold cannot receive any seat.

Our experimental analysis confirmed that rules based on rankings could reduce the number of unrepresented voters, while allowing voters to vote less strategically. This is particularly true for high thresholds. Thus, countries that place substantial importance on *governability* could in principle use methods like the ones we studied to combine proportional representation with a high threshold (perhaps as high as 10%) without causing unacceptable amounts of wasted votes.

Further Work

Our theoretical results operate within social choice theory and the axiomatic method. It would be interesting to study this setting from the perspective of strategic candidacy, evaluating the rules' impact on party formation and political innovation. Our experiment could also be productively repeated in other countries, to better understand the robustness of our conclusions. We could also run similar experiments on synthetic data.

As we mentioned in the introduction, the model of *party selection rules* is also relevant in the context of group activity selection (or facility location) with group size constraints. In this context, the goals might be different from ones we focused on in this chapter. In particular, MaxP and MaxR become more relevant, as inclusion of direct winners is less crucial to satisfy, and set-maximality is probably more important. Thus, it would be interesting to conduct experiments using several datasets of truncated rankings, either real or synthetic, to see how the different rules perform in this context, and how similar they are.

Implementation in Practice

Finally, it would be interesting to investigate whether the electorate would accept or welcome a switch to one of these rules. In our online experiments, we asked some closed questions to the participants, and additionally gave them the possibility to write comments.[11] When asked which voting system they preferred between the current one, a system using two-choice votes, and a system using rankings, 46% of the participants in the representative sample said they preferred one of the alternative methods. This value is much higher (77%) for the self-selected sample (even after assigning weights to the voters), as many participants in this sample are interested in the topic of alternative voting methods. We observe that the share of participants who appreciated alternative methods is higher among voters who voted strategically at the election (as described in Table 5.2): 53% in the representative sample and 89% in the self-selected sample. It also appears that left-leaning voters are more open to alternative voting methods (58% in the representative sample and 91% in the self-selected one). Moreover, some participants appreciated having the possibility to give more expressive preferences, and the fact they did not have to choose between their favorite party and a safe party. Some of them also argued in favor of limiting the number of parties allowed in the rankings to 3 or 4.

Some participants suggested that instead of dealing with the issues caused by the threshold, the best solution would be to simply delete it. It would indeed be easier, however it appears that this proposal is not as unanimously popular as it seems, at least in France. In our representative sample, only 37% of the participants answered positively regarding the *deletion* of the threshold, and 36% negatively. 42% answered positively regarding a *reduction* of the threshold, and 31% negatively. The percentage however increases to 55% of participants in favor of reducing the threshold (and 32% against) in the self-selected sample. The recurring argument in favor of keeping the threshold

[11] Some of the feedbacks were very interesting. To read these comments and our answers, see the following blog post in French language: `https://www.lamsade.dauphine.fr/vote/questions.html`.

among the participants is the need to reduce the number of parties in the parliament, to facilitate the formation of a coalition.

Finally, the question of the practical implementation of the rules, and of the counting of the votes, was raised by some participants. The counting would indeed take more time than is required for the current method, but probably as much as is needed in countries that use STV and IRV for major elections (such as Ireland and Australia, in which ballots are still counted by hand). Ayadi et al. (2019) proposed an efficient procedure for counting ballots with STV that could be adapted to our context, and which is similar to the one discussed in the previous chapter for counting ballots with Approval-IRV (Section 4.5). An even easier method would be to simply centralize all ballots numerically, and apply the rule directly on the obtained preference profile. However, it would require that people trust 'the algorithm', as well as the whole process of recording votes in a database.

Part II

Expressive Ballots for Political Analysis

Chapter 6

Learning Candidate Axes from Approval Data

6.1 Introduction

In the first part of this thesis, we have seen several voting contexts in which increasing the expressiveness level of voters' preferences could help solve actual issues, by allowing the use new voting rules, which generally satisfy interesting theoretical properties. In all of the problems we considered so far, the preferences of the voters were used to select a winner, or a set of winners, in other words, to make a *social choice*. In this second part, we consider problems in which we use methods and concepts from social choice theory to *learn* the hidden structure of the sets of voters and candidates, based on the preferences of the voters. This line of research includes for instance the problem of identifying exact or approximate structured preferences, such as single-peakedness (see Section 2.2.6), or comparing preference profiles, for instance by constructing *maps of elections*, that we discussed in Section 2.5.1.

On a formal level, these problems look exactly like the ones we have seen so far: we have a set of voters, a set of candidates, a profile of preferences, and we want to find the best outcome. We define several rules, and we evaluate them based on their theoretical properties (by doing an axiomatic analysis), and on their performance on real or synthetic data (by doing an experimental analysis). The main difference resides in that the outcome is not the output of a collective decision, but a structured representation of the preferences of the voters based on a given criterion. For instance, if we are using *approval* preferences, we can identify some links between the candidates: if two candidates are approved by the same voters and disapproved by the same voters, then they are likely quite similar. In contrast, if we are using *uninominal* preferences, such an analysis cannot be done, and we cannot learn anything about the structure of the electorate or the candidates. In that sense, the expressivity gain brought by approval preferences is not only a way to select better winners, but also a way to better understand the electorate and the candidates.

The Left-Right Axis

To represent the similarities between the candidates, and the structure of the political landscape, it is common to use an ordering of the candidates, which we interpret as an *axis*. Intuitively, candidates that are next to each other on the axis have overlapping sets of supporters, and candidates that are further away from each other on the axis have very different sets of supporters.

In the political context, it is very common to assume that candidates have a position on an ideological axis, that represents some *left-right* political spectrum. This idea that there exists such an axis of the candidates is quite strong, with the left being generally associated to the more progressive and social policies, and the right to the more conservative and economically liberal ones. A lot of political parties (especially in European countries) even have names that directly refer to the fact that they are from the *left* (*The Left* in the European parliament, *Die Linke* in Germany, *GroenLinks* in the Netherlands, *Vänsterpartiet* in Sweden, etc.), the *center* (*Centrist Party of Canada* in Canada, *Les Centristes* in France, *The Centre* in Switzerland, etc.) or the *right* of the spectrum (*Kongres Nowej Prawicy* in Poland), though generally the parties claiming to be from the right are actually from the far-right (*Union des Droites pour la République* in France, *La Destra* in Italy, *Die Rechte* in Germany, etc.). Political axes are also used in the United States, for instance to order the different justices of the Supreme Court.[1]

These political axes find many applications. First of all, they are directly used for assigning seats to the parliament members. Second, political axes are frequently used in the medias, for instance when results of a poll for an election are discussed. Figure 6.1[2] illustrates this: parties are displayed in a particular left-right order inside the parliament, which more or less matches their actual arrangement. Intuitively, we could think that the axes used in these cases are based on the analyses of political experts who ordered the candidates based on the policies that they proposed. While it might be the case for some of them, when we asked a polling institute representative how they decide on which axis to use, they replied that they simply select an ordering that looks reasonable and on par with the perceived position of each party on the political spectrum.

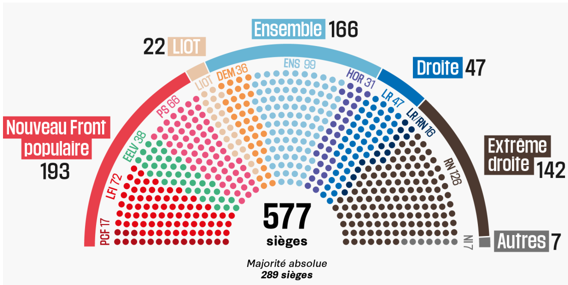


Figure 6.1: Picture of the seat distribution in the national Assembly used by the newspaper *Liberation* to show the results of the 2024 parliamentary election in France.

Discovering the Hidden Axes

Nonetheless, the problem of placing political parties on an axis is well studied in the political science literature, and is not restricted to the *left-right* axis. For instance, there exists axes focusing on the parties' economic policies, or on their sentiment towards EU. To decide on the relative positions of parties on these axes, one of the most common methods is to use the party manifesto and the policies it contains to estimate the position of the party. In particular, this enables to see how the position of a political party evolve over time. The most famous database of political parties using this method is the *Comparative Manifesto Project* (Lehmann et al., 2024). However, the positions returned by this method are not always reliable: for instance, in 2018 it placed the French communist party further to the right than the centrist party *Mouvement Democrates*. Other widespread methods for placing parties on a left-right axis are to aggregate opinions of experts (see for instance the Chapel Hill Expert Survey, Jolly et al. (2022)), or the opinions of a representative sample of the population (see for instance The Comparative Study of Electoral Systems (2023), or the European Election Studies, Popa et al. (2024)).

[1]See for instance *The political leanings of the Supreme Court justices, Axios (Oriana González, Danielle Alberti)*, https://www.axios.com/2019/06/01/supreme-court-justices-ideology

[2]This picture was taken from the website of the newspaper *Liberation*, https://web.archive.org/web/20250228053455/https://www.liberation.fr/resultats-elections/.

In this chapter however, we propose novel methods that use the preferences of the voters to construct the ordering of the candidates. In that sense, these methods are *endogenous* to the preferences, while the other methods we mentioned are *exogenous*. Obviously, the possible methods we can use depend on the format of the preferences. For instance, the problem is well-studied in the case of ranking preferences (Niemi, 1969; Elkind and Lackner, 2014; Faliszewski et al., 2014; Escoffier et al., 2021), and we gave an overview of the main results in Section 2.3.3. In this chapter, we focus on the case of approval preferences. There are several reasons why approval preferences are interesting for this problem. In particular, it is easier to collect approval preferences than rankings and approval preferences intuitively seem more natural for building an axis. Despite this, there has been little work on this problem in the social choice literature. The main exception is the work of Lebon et al. (2017); Baujard and Lebon (2022), who used the approval preferences of participants in the *Voter Autrement* experiments (see Section 2.5.4) to construct an axis of the candidates in the French presidential elections of 2012 and 2017. We include the rule they used in our study, as well as several rules inspired by the literature on ranking preferences.

Note that approval preferences have also been used to describe the political landscape in other ways than constructing an axis. Laslier (2006) proposed a method to place candidates *and voters* on a multi-dimensional space using approval preferences, and tested it on the *Voter Autrement* data of the 2002 French presidential election (see also Laslier (2010a)). In their model, the probability for a voter to approve a candidate depends on the distance between them, and on the overall popularity of the candidate. In that sense, their model is closer to the Euclidean preference model (Section 2.5.1). Baujard et al. (2011) repeated this analysis on the data of the 2007 French presidential election. Alós-Ferrer and Granić (2015) conducted a similar study on the data of a 2012 experiment in Germany. These studies often conclude that one dimension is not enough to faithfully represent the political landscape. Finally, this problem is not restricted to the political context: Peress and Spirling (2010) propose for instance a way to embed movies in a multi-dimensional space based on the preferences (here, ratings) of the users on the movies.

Interval Approval Ballots

Intuitively, an axis perfectly depicts the data if every voter approves an *interval* of the axis. In other words, if a voter approves two candidates on the axis, then they must approve all candidates that are placed between them. This corresponds to *interval approval preferences*, that we introduced and discussed in Section 2.2.6. We say that the associated approval profile satisfies the *Candidate Interval* (CI) property (Elkind and Lackner, 2015), or that it is *linear*. Of course, it is very unlikely that real datasets will satisfy this property, and that there exists an axis that *perfectly* depicts the preferences. Thus, the methods we propose aim at finding an axis that best *approximate* the interval structure, and thereby provide a good ordinal one-dimensional embedding of the profile.

The most natural method, inspired by the *Voter Deletion* rule defined for ordinal preferences (see Section 2.3.3), selects the axes which minimize the number of voters whose ballots need to be removed from the profile so that all remaining votes are intervals of the axis. For their analysis of the French political landscape, Lebon et al. (2017) used another method, which selects the axes which minimize the total number of candidates that need to be added to approval ballots so that all ballots become intervals of the axis. These are two examples of the five objective functions we introduce and study in this chapter.

Algorithmically, the task of finding an axis optimizing a particular objective function is well studied. First of all, to check whether a perfect axis exists (i.e., an axis in which every approval ballot is an interval), one simply needs to check whether the 0–1 approval matrix satisfies the *consecutive ones property* (C1P), which can be done in linear time (Booth and Lueker, 1976). The problem of finding an axis that makes as many votes as possible into an interval (the *voter deletion* method) is NP-hard (Garey and Johnson, 1979, Problem SR14). Similarly, the problem of minimizing the number of "1" to add to the matrix to satisfy C1P (corresponding to the method used by Lebon et al. (2017)) is also known to be NP-hard (Problem SR16). More generally, the study of the algorithmic problem of recognizing profiles that are *nearly* C1P has received thorough attention in the literature (e.g., Hajiaghayi and Ganjali, 2002; Tan and Zhang, 2007; Chauve et al., 2009; Dom et al., 2010; Narayanaswamy and Subashini, 2015). We refer to Dom (2009) for an extensive survey on the consecutive ones property. However, these complexity-theoretic works do not tell us which of these objective functions "work best", which is what we want to uncover in this chapter.

Other Applications

Our problem finds applications beyond ordering political candidates with voters' approval preferences. Looking at it more mathematically, we can summarize the problem as follows: given a matrix of 0 and 1 (e.g., encoding approval ballots) that links *row* items (e.g., voters) to *column* items (e.g., candidates), find an ordering of the column items such that similar items are close to each other on this ordering, in the sense that they are associated with similar row items.

There exist many applications in which axes can help understanding and representing data, as well as direct use-cases where the axis itself plays a key role. If we first stay in the political context, we could use some other information than the approval preferences of voters to obtain an ordering of parties (or candidates), as long as this information can be represented by a binary matrix. For instance, we can interpret votes of the different parties that form a parliament on the bills that are proposed to them as 0 if they vote against a bill, and 1 if they vote in favor, and we could use our methods to find an ordering of the parties (here, the bills would be the "voters" that "approve" the parties). There also exist several other use-cases beyond political ones. In particular, *archaeological seriation* is a well-established approach in archaeology for ordering artefacts by their age. The idea is to let the features that were temporarily "in fashion" (e.g., drawing styles) approve artefacts on which they are found (Petrie, 1899; Baxter, 2003). In the true ordering by age, each feature is likely to induce an interval, as artefacts from the same period generally have similar features. Thus, a good axis rule will produce an ordering of artefacts by their age with few errors. More generally, these methods can be used for *relative dating* of geological or biological events. A typical example is if we want to order geological strata by chronological order, based on which kinds of ammonite fossils can be found in each stratum (here, the ammonite fossils "approve" the strata). This works well because such fossils have a very short lifespan as a species, but at the same time they generally cover a large geographical territory. Finally, to stay on the theme of *temporal* axes, our methods can also be used for *scheduling* purposes. Consider for instance a conference in which the organizers ask attendees which talks they wish to see, and then use our methods to arrange the talks so attendees can join for consecutive talks. In that sense, our problem is slightly related to *collective scheduling* (Pascual et al., 2018), where the goal is to output a collective schedule based on voters' preferences over the possible schedules (the model and the existing methods are however quite different). A different way of applying our methods (without the need to ask for attendees preferences) is for keywords to "approve" the talks/papers that mention them, leading to a thematically coherent ordering of the talks.

Outline of the Chapter

In this chapter, we provide a framework for answering the question of what is the "best" objective function, via the axiomatic analysis and the experimental analysis, as we did in the previous chapters. We interpret different objective functions as *rules* that take an approval profile as input and return an axis or a set of tied axes. In particular, we focus on the following five rules, which we define more precisely in Section 6.2:

- *Voter Deletion* (VD) selects the axes which minimize the number of voters whose ballots are not intervals.
- *Minimum Flips* (MF) selects the axes which minimize the number of approvals that need to be added or removed from ballots to make all ballots intervals of the axis.
- *Ballot Completion* (BC) selects the axes which minimize the number of approvals that need to be added to ballots to make all ballots intervals of the axis (but we cannot remove approvals).
- *Minimum Swaps* (MS) selects the axes which minimize the average number of swaps within the axes that are needed to turn ballots into intervals of the axis.
- *Forbidden Triples* (FT) selects the axes which minimize the total size of holes in a ballot, weighted by how many approved candidates they separate.

Note that all these rules aim to find an axis that minimizes some function. In that sense, all these rules are part of the *scoring rules* family, on which we will be focusing in this chapter.

On a high level, these five rules can be ordered on a spectrum based on how much ballot information they use to select the axes. This ranges from VD, which uses the least information by only considering whether a ballot represents an interval on the axis, to FT, which uses the most, as the placement of non-approved candidates among approved ones is a critical factor. Our axiomatic analysis, in Section 6.3, shows that the rules that use less information are generally more robust to changes. In particular, among the discussed rules, only VD and BC satisfy a desirable monotonicity property. Moreover, VD is the only scoring rule that satisfies this monotonicity property and *independence of clones* (Section 2.4.4), adapted to this context. On the other hand, the rules that use more information seem to have more interesting behaviors in practice. For instance, they tend to push disliked candidates towards the extremes of the axis. Moreover, FT is the only one of the five rules to satisfy the *clone-proximity* axiom, which says that two candidates that are approved by exactly the same voters should always be next to each other on the axis. Interestingly, we can show an incompatibility between our two clone-related axioms for scoring rules.

We complete the analysis in Section 6.4 by applying our rules to different datasets, including *Voter Autrement* datasets, votes of the justices of the US Supreme Court, tierlist datasets, and synthetic datasets. The simulations show how our rules differ, which perform best, and how they compare to rules that are based on rankings rather than approvals.

6.2 Axis Rules

In this chapter, we assume that we have a profile $P = (A_1, \dots, A_n)$ of approval ballots, and that we want to find an ordering of the candidates, called *axis*. We already introduced this structure in Section 2.3.3, but we recall here the main notations and definitions. An *axis* $\triangleleft \in \mathcal{L}(C)$ is a strict linear order of the candidates, so that $a \lhd b$ means that candidate a is strictly on the left of b on the axis. We write $a \unlhd b$ if $a \lhd b$ or $a = b$. For brevity, we sometimes omit the $\lhd$ and write for instance abc for the axis $a \lhd b \lhd c$. The direction of an axis is irrelevant, so we informally treat the axes $\lhd = abcd$ and $\overleftarrow{\lhd} = dcba$ as being the same axis.

We say that an approval ballot A is an *interval* of an axis $\lhd$ if for all pairs of approved candidates $a, b \in A$ and every c such that $a \lhd c \lhd b$, we have $c \in A$. If instead there exist $c \notin A$ and $a, b \in A$ such that $a \lhd c \lhd b$, we say that c is an *interfering candidate* on ballot A. A profile P is *linear* if there exists an axis $\lhd$ such that all approval ballots in P are intervals of $\lhd$. We also say that this axis $\lhd$ is *consistent* with the profile P. We write $\text{con}(P) \subseteq \mathcal{L}(C)$ for the set of all axes consistent with P. For instance, if $P = (\{a, b\}, \{b, c\}, \{c, d\})$, the axis $\lhd = abcd$ is consistent with P, as all approval ballots in P are intervals of $\lhd$.

For an approval ballot A and an axis $\lhd = c_1 c_2 \cdots c_m$ with candidates relabeled by their position on the axis, we denote by $x_{A,\lhd} = (x^1_{A,\lhd}, \dots, x^m_{A,\lhd})$ the *approval vector* where $x^j_{A,\lhd} = 1$ if $c_j \in A$, and 0 otherwise. For instance, for the axis $\lhd = abcd$ and the ballot $A = \{b, c\}$, we get the approval vector $(0, 1, 1, 0)$, while $A' = \{a, d\}$ gives the approval vector $(1, 0, 0, 1)$ (which has two interfering candidates b and c). The *approval matrix* of a profile $P = (A_1, \dots, A_n)$ has $x_{A_i,\lhd}$ as its ith row. Thus, its (i, j)-entry is equal to 1 if $c_j \in A_i$ and equal to 0 if $c_j \notin A_i$. Note that a profile is linear if and only if its approval matrix (derived from an arbitrary axis $\lhd$) satisfies the *consecutive ones property* (C1P), i.e., its columns can be reordered such that in each row, the "1"s form an interval.

An *axis rule* f is a function that takes as input an approval profile P and returns a non-empty set of axes $f(P) \subseteq \mathcal{L}(C)$, such that for each axis $\lhd$ in $f(P)$ its *reverse* axis $\overleftarrow{\lhd}$ is also in $f(P)$, encoding the idea that the direction of the axis does not matter. In this chapter, we focus on the family of *scoring rules*, which we define in analogy to other social choice settings (Myerson, 1995; Pivato, 2013a). Let $\mathsf{cost} : 2^C \times \mathcal{L}(C) \to \mathbb{R}_{\geqslant 0}$ be a *cost function*, indicating the cost $\mathsf{cost}(A_i, \lhd)$ that a ballot $A_i \in P$ incurs when the axis $\lhd$ is chosen. By summing up these costs, we get the cost of an axis $\lhd$ for the profile P: $\mathsf{cost}(P, \lhd) = \sum_{i \in V} \mathsf{cost}(A_i, \lhd)$. An axis rule f is a *scoring rule* if there is a cost function cost_f such that $f(P) = \text{argmin}_{\lhd \in \mathcal{L}(C)} \mathsf{cost}_f(P, \lhd)$ for all profiles P.

A focus on this class of scoring rules can be justified as an analogue to scoring rules in voting theory, in that every scoring rule satisfies the *reinforcement* axiom (Section 2.4.3), which says that if f chooses at least one common axis $\lhd$ in two disjoint profiles P_1 and P_2, so that $f(P_1) \cap f(P_2) \neq \emptyset$, then the axes it chooses in the combined profile $P_1 + P_2$ are exactly the common axes, i.e., $f(P_1 + P_2) = f(P_1) \cap f(P_2)$. Another motivation for scoring rules is their natural intepretation as *maximum likelihood estimators* when there is a ground truth axis (similarly to what was observed by Conitzer et al. (2009) and Pivato (2013b) in the voting setting). To see the connection, let $\lhd$ be the ground truth axis, and suppose voters obtain their approval ballots A_i i.i.d. from a probability distribution $\mathbb{P}(A_i \mid \lhd)$ (where intuitively ballots are more likely the closer they are to forming an interval of $\lhd$). Then, the likelihood of a profile P is $\mathbb{P}(P \mid \lhd) = \prod_i \mathbb{P}(A_i \mid \lhd)$. To find the axis inducing maximum likelihood, we solve:

$$\text{MLE}(P) := \underset{\lhd}{\text{argmax}}\, \mathbb{P}(P \mid \lhd) = \underset{\lhd}{\text{argmin}} \left(-\sum_i \log(\mathbb{P}(A_i \mid \lhd)) \right),$$

which is a scoring rule f with costs $\mathsf{cost}_f(A_i, \lhd) = -\log(\mathbb{P}(A_i \mid \lhd))$.

Rules

We now introduce five scoring rules. Many are inspired by objective functions proposed for near single-peakedness (that we discussed in Section 2.3.3), in which preferences are rankings (Faliszewski et al., 2014; Escoffier et al., 2021). The first rule is called *Voter Deletion* (VD).

Voter Deletion
This rule returns the axes that minimize the number of ballots to delete from the profile P in order to become consistent with it. This rule is a scoring rule based on the cost function $\mathsf{cost}_{\mathrm{VD}}$ such that $\mathsf{cost}_{\mathrm{VD}}(A, \lhd) = 0$ if A is an interval of $\lhd$ and 1 otherwise.

The idea behind this rule is that perhaps some "maverick" voters are "irrational", and should hence be disregarded. The aim is to delete as few maverick voters as possible. Figure 6.2 shows the costs of some ballots under VD. Observe that the rule gives the same cost to all non-interval ballots. An intuitive shortcoming of VD is that it does not measure the *degree of incompatibility* of a given vote with an axis. For example, VD does not distinguish ballots that miss just one candidate to be an interval, and an approval ballot in which only the two extreme candidates of the axis are approved. For this reason, more gradual rules might do better.

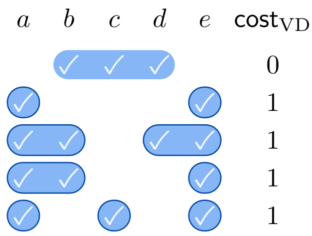


Figure 6.2: Costs of some ballots under VD.

The first rule in this direction is *Minimum Flips* (MF) which changes ballots by removing and adding candidates. The idea behind this rule is that the approval ballots of the voters are "noisy" versions of intervals of the axis, in the sense that some approvals 'bits' have been flipped (i.e., some candidates that should have been approved are not, and some candidates that should not have been approved are). The rule aims to find the axes that minimize the number of flips needed in each ballot to make the profile linear.

Minimum Flips
This rule returns the axes that minimize the number of candidates that need to be removed from or added to approval ballots in order to make the profile linear. It is the scoring rule based on:

$$\mathsf{cost}_{\mathrm{MF}}(A, \lhd) = \min_{x,y \in A : x \trianglelefteq y} |\{z \in A : z \lhd x \text{ or } y \lhd z\}| + |\{z \notin A : x \lhd z \lhd y\}|$$

The definition of $\mathsf{cost}_{\mathrm{MF}}$ optimizes the choice of the left and right-most candidates x and y in the ballots *after* removing and adding candidates, and then counts the number of candidates that need to be removed (first term of the sum) and added (second term) to the ballot. We can equivalently view MF as finding for each vote A_i the interval ballot closest to A_i in Hamming distance, with that distance being the cost of $\lhd$. In another equivalent view, MF finds the linear profile of minimum total Hamming distance to the input profile, and returns the axes consistent with this profile. Figure 6.3 shows the costs of some ballots under the MF rule. Observe that we can obtain an interval by only removing candidates (second ballot), by only adding candidates (third ballot), or by both

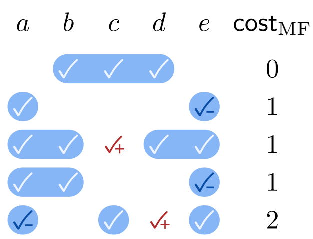


Figure 6.3: Costs of some ballots under MF. Candidates that need to be added are represented by red ticks, candidates that need to be removed by blue ticks.

removing and adding candidates (last ballot).

In many applications, adding approvals seems better motivated than removing them. For example, a voter i might not approve a candidate c because they do not know who c is; fixing this error corresponds to adding a candidate. On the other hand, approving a candidate by accident seems less likely. The *Ballot Completion* (BC) rule implements this thought.

Ballot Completion
This rule returns the axes that minimize the number of candidates to add to approval ballots to make the profile consistent with it. It is the scoring rule based on:

$$\mathsf{cost}_{\mathrm{BC}}(A, \lhd) = |\{b \notin A : a \lhd b \lhd c \text{ for some } a, c \in A\}|$$

Thus, given a ballot A and an axis $\lhd$, this rule counts all interfering candidates with respect to A and $\lhd$. To see the difference between MF and BC, observe that $\mathsf{cost}_{\mathrm{BC}}(\{a, d\}, abcd) = 2$, as we need to add b and c to obtain an interval, while $\mathsf{cost}_{\mathrm{MF}}(\{a, d\}, abcd) = 1$, as we can just remove a. Figure 6.4 shows the costs of some ballots under the BC rule.

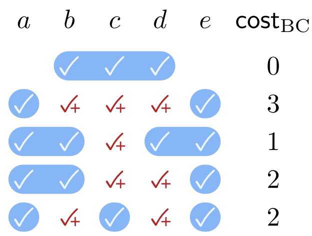


Figure 6.4: Costs of some ballots under the BC rule. Candidates that need to be added are represented by red ticks.

In the approval context, BC is the only rule we know of that has already been used in the literature to find an underlying political axis of voters, on the data of *Voter Autrement* experiments conducted during the 2012 and 2017 French presidential elections (Lebon et al., 2017; Baujard and Lebon, 2022). The axes found by BC were close to the orderings commonly discussed in the media.

In all the rules introduced so far, we modify the ballots (by adding and/or removing candidates) so that they become intervals of the optimal axis, and find the axis that require the minimal amount of changes. The next rule, called *Minimum Swaps* (MS), modifies the *axis* rather than the ballots. Given an approval ballot A, MS asks how many candidate swaps we need to perform in an axis $\lhd$ until A becomes an interval of it: the cost $\mathsf{cost}_{\mathrm{MS}}$ is the minimum swap distance between $\lhd$ and an axis $\lhd'$ such that A is an interval of $\lhd'$ (see Section 2.3.1 for a definition of the swap/Kendall-tau distance). For instance, $\mathsf{cost}_{\mathrm{MS}}(\{a, d\}, abcd) = 2$ because we need to have a next to d on any axis consistent with $\{a, d\}$ and we need at least two swaps to obtain this.

Minimum Swaps
This scoring rule is based on the cost function:

$$\mathsf{cost}_{\mathrm{MS}}(A, \lhd) = \sum_{x \notin A} \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\}|).$$

To see why this formula implements our swapping description of $\mathsf{cost}_{\mathrm{MS}}(A, \lhd)$, note that to modify the axis $\lhd$ such that A becomes an interval of it, we need to "push outside" all interfering candidates ($x \notin A$ such that there exists $y, z \in A$ with $y \lhd x \lhd z$). Let us prove it more formally. Fix some ballot A and some axis $\lhd$. Let $\lhd'$ be an axis of minimum swap distance to $\lhd$ and such that A is an interval of $\lhd'$. Write $\mathrm{KT}(\lhd, \lhd')$ this minimum swap distance (or Kendall-tau distance). It is clear that $\lhd$ and $\lhd'$ agree on the ordering of the approved candidates in A: if they ordered some pair of approved alternatives in different ways, we could swap them in one of the axes and thereby reduce the swap distance.

We first show that $\mathrm{KT}(\lhd, \lhd') \geqslant \sum_{x \notin A} \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\})$. Because A is an interval of $\lhd'$, every non-approved candidates $x \notin A$ must appear either to the left or to the right of all approved candidates in $\lhd'$. Thus, x must have been swapped with at least all candidates $y \in A$ to its right or to its left, giving us the lower bound. To see that $\mathrm{KT}(\lhd, \lhd') \leqslant \sum_{x \notin A} \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\})$, partition the set $C \setminus A$ of non-approved candidates into two parts, corresponding to those candidates for which it is cheaper to push them to the left or to the right, respectively:

$$L = \{x \in C \setminus A : |\{y \in A : y \lhd x\}| < |\{y \in A : x \lhd y\}|\},$$
$$R = \{x \in C \setminus A : |\{y \in A : y \lhd x\}| \geqslant |\{y \in A : x \lhd y\}|\}.$$

Consider the axis $\lhd'' = \lhd_L \lhd_A \lhd_R$ obtained by placing the candidates of L on the left, the candidates of A in the center and the candidates of R on the right, but keeping the same relative ordering of candidates as $\lhd$ within each set L, A, and R. By construction, A is an interval of $\lhd''$, and it is easy to compute that the swap distance between $\lhd$ and $\lhd''$ is precisely $\mathrm{KT}(\lhd, \lhd'') = \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\})$, which by minimality of $\mathrm{KT}(\lhd, \lhd')$ must be at least as large as $\mathrm{KT}(\lhd, \lhd'')$.

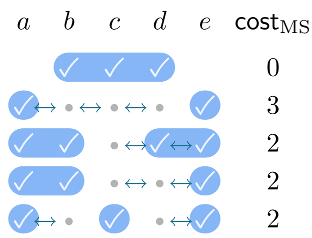


Figure 6.5: Costs of some ballots under the MS rule. Arrows indicate the swaps needed to make the vote an interval.

Figure 6.5 shows the costs of some ballots under the MS rule. Note that the order in which the swaps are performed matters. For instance, we need to swap the same pairs of candidates for the third and the fourth ballots ($\{c, d\}$ and $\{d, e\}$), but we start by swapping c and d in the third ballot and we start with d and e in the fourth ballot.

Our last rule *Forbidden Triples* (FT), is inspired by a proposal for rankings by Escoffier et al. (2021). It is defined by counting the number of violations of the interval condition.

Forbidden Triples
This scoring rule is based on the cost function:

$$\mathsf{cost}_{\mathrm{FT}}(A, \lhd) = |\{(x, y, z) : x, z \in A, y \notin A, x \lhd y \lhd z\}|.$$

Note that there is one forbidden triple for each combination of an interfering candidate and a pair of candidate lying on its left and its right, respectively. Thus, we also have:

$$\mathsf{cost}_{\mathrm{FT}}(A, \lhd) = \sum_{x \notin A} |\{y \in A : y \lhd x\}| \times |\{y \in A : x \lhd y\}|.$$

For instance, we have $\mathsf{cost}_{\mathrm{FT}}(\{a, b, d, e\}, abcde) = 2 \times 2 = 4$ while $\mathsf{cost}_{\mathrm{FT}}(\{a, b, c, e\}, abcde) = 3 \times 1 = 3$. Intuitively, this rule looks at the holes in a vote, with larger holes separating many approved candidates counting more. Figure 6.6 shows the costs of some ballots under the FT rule.

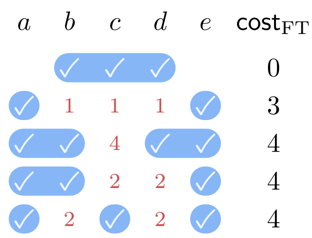


Figure 6.6: Costs of some ballots under the FT rule. The number of forbidden triples involving each interfering candidate is shown in red.

Observe that these rules are *distance-rationalizable* (as defined in Section 2.3.3), if we consider the consensus class of *linear profiles*. In particular, the distance for VD is the discrete distance, and the one for MF is the Hamming distance.

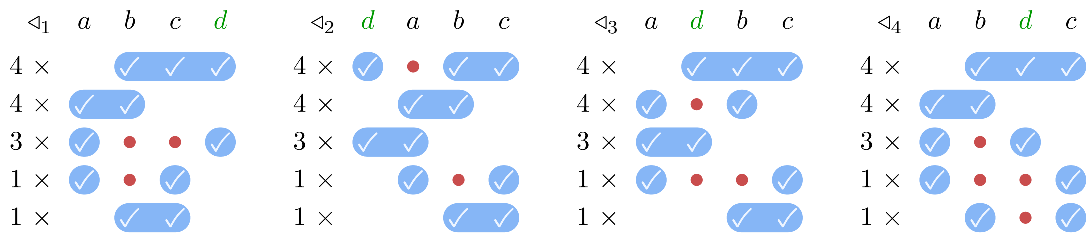


Figure 6.7: Profile of Example 6.1 on 4 different axes. Red circles indicate interfering candidates.

Linking the Rules

The cost functions of our five scoring rules can be related via a chain of inequalities, suggesting that they form a natural collection of rules to study.

Proposition 6.1
For all ballots A and axes $\lhd$, we have:

$$\mathsf{cost}_{\mathrm{VD}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{MF}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{BC}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{MS}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{FT}}(A,\lhd).$$

Proof. To see $\mathsf{cost}_{\mathrm{VD}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{MF}}(A,\lhd)$, note that if A is not an interval of $\lhd$ then $\mathsf{cost}_{\mathrm{VD}}(A,\lhd) = 1$ and at least one candidate must be flipped to make A an interval of $\lhd$, so $\mathsf{cost}_{\mathrm{MF}}(A,\lhd) \geqslant 1$. If A is an interval then $\mathsf{cost}_{\mathrm{MF}}(A,\lhd) = \mathsf{cost}_{\mathrm{VD}}(A,\lhd) = 0$.

We have $\mathsf{cost}_{\mathrm{MF}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{BC}}(A,\lhd)$ because in MF we can add and remove approvals, but in BC we can only add approvals. We can also see this from the formal definitions: $\mathsf{cost}_{\mathrm{MF}}(A,\lhd)$ is a minimum, and if x and y are the left-most and right-most approved candidates in A, we obtain the value of $\mathsf{cost}_{\mathrm{BC}}(A,\lhd)$, which must thus be at least as high as the minimum over x and y.

Finally, observe that for any interfering candidate x on A, $\min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\}|) \geqslant 1$. Moreover, as these are all natural numbers, $\min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\}|) \leqslant |\{y \in A : y \lhd x\}| \times |\{y \in A : x \lhd y\}|$. Thus, $\mathsf{cost}_{\mathrm{BC}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{MS}}(A,\lhd) \leqslant \mathsf{cost}_{\mathrm{FT}}(A,\lhd)$ by the definitions of these rules. □

Note that the rules we consider here are invariant to rescaling the costs. Thus, we could have different cost functions for the same rules (by multiplying all costs by some constant factor), and they would not necessarily satisfy the inequalities in Proposition 6.1. However, the five cost functions we consider in this result are all scaled such that the minimal cost of a ballot that is not an interval of the axis is 1. Thus, it is a natural scaling for comparing the different rules.

We say that two axis rules f_1 and f_2 are *equivalent* if for all profiles P we have $f_1(P) = f_2(P)$. Note that if $n \leqslant 2$ or $m \leqslant 2$, every profile is linear. Moreover, if there are $m = 3$ candidates, all the rules defined in this section are equivalent (as there is only one non-interval approval vector, so the only possible costs are 0 and 1). If there are $m = 4$ candidates, VD and MF are equivalent and BC and MS are equivalent, because their respective cost functions coincide. This does not remain true for $m \geqslant 5$, for which the rules are pairwise non-equivalent. Example 6.1 discusses a profile with $m = 4$ candidates for which VD, BC, and FT all select different axes.

Example 6.1
Consider the following profile P

$$4 : \{b, c, d\} \qquad 4 : \{a, b\} \qquad 3 : \{a, d\} \qquad 1 : \{a, c\} \qquad 1 : \{b, c\}$$

On this profile, all rules agree that $a \lhd b \lhd c$, but they disagree on the position of d. Indeed, $\lhd_1 = abc\underline{d}$ is optimal for VD and MF, $\lhd_2 = \underline{d}abc$ for BC and MS, and $\lhd_3 = a\underline{d}bc$ and $\lhd_4 = ab\underline{d}c$ for FT. Figure 6.7 shows the profile aligned according to the four possible axes. One can easily see that among these axes (1) the axis $\lhd_1$ on the left minimizes the VD cost with only 4 non-interval ballots, (2) the axis $\lhd_2$ in the center-right minimizes the BC cost with 5 interfering candidates (red circles) and (3) the axes $\lhd_3$ and $\lhd_4$ on the right minimize the FT cost with 6 forbidden triplets.

Computational Complexity

Problems about recognizing matrices that are almost C1P have long been known to be NP-hard. Hardness of VD and BC is explicitly known, as computing VD is equivalent to the *consecutive ones submatrix* problem (Booth, 1975, Theorem 4.24), and BC to the *consecutive ones matrix completion* problem (Booth, 1975, Theorem 4.19), which are both NP-hard. Moreover, the reductions only use approval ballots of size at most two (i.e., $\max_i |A_i| = 2$). The result for all the rules follows from the fact that if approval ballots have size at most two, MF is equivalent to VD and MS and FT are equivalent to BC.

Theorem 6.2
The VD, MF, BC, MS, and FT rules are NP-hard to compute, even for profiles in which every ballot approves at most 2 candidates.

6.3 Axiomatic Analysis

In this section, we conduct an axiomatic analysis of the rules we introduced. We start with some basic axioms that all these rules satisfy. The first two are the classical symmetry axioms (Section 2.4.1): a rule f is *anonymous* if whenever two profiles P and P' are such that every ballot appears exactly as often in P as in P', then $f(P) = f(P')$. It is *neutral* if for every profile P, renaming the candidates in P leads to the same renaming in $f(P)$. All scoring rules are by definition anonymous, as they compute a sum of costs over all voters, regardless of which voters have which preferences. The third basic property fundamentally captures the aim of an axis rule: if there are perfect axes for the profile, then the rule should return exactly those.

Consistency with linearity
An axis rule f is *consistent with linearity* if $f(P) = \mathrm{con}(P)$ for all linear profiles P.

If f is a scoring rule and it satisfies these three axioms, this means that f can be associated with a cost function that has a certain structure. In particular, the cost function attains its minimum value for intervals, it is invariant under reversing the axis, and it is symmetric.

Lemma 6.3
Let f be a scoring rule. Then f is neutral and consistent with linearity if and only if there exists a cost function cost_f that induces it and that satisfies:

(1) for all approval ballots A and all axes $\lhd$, we have $\mathsf{cost}_f(A, \lhd) \geqslant 0$, and $\mathsf{cost}_f(A, \lhd) = 0$ if and only if A is an interval of $\lhd$,

(2) for all approval ballots A and all axes $\lhd$, we have $\mathsf{cost}_f(A, \lhd) = \mathsf{cost}_f(A, \overleftarrow{\lhd})$, and

(3) there exists a function $g : \{0,1\}^m \to \mathbb{R}_{\geqslant 0}$ such that for all approval ballots A and all axes $\lhd$, we have $\mathsf{cost}_f(A, \lhd) = g(x_{A,\lhd}) = g(x_{A,\overleftarrow{\lhd}})$ (in other words, cost_f depends only on the induced approval vector $x_{A,\lhd}$).

Proof. Let f be a scoring rule induced by some cost function cost. It is straightforward to check that if cost satisfies all three conditions, then f is neutral (due to (3)) and consistent with linearity (due to (1)).

For the other direction, consider the cost function cost' such that for all approval ballot A and axis $\lhd$, we have:

$$\mathsf{cost}'(A, \lhd) = \mathsf{cost}(A, \lhd) - \min_{\lhd' \in \mathcal{L}(C)} \mathsf{cost}(A, \lhd').$$

It is clear that cost' still induces f as for all profiles P and axis $\lhd$, we have $\mathsf{cost}'(P, \lhd) = \mathsf{cost}(P, \lhd) - \sum_{A \in P} \min_{\lhd' \in \mathcal{L}(C)} \mathsf{cost}(A, \lhd')$, and the second term of the subtraction is constant for all axes $\lhd$. Thus, the axes minimizing cost are exactly the ones that minimize cost'. Moreover, we have $\min_{\lhd} \mathsf{cost}'(A, \lhd) = 0$. We will show that this cost function cost' satisfies (1) and (2), and afterwards we will derive another cost function that also induces f, and that additionally satisfies (3).

We first show (1). That we always have $\mathsf{cost}'(A, \lhd) \geqslant 0$ is clear from our choice of cost function. Assume for a contradiction that there is an axis $\lhd$ and a ballot A such that $\mathsf{cost}'(A, \lhd) = 0$ but A is not an interval of $\lhd$. Then, on the linear profile $P = \{A\}$, we have $\lhd \in f(P)$, which is a contradiction with consistency with linearity. Similarly, if A is an interval of $\lhd$ but $\mathsf{cost}'(A, \lhd) > 0$, then on the linear profile $P = \{A\}$, we have $\lhd \notin f(P)$ while $\lhd$ is consistent with P, a contradiction with consistency with linearity.

We now show (2). If A is an interval of $\lhd$, it is also an interval of $\overleftarrow{\lhd}$, so from (1) we clearly have $\mathsf{cost}'(A, \lhd) = \mathsf{cost}'(A, \overleftarrow{\lhd})$. Assume now that A is not an interval of $\lhd$. Thus, $y = \mathsf{cost}'(A, \lhd) > 0$ and $y' = \mathsf{cost}'(A, \overleftarrow{\lhd}) > 0$. Assume for a contradiction that $y \neq y'$, and without loss of generality that $y < y'$. Let us denote the candidates $c_1, \dots, c_m$ such that $\lhd = c_1 c_2 \dots c_m$. Moreover, let $z > 0$ be the minimum value of $\mathsf{cost}'(A', \lhd)$ over all ballots A' that are not intervals of $\lhd$. Take $q \in \mathbb{N}$ such that $q > y/z$ and consider the profile P which contains A and q ballots $\{c_i, c_{i+1}\}$ for each $i \in [1, m-1]$. Clearly, any axis $\lhd' \notin \{\lhd, \overleftarrow{\lhd}\}$ is breaking at least one pair, inducing a cost greater than $q \cdot z > y$. The cost of $\lhd$ is y and the cost of $\overleftarrow{\lhd}$ is $y' > y$. Thus, $f(P) = \{\lhd\}$ which contradicts the definition of axis rules (which requires that whenever an axis is selected, then so is its reverse axis). Therefore, $y = y'$ and $\mathsf{cost}'(A, \lhd) = \mathsf{cost}'(A, \overleftarrow{\lhd})$.

For (3), we show that f is induced by a cost function cost^* such that $\mathsf{cost}^*(A, \lhd)$ only depends on $x_{A,\lhd}$. Let Π be the set of all permutations of the candidates. For a permutation $\pi \in \Pi$, for each ballot A we write $\pi(A) = \{\pi(a) : a \in A\}$, and for each axis $\lhd = c_1 \dots c_m$ we write $\pi(\lhd) = \pi(c_1) \dots \pi(c_m)$. Then we define:

$$\mathsf{cost}^*(A, \lhd) = \sum_{\pi \in \Pi} \mathsf{cost}'(\pi(A), \pi(\lhd)) \text{ for all } A \text{ and } \lhd.$$

We will show that this cost function still induces f, and that it satisfies conditions (1)–(3).

To show that f is still induced by this cost function, let $\lhd \in f(P)$ be an optimal axis for profile P. Then, by neutrality, $\pi(\lhd) \in f(\pi(P))$ for all $\pi \in \Pi$. This implies that $\mathsf{cost}(\pi(\lhd), \pi(P)) \leqslant \mathsf{cost}(\pi(\lhd'), \pi(P))$ for all axes $\lhd' \in \mathcal{L}(C)$. Since this inequality carries over to the sum over all $\pi \in \Pi$, this implies $\mathsf{cost}^*(\lhd, P) \leqslant \mathsf{cost}^*(\lhd', P)$ for all $\lhd'$. For the other direction, let $\lhd' \notin f(P)$ and fix some $\lhd \in f(P)$. With the same argument, we obtain $\mathsf{cost}^*(\lhd, P) < \mathsf{cost}^*(\lhd', P)$, which shows that, for all profiles, an axis $\lhd$ has minimal cost w.r.t. cost^* if and only if it is chosen by f.

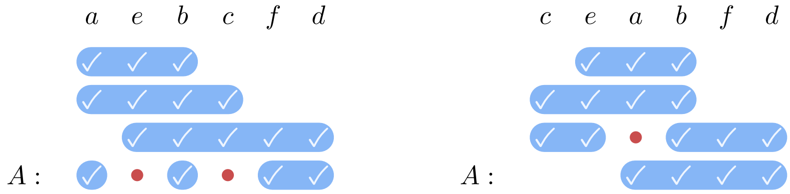


Figure 6.8: Profile P and ballot A (the last row) with the axes $aebcfd$ and $ceabfd$ in the proof of Proposition 6.4. Red circles indicate interfering candidates.

Finally, we check that cost^* satisfies the conditions of the lemma. Because cost' satisfies conditions (1) and (2), it is clear that cost^* also satisfies conditions (1) and (2). For condition (3), take any $A, \lhd$ and $A', \lhd'$ with the same approval vector, i.e., $x_{A,\lhd} = x_{A',\lhd'}$. Then, there exists a permutation $\tau \in \Pi$ with $\tau(A) = A'$ and $\tau(\lhd) = \lhd'$. Thus, we obtain that $\mathsf{cost}^*(A', \lhd') = \mathsf{cost}^*(\tau(A), \tau(\lhd)) = \sum_{\pi \in \Pi} \mathsf{cost}'(\pi(\tau(A)), \pi(\tau(\lhd))) = \sum_{\pi' \in \Pi} \mathsf{cost}'(\pi'(A), \pi'(\lhd)) = \mathsf{cost}^*(A, \lhd)$. □

Stability and Monotonicity

Some rules are more sensitive to changes than others. Intuitively, Voter Deletion rarely reacts to changes in the profile, as it only checks whether the ballots are intervals of the axis or not. Thus, a single voter will have little effect on the axes selected. Indeed, for VD, adding a new ballot to the profile cannot completely change the set of optimal solutions. For other rules, this is not the case. We can formalize this behavior with the following axiom.

Stability
An axis rule f satisfies *stability* if for every profile P and every approval ballot A, we have $f(P) \cap f(P + \{A\}) \neq \emptyset$.

This axiom is also considered by Tydrichová (2023, Sec. 4.4.2) in the context of rankings, and a similar axiom is used by Ceron and Gonzalez (2021) to characterize approval voting among approval-based single-winner voting rules.

Proposition 6.4
Stability is satisfied by VD, but not by MF, BC, MS and FT.

Proof. Let us prove that VD satisfies stability. Let P be a profile and A an approval ballot. If $\text{VD}(P + \{A\}) \subseteq \text{VD}(P)$, then clearly $\text{VD}(P) \cap \text{VD}(P + \{A\}) \neq \emptyset$. Otherwise, let $\lhd_1 \in \text{VD}(P + \{A\}) \setminus \text{VD}(P)$ and $\lhd_2 \in \text{VD}(P)$. Then, $\mathsf{cost}_{\text{VD}}(P, \lhd_2) \leqslant \mathsf{cost}_{\text{VD}}(P, \lhd_1) - 1$. Moreover, by definition of VD, $0 \leqslant \mathsf{cost}_{\text{VD}}(A, \lhd) \leqslant 1$ for all axes $\lhd$. Put together, this gives:

$$\begin{aligned}
\mathsf{cost}_{\text{VD}}(P + \{A\}, \lhd_2) &= \mathsf{cost}_{\text{VD}}(P, \lhd_2) + \mathsf{cost}_{\text{VD}}(A, \lhd_2) \\
&\leqslant (\mathsf{cost}_{\text{VD}}(P, \lhd_1) - 1) + 1 \\
&\leqslant \mathsf{cost}_{\text{VD}}(P, \lhd_1) + \mathsf{cost}_{\text{VD}}(A, \lhd_1) \\
&= \mathsf{cost}_{\text{VD}}(P + \{A\}, \lhd_1)
\end{aligned}$$

Therefore, $\mathsf{cost}_{\text{VD}}(P + \{A\}, \lhd_2) \leqslant \mathsf{cost}_{\text{VD}}(P + \{A\}, \lhd_1)$ and thus, because $\lhd_1 \in \text{VD}(P + \{A\})$, we must also have $\lhd_2 \in \text{VD}(P + \{A\})$, and thus $\text{VD}(P) \cap \text{VD}(P + \{A\}) \neq \emptyset$, as required.

For $f \in \{\mathrm{MF}, \mathrm{BC}, \mathrm{MS}, \mathrm{FT}\}$, let us consider the following profile P:

1: $\{a, b, e\}$ 1: $\{a, b, c, e\}$ 1: $\{b, c, d, e, f\}$

By consistency with linearity, $f(P) = \{aebcfd, aebcdf, abecdf, abecfd\}$ (up to the reverse axes). Now, consider the ballot $A = \{a, b, d, f\}$. For every $\lhd \in f(P)$, we have $\mathsf{cost}_{\mathrm{MF}}(P + \{A\}, \lhd) = \mathsf{cost}_{\mathrm{BC}}(P + \{A\}, \lhd) = 2$, $\mathsf{cost}_{\mathrm{MS}}(P + \{A\}, \lhd) \in \{3, 4\}$ and $\mathsf{cost}_{\mathrm{FT}}(P + \{A\}, \lhd) \in \{7, 8\}$.

However, let us consider the axis $\lhd' = ceabfd \notin f(P)$. As one can see on Figure 6.8, the only ballot in $P + \{A\}$ that is not an interval of $\lhd'$ is $\{b, c, d, e, f\}$, and thus we can calculate that $\mathsf{cost}_{\mathrm{MF}}(P+\{A\}, \lhd') = \mathsf{cost}_{\mathrm{BC}}(P+\{A\}, \lhd') = 1$, $\mathsf{cost}_{\mathrm{MS}}(P+\{A\}, \lhd') = 2$ and $\mathsf{cost}_{\mathrm{FT}}(P+\{A\}, \lhd') = 6$. Therefore, none of the axes in $f(P)$ is optimal for the profile $P+\{A\}$, and hence $f(P) \cap f(P+\{A\}) = \emptyset$ for $f \in \{\mathrm{MF}, \mathrm{BC}, \mathrm{MS}, \mathrm{FT}\}$. Thus, these rules do not satisfy stability. $\square$

Whether stability is a desirable property depends on the context: while it implies that the rule is robust, it also means that the rule might disregard too much information. In some cases, it is clearer that robustness to changes is desirable. In particular, monotonicity axioms say that if the input changes so as to more strongly support the current outcome, then the outcome should remain selected after the change (see Section 2.4.5). In our setting, monotonicity says that if an axis $\lhd$ is selected and some voters *complete* their ballots by approving all interfering candidates with respect to $\lhd$, then $\lhd$ should continue being selected.

Ballot monotonicity
An axis rule f satisfies *ballot monotonicity* if for every profile P, ballot $A \in P$ and axis $\lhd \in f(P)$ such that A is not an interval of $\lhd$, we still have $\lhd \in f(P')$ for the profile P' obtained from P by replacing A by the ballot $A' = \{x \in C : \exists y, z \in A \text{ s.t. } y \trianglelefteq x \trianglelefteq z\}$, which is an interval of $\lhd$.

Both VD and BC satisfy this axiom. To see why, observe that for a profile P and $\lhd \in f(P)$, by changing the ballot A to A', we decrease the VD cost of $\lhd$ by exactly 1 (because A' is an interval but not A), and the VD cost of all other axes by at most 1, so $\lhd$ is still selected. Similarly, suppose that the BC cost of A for $\lhd$ is $k > 0$, this means that A' contains k more candidates than A. Then, the change to A' reduces the BC cost of $\lhd$ by exactly k, and the cost of all other axes by at most k, so $\lhd$ is still selected. On the other hand, MF, MS and FT fail this axiom.

Proposition 6.5
Ballot monotonicity is satisfied by VD and BC, but not by MF, MS and FT.

Proof. We first prove the result more formally for VD and BC. Let P be a profile and $\lhd \in f(P)$ an optimal axis. Let $A \in P$ be a ballot that is not an interval of $\lhd$, $A' = \{x \in C : \exists y, z \in A \text{ s.t. } y \trianglelefteq x \trianglelefteq z\}$ the completion of A, and P' the profile obtained from P by replacing A by A'.

For VD, we have that $\mathsf{cost}_{\mathrm{VD}}(A, \lhd) = 1$ and $\mathsf{cost}_{\mathrm{VD}}(A', \lhd) = 0$. For every axis $\lhd'$, we have $\mathsf{cost}_{\mathrm{VD}}(A, \lhd') \leqslant 1$ and $\mathsf{cost}_{\mathrm{VD}}(A', \lhd') \geqslant 0$. This gives the following.

$$\mathsf{cost}_{\mathrm{VD}}(P', \lhd) = \mathsf{cost}_{\mathrm{VD}}(P, \lhd) - \mathsf{cost}_{\mathrm{VD}}(A, \lhd) + \mathsf{cost}_{\mathrm{VD}}(A', \lhd) = \mathsf{cost}_{\mathrm{VD}}(P, \lhd) - 1, \text{ and}$$
$$\mathsf{cost}_{\mathrm{VD}}(P', \lhd') = \mathsf{cost}_{\mathrm{VD}}(P, \lhd') - \mathsf{cost}_{\mathrm{VD}}(A, \lhd') + \mathsf{cost}_{\mathrm{VD}}(A', \lhd') \geqslant \mathsf{cost}_{\mathrm{VD}}(P, \lhd') - 1 \text{ for all } \lhd'.$$

Since $\lhd \in f(P)$, we have $\mathsf{cost}_{\mathrm{VD}}(P, \lhd) \leqslant \mathsf{cost}_{\mathrm{VD}}(P, \lhd')$ and thus $\mathsf{cost}_{\mathrm{VD}}(P', \lhd) \leqslant \mathsf{cost}_{\mathrm{VD}}(P', \lhd')$ for all axes $\lhd'$. Therefore, $\lhd \in f(P')$, and VD satisfies ballot monotonicity.

We use a similar reasoning for BC. Let $k = \mathsf{cost}_{\mathrm{BC}}(A, \lhd)$ and $\mathsf{cost}_{\mathrm{BC}}(A', \lhd) = 0$. This means that we add $k > 0$ candidates to ballot A, thereby obtaining a ballot A' that forms an interval of

$\lhd$. As before, since we added only k candidates, the cost of any other axis $\lhd'$ decreases by at most k, i.e., $\mathsf{cost}_{\mathrm{BC}}(A, \lhd') - \mathsf{cost}_{\mathrm{BC}}(A', \lhd') \leqslant k$. Thus,

$$\mathsf{cost}_{\mathrm{BC}}(P', \lhd) = \mathsf{cost}_{\mathrm{BC}}(P, \lhd) - k + 0 = \mathsf{cost}_{\mathrm{BC}}(P, \lhd) - k, \text{ and}$$
$$\mathsf{cost}_{\mathrm{BC}}(P', \lhd') = \mathsf{cost}_{\mathrm{BC}}(P, \lhd') - \mathsf{cost}_{\mathrm{BC}}(A, \lhd') + \mathsf{cost}_{\mathrm{BC}}(A', \lhd') \geqslant \mathsf{cost}_{\mathrm{BC}}(P, \lhd') - k \text{ for all } \lhd'.$$

Since $\lhd \in f(P)$, we have $\mathsf{cost}_{\mathrm{BC}}(P, \lhd) \leqslant \mathsf{cost}_{\mathrm{BC}}(P, \lhd')$ and thus $\mathsf{cost}_{\mathrm{BC}}(P', \lhd) \leqslant \mathsf{cost}_{\mathrm{BC}}(P', \lhd')$ for all $\lhd'$. Therefore, $\lhd \in f(P')$, and BC satisfies ballot monotonicity.

To see that the other rules do not satisfy ballot monotonicity, consider $f \in \{\mathrm{MF}, \mathrm{MS}, \mathrm{FT}\}$, a set of 6 candidates $C = \{a, b, c, d, e, f\}$, and the profile P containing each of the $\binom{6}{4}$ possible ballots of 4 candidates once. As f satisfies neutrality and the profile is symmetric, all axes are chosen, and there is $x \in \mathbb{R}$ such that $\mathsf{cost}_f(\lhd, P) = x$ for all $\lhd$. Consider now the axis $\lhd_1 = abcdef$, and the ballot $A = \{a, b, c, f\} \in P$. Let P' be the profile obtained from P in which A is replaced by $A' = \{a, b, c, d, e, f\}$. Since $\lhd_1 \in f(P)$, it suffices to show that $\lhd_1 \notin f(P')$. Let $\lhd_2 = abdefc$. For every axis $\lhd$, we have $\mathsf{cost}_f(\lhd, A') = 0$, and thus by definition of P', we have

$$\mathsf{cost}_f(\lhd, P') = \mathsf{cost}_f(\lhd, P) + \mathsf{cost}_f(\lhd, A') - \mathsf{cost}_f(\lhd, A) = x - \mathsf{cost}_f(\lhd, A).$$

Additionally, note that for MF, MS, and FT, the cost of A on $\lhd_1$ is respectively 1, 2, and 6, while the cost on $\lhd_2$ is respectively 2, 4, and 8. Therefore, we have $\mathsf{cost}_f(\lhd_2, A) > \mathsf{cost}_f(\lhd_1, A)$, which implies $\mathsf{cost}_f(\lhd_2, P') < \mathsf{cost}_f(\lhd_1, P')$, and so $\lhd_1 \notin f(P')$, as required. □

Centrists and Outliers

On a high level, good axes should place less popular candidates towards the extremes, where they are less likely to destroy intervals. Conversely, popular candidates are safer to place in the center. We define two axioms that identify profiles where this expectation is strongest, and that require candidates to be accordingly placed in center or extreme positions.

Our first axiom considers the placement of highly unpopular candidates. The axiom is easiest to satisfy by placing them at the extremes, but it does not require doing so in all cases.

Clearance
An axis rule f satisfies *clearance* if for every profile P in which some candidate x is never approved, all $\lhd \in f(P)$ are such that there is no $A \in P$ with $y, z \in A$ and $y \lhd x \lhd z$.

Thus, under clearance, never-approved candidates cannot be interfering.

Proposition 6.6
Clearance is satisfied by BC, MS and FT, but not by VD and MF.

Proof. Let $f \in \{\mathrm{BC}, \mathrm{MS}, \mathrm{FT}\}$. We show that f satisfies clearance. Let P be a profile with a never-approved candidate x and let $\lhd$ be an axis such that there is a ballot A in P with x interfering A on $\lhd$. We will show that $\lhd \notin f(P)$. Consider the axis $\lhd'$ identical to $\lhd$ but in which x was moved to the left extreme. As x is interfering A on $\lhd$, we have

$$\mathsf{cost}_{\mathrm{BC}}(A, \lhd') = \mathsf{cost}_{\mathrm{BC}}(A, \lhd) - 1,$$
$$\mathsf{cost}_{\mathrm{MS}}(A, \lhd') = \mathsf{cost}_{\mathrm{MS}}(A, \lhd) - \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\}|), \text{ and}$$
$$\mathsf{cost}_{\mathrm{FT}}(A, \lhd') = \mathsf{cost}_{\mathrm{FT}}(A, \lhd) - |\{y \in A : y \lhd x\}| \cdot |\{y \in A : x \lhd y\}|.$$

In each case, we have $\mathsf{cost}_f(A, \lhd') < \mathsf{cost}_f(A, \lhd)$. This is true for all ballots A for which x is an interfering candidate on $\lhd$. For all other ballots $A \in P$ for which x is not interfering, note that already in $\lhd$, candidate x is placed outside of all candidates approved by A (either to the left of the left-most approved candidate, or to the right of the right-most approved candidate). In $\lhd'$, we have moved x to an even more extreme position, but it follows from the rules' definitions that this does not change the cost, i.e., $\mathsf{cost}_f(A, \lhd') = \mathsf{cost}_f(A, \lhd)$. Since there exists at least one ballot for which x is interfering on $\lhd$, we have that $\mathsf{cost}_f(P, \lhd') < \mathsf{cost}_f(P, \lhd)$ and $\lhd \notin f(P)$, as required.

Now let $f \in \{\text{VD}, \text{MF}\}$. Consider the profile $P = (\{a, b\}, \{a, c\}, \{a, d\})$ on the set of candidates $C = \{a, b, c, d, e\}$. This profile is not linear because at most two of the candidates b, c, d can be placed next to a on the axis. Thus, for each $\lhd \in f(P)$, we have $\mathsf{cost}_f(P, \lhd) \geqslant 1$. Consider the axis $\lhd = baced$. We have $\mathsf{cost}_f(P, \lhd) = 1$, so $\lhd \in f(P)$. But e is never approved and it interferes with the ballot $\{a, d\}$ on $\lhd$. Hence f does not satisfy clearance. $\square$

While VD and MF always choose *some* axis that satisfies the clearance condition, they can additionally choose axes which violate this condition, and hence they fail the axiom.

For another way of formalizing the intuition that unpopular candidates should be placed at the extremes, we consider *veto profiles*, in which every ballot has size $m - 1$, i.e., each voter approves all candidates *but one*. An example of a veto profile is given in Figure 6.9. In a veto profile, the only voters whose ballots are intervals of an axis are those who veto a candidate at one extreme of the axis. Thus, the best candidates to put at the left and right end of the axis are the two most vetoed ones. All of our rules indeed choose only such outcomes.

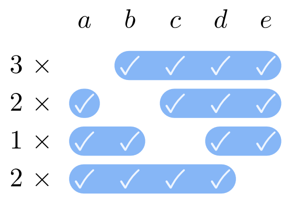


Figure 6.9: Example of a veto profile.

We can extend this intuition to say that candidates that are vetoed more frequently should be placed at positions closer to the extremes. This would imply that the *least* vetoed candidate should be placed in the center, so that as few ballots as possible have holes in the center.

Veto winner centrism
An axis rule f satisfies *veto winner centrism* if for every veto profile P, the median candidate (or one of the two median candidates if the number of candidates is even) of every axis $\lhd \in f(P)$ has the highest approval score.

Among the rules studied in this chapter, only MS and FT satisfy veto winner centrism.

Proposition 6.7
Veto winner centrism is satisfied by MS and FT, but not by VD, MF, and BC.

Proof. Let P be a veto profile. Let us denote the candidates by $c_1, c_2, \ldots, c_m$, and let A_{-i} be the ballot approving all candidates but c_i. For each $i \in \{1, \ldots, m\}$, we denote by n_i the number of occurrences of A_{-i} in P. Since P is a veto profile, $n = n_1 + n_2 + \ldots + n_m$.

Let us now prove that FT and MS satisfy veto winner centrism. For simplicity, we assume that m is odd, and so there is only one median candidate on the axis, at position $\mathsf{med} := (m+1)/2$ (however, note that the reasoning below also works for m even, with only a slight straightforward modification).

Regarding FT, given an axis $\lhd$, let us denote by $k_i \in \mathbb{N}$ the position of c_i on $\lhd$ (for instance, the left-most candidate has position 1, and the rightmost position m). Then, each copy of A_{-i} in P creates $t_{k_i} = (k_i - 1) \cdot (m - k_i)$ forbidden triples and $\mathsf{cost}_{\text{FT}}(P, \lhd) = \sum_{i=1}^{m} n_i \cdot t_{k_i}$. Note that t_{k_i} is maximal when $k_i = \mathsf{med}$ (i.e., when c_i is the median candidate of $\lhd$). Without loss of generality, let c_1 be a most approved candidate and let c_2 be a candidate with strictly fewer approvals, i.e., $n_2 > n_1$. Consider any axis $\lhd$ for which c_2 is the median candidate (thus, $t_{k_2} = t_{\mathsf{med}}$), and let $\lhd'$ be an axis obtained from $\lhd$ by swapping the positions of c_1 and c_2. We claim that $\mathsf{cost}_{\mathrm{FT}}(P, \lhd') < \mathsf{cost}_{\mathrm{FT}}(P, \lhd)$. For each $i \neq 1, 2$, the number of forbidden triples induced by ballots of type A_{-i} is the same for both axes, as this number only depends on the position of c_i on the axis. Hence, $\mathsf{cost}_{\mathrm{FT}}(P, \lhd')$ and $\mathsf{cost}_{\mathrm{FT}}(P, \lhd)$ only differ in triples caused by ballots of type A_{-1} and A_{-2}. Thus,

$$\mathsf{cost}_{\mathrm{FT}}(P, \lhd') - \mathsf{cost}_{\mathrm{FT}}(P, \lhd) = (n_1 t_{\mathsf{med}} + n_2 t_{k_1}) - (n_1 t_{k_1} + n_2 t_{\mathsf{med}}) = (n_1 - n_2)(t_{\mathsf{med}} - t_{k_1}) < 0,$$

as $n_1 < n_2$ and $t_{\mathsf{med}} > t_{k_1}$. This implies that no axis whose center candidate has fewer than the maximum number of approvals can be optimal, and thus FT satisfies veto winner centrism.

Regarding MS, we proceed similarly, by noting that each copy of A_{-i} in P generates $t_{k_i} = \min\{k_i - 1, m - k_i\}$ swaps. Indeed, c_i is the unique non-approved candidate in A_{-i}, so it needs to be swapped with all the candidates on its left, or its right. This value is maximal if c_i is the median candidate of the axis, i.e., if $k_i = \mathsf{med} = (m+1)/2$. It is now easy to see that the previous argument also works for MS.

Regarding VD, MF, and BC, note that these three rules are all equivalent on veto profiles, since their cost functions equal 1 for every ballot A_{-i} that is not an interval of a given axis. Thus, each axis $\lhd$ with left- and rightmost candidates c_l and c_r has a cost of $n - n_l - n_r$ according to these rules. It follows that an axis is optimal if and only if its two outermost candidates correspond to the two least approved candidates. In particular, the optimality of a solution is independent of the position of the most approved candidate (provided it is not placed at the extremes). Hence, if $m \geqslant 5$, for all veto profiles P ,there exists an optimal axis such that the most approved candidate is not the median candidate. Therefore, VD, MF, and BC fail veto winner centrism. □

In fact, on veto profiles, MS and FT always return axes on which the approval scores of the candidates are single-peaked. By this, we mean that when going from left to right on the axis, the approval score of candidates is always increasing until it reaches a peak, after what it is always decreasing: thus, the distribution of approval scores along the axis has a single peak.

Of course, veto profiles, as well as never-approved candidates, are not likely to naturally arise from preferences. However, clearance and veto winner centrism suggest that MS and FT use the information in a profile accurately by correctly placing popular and unpopular candidates. Their tendency to put low-approval candidates towards the extremes is also confirmed by our experiments in Section 6.4. While this generally seems sound, in the political context it can sometimes lead to wrong answers: for instance, there can be ideologically centrist candidates who don't get many votes due to not being well-known, and should not be placed at the extremes.

Clones

We now focus on the behavior of rules in the presence of essentially identical candidates. We recall that $a, b \in C$ are *clones* if for each voter $i \in V$, $a \in A_i$ if and only if $b \in A_i$. While perfect clones are rare, two candidates may have very similar sets of supporters, and studying clones gives insights for how rules handle such similar candidates. Intuitively, one would expect clones to be next to each other on any optimal axis. This is captured by the following axiom:

Clone-proximity

An axis rule f satisfies *clone-proximity* if for every profile P in which $a, a' \in C$ are clones, for every axis $\lhd \in f(P)$, every candidate x with $a \lhd x \lhd a'$ or $a' \lhd x \lhd a$, and every $A \in P$, we have $x \in A$ whenever $a, a' \in A$.

Informally, this axiom says that if there is a candidate x between two clones a and a' on an axis $\lhd$, then x must *also* be approved by all voters who approve a and a'. However, x is not necessarily a clone of a and a', because it can be approved by voters who do not approve a and a'.

Surprisingly, only FT satisfies clone-proximity. All of our rules choose at least one axis where the clones are next to each other, but the rules other than FT may choose extra axes which violate clone-proximity, as we show in the following result.

Proposition 6.8

Clone-proximity is satisfied by FT, but not by VD, MF, BC, and MS.

Proof. We first prove that FT satisfies clone-proximity. Let $P = (A_1, \dots, A_n)$ be a profile where a and a' are clones. For an axis $\lhd$, we denote by $T_\lhd$ the set of all forbidden triples $(i, l, c, r) \in V \times C^3$ such that $l \lhd c \lhd r$ and $l, r \in A_i$ but $c \notin A_i$. Then, $\mathsf{cost}_{\mathrm{FT}}(P, \lhd) = |T_\lhd|$.

First note that we cannot have a forbidden triple $(i, l, c, r) \in T_\lhd$ with one of the clones as c (in the center) and the other on one side (l or r), as a and a' are always approved together. Thus, the only triples in $T_\lhd$ involving both a and a' are those for which both sides l and r are one of the clones, i.e., triples of form $(\cdot, a, \cdot, a')$ and $(\cdot, a', \cdot, a)$. For an axis $\lhd$, let us denote by $S_\lhd^{\{a,a'\}}$ the number of such triples in $T_\lhd$. Moreover, let $S_\lhd^a$ be the number of triples involving a and not a' and let $S_\lhd^{a'}$ be the number of triples involving a' and not a. Finally, let $S_\lhd^0$ be the number of triples involving neither a nor a'. For every axis $\lhd$, we have $\mathsf{cost}_{\mathrm{FT}}(P, \lhd) = S_\lhd^{\{a,a'\}} + S_\lhd^a + S_\lhd^{a'} + S_\lhd^0$.

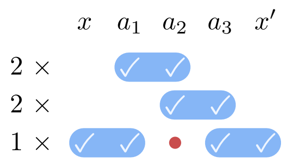
Now take any $\lhd$ where the clones are not next to each other, i.e., there exists $x \in C$ such that $a \lhd x \lhd a'$ or $a' \lhd x \lhd a$ and a ballot $A_i \in P$ such that $a, a' \in A_i$ and $x \notin A_i$. We will show that $\lhd \notin \mathrm{FT}(P)$. Note that by choice of $\lhd$, we have $S_\lhd^{\{a,a'\}} \geqslant 1$. Assume without loss of generality that $S_\lhd^a \leqslant S_\lhd^{a'}$. Let us consider the axis $\lhd'$ obtained by moving a' next to a on $\lhd$, i.e., there is no $x \in C$ such that $a \lhd x \lhd a'$ or $a' \lhd x \lhd a$. Thus, we have $S_{\lhd'}^{\{a,a'\}} = 0$, $S_{\lhd'}^0 = S_\lhd^0$, and $S_{\lhd'}^{a'} = S_{\lhd'}^a = S_\lhd^a$, as all triples that do not involve a' will not be affected by the move, and the triples involving a' will be the same as those involving a now that they are next to each other. Thus, we have the following:

$$\begin{aligned}\mathsf{cost}_{\mathrm{FT}}(P, \lhd) &= S_\lhd^{\{a,a'\}} + S_\lhd^a + S_\lhd^{a'} + S_\lhd^0 \\ &\geqslant \quad 1 \quad\;\; + S_\lhd^a + S_\lhd^a + S_\lhd^0 \\ &> \quad 0 \quad\;\; + S_{\lhd'}^a + S_{\lhd'}^{a'} + S_{\lhd'}^0 = \mathsf{cost}_{\mathrm{FT}}(P, \lhd').\end{aligned}$$

Since $\lhd'$ has a lower FT cost than $\lhd$, we have $\lhd \notin \mathrm{FT}(P)$. Hence FT satisfies clone-proximity.

We now show that other rules do not satisfy clone-proximity. For VD, BC, and MF, consider the following profile:

$2 \times \{a_1, a_2\}$,
$2 \times \{a_2, a_3\}$,
$1 \times \{x, x', a_1, a_3\}$.



Because of the cycle among a_1, a_2, a_3, this profile is not linear and so all axes have cost at least

1. Now observe that the axis $x \lhd a_1 \lhd a_2 \lhd a_3 \lhd x'$ has cost 1 for VD, BC, and MF. On this axis, a_2 is between the clones x and x' but is never approved with them. Thus, these three rules fail clone-proximity. (However, note that all these rules also choose the compliant axis $x \lhd x' \lhd a_1 \lhd a_2 \lhd a_3$.)

For MS, consider the following profile:

$$1 : \{a, a', b, b'\}, \quad 1 : \{b, b', x, x'\}, \quad 1 : \{x, x', a, a'\}$$

By neutrality of MS, the cost of all axes in which clones are next to each other is the same as the cost of $a \lhd a' \lhd x \lhd x' \lhd b \lhd b'$, which is 4. However, another axis has cost 4 for MS: $x \lhd a \lhd a' \lhd x' \lhd b \lhd b'$. On this axis, a is between the clones x and x' but a is not approved in the ballot $\{b, b', x, x'\}$, containing x and x'. Thus, MS also fails clone-proximity. □

We can also adapt the *independence of clones* axiom from voting theory (Tideman, 1987) that we introduced in Section 2.4.4 and that we discussed in the previous chapters of this thesis. In our model, it would say that adding or removing a clone of a candidate from an approval profile should not change the relative order of the other candidates on any optimal axis returned by the rule, and conversely removing a clone from a profile should not change the relative order of the other candidates either. To formally define it, we recall some notations. For a profile P defined on a set C of candidates, we denote by $P_{C'}$ the restriction of P to a subset of candidates $C' \subseteq C$. We also denote by P_{-c} the restriction of the profile to $C \setminus \{c\}$ where $c \in C$ is a given candidate. We similarly define $\lhd_{C'}$ and $\lhd_{-c}$. We can now state the axiom:

Independence of clones

An axis rule f is *independent of clones* if for every profile P in which $a, a' \in C$ are clones,

(1) for all axes $\lhd \in f(P)$, we have $\lhd_{-a} \in f(P_{-a})$, and

(2) for all axes $\lhd^* \in f(P_{-a})$, there is an axis $\lhd \in f(P)$ with $\lhd_{-a} = \lhd^*$.

Among the rules studied in this chapter, only VD is independent of clones.

Proposition 6.9

Independence of clones is satisfied by VD, but not by MF, BC, MS, and FT.

Proof. We start by proving that f = VD satisfies independence of clones. We first check (1). Let $\lhd \in f(P)$; we need to show that $\lhd_{-a} \in f(P_{-a})$. Because all interval ballots of P on $\lhd$ will remain interval ballots of P_{-a} on $\lhd_{-a}$, we have $\mathsf{cost}_{\text{VD}}(\lhd_{-a}, P_{-a}) \leqslant \mathsf{cost}_{\text{VD}}(\lhd, P)$. Now, assume for a contradiction that $\lhd_{-a} \notin f(P_{-a})$ and instead some axis $\lhd' \in f(P_{-a})$ is optimal, with $\mathsf{cost}_{\text{VD}}(\lhd', P_{-a}) < \mathsf{cost}_{\text{VD}}(\lhd_{-a}, P_{-a})$. Consider the axis $\lhd'_{+a}$ obtained from $\lhd'$ by placing a next to a'. Because a and a' are clones, an approval ballot of P is an interval of $\lhd'_{+a}$ if and only if its restriction in P_{-a} is an interval of $\lhd'$. Thus, $\mathsf{cost}_{\text{VD}}(\lhd'_{+a}, P) = \mathsf{cost}_{\text{VD}}(\lhd', P_{-a})$. Combining all of this, we have:

$$\mathsf{cost}_{\text{VD}}(\lhd'_{+a}, P) = \mathsf{cost}_{\text{VD}}(\lhd', P_{-a}) < \mathsf{cost}_{\text{VD}}(\lhd_{-a}, P_{-a}) \leqslant \mathsf{cost}_{\text{VD}}(\lhd, P)$$

which contradicts the optimality of $\lhd$ for P. Hence $\lhd_{-a} \in f(P_{-a})$.

Next, we check (2) using the same reasoning. Let $\lhd \in f(P_{-a})$ and let $\lhd_{+a}$ be the axis obtained from $\lhd$ by placing a next to a'. We will show that $\lhd_{+a} \in f(P)$. Again, we have $\mathsf{cost}_{\text{VD}}(\lhd_{+a}, P) = \mathsf{cost}_{\text{VD}}(\lhd, P_{-a})$. Now assume for a contradiction that there is $\lhd' \in f(P)$ with a lower cost than $\lhd_{+a}$, i.e., $\mathsf{cost}(\lhd', P) < \mathsf{cost}(\lhd_{+a}, P)$. As explained above, we have $\mathsf{cost}_{\text{VD}}(\lhd'_{-a}, P_{-a}) \leqslant \mathsf{cost}_{\text{VD}}(\lhd', P)$.

Figure 6.10: Profile P with the axes $aa'bcd$ and $daa'bc$ in the proof of Proposition 6.9. Red circles indicate interfering candidates, and the clone a' of a is grayed out.

Combining these three inequalities gives $\mathsf{cost}_{\mathrm{VD}}(\lhd'_{-a}, P_{-a}) < \mathsf{cost}_{\mathrm{VD}}(\lhd, P_{-a})$, which contradicts the optimality of $\lhd$. Hence $\lhd_{+a} \in f(P)$, as required.

To prove that BC does not satisfy independence of clones, we consider the following profile P:

$$3 \times \{b, a, a'\} \qquad 4 \times \{c, a, a'\} \qquad 2 \times \{b, c\}$$

It is easy to check that the unique optimal axis (up to reversal, and up to permutation of a and a') is $\lhd = bcaa'$ with $\mathsf{cost}_{\mathrm{BC}}(P, \lhd) = 3$. Indeed, if a and a' are not next to each other, at least two types of ballot will not be interval of the axis, which will yield a BC cost of at least 5, and the axes on which b and c are the extremities have a BC cost of at least 4 due to ballots $\{b, c\}$. This leaves $bcaa'$ and $cbaa'$ with cost 3 and 4 respectively. However, if we remove the candidate a', the BC cost of $\lhd_{-a'} = bca$ is 3. It is hence no longer optimal, as the axis $\lhd^* = bac$ achieves a lower BC cost of 2.

We use a very similar idea to prove that MF does not satisfy independence of clones. We consider the following profile P:

$$\begin{array}{lll} 2 \times \{b, a, a'\} & 1 \times \{b, d\} & 1 \times \{b, c, d, e\} \\ 2 \times \{c, a, a'\} & 1 \times \{c, e\} & \end{array}$$

We can check that the axis $\lhd = dbecaa'$ is optimal for MF with $\mathsf{cost}_{\mathrm{MF}}(P, \lhd) = 2$. If a and a' are not next to each other on the axis, at least two of the ballot types $\{b, a, a'\}$, $\{c, a, a'\}$ and $\{b, c, d, e\}$ are not intervals, which yields a MF cost of at least 3. Any axis such that two candidates are on the left of a and a' and two candidates on their right has a MF cost greater or equal than 2 because of ballots $\{b, c, d, e\}$. Any axis with one candidate on one side of a and a' and three candidates on their right has a MF cost of at least 2: the ballots $\{b, c, d, e\}$ generates at least 1 flips, and at least one of the ballots $\{b, d\}, \{c, e\}$ is not an interval either. Now, if a and a' are on an extremity, only one of $\{b, a, a'\}$ and $\{c, a, a'\}$ can be an interval, imposing a MF cost of at least 2. Thus, $\lhd$ is indeed minimizing the MF cost. However, $\lhd_{-a'} = dbeca$ is not optimal for $P_{-a'}$: $\mathsf{cost}_{\mathrm{MF}}(P_{-a'}, \lhd_{-a'}) = 2$. The axis $\lhd^* = dbace$ has a lower cost of 1. This contradicts independence of clones.

To prove that FT and MS do not satisfy independence of clones, let us consider the following profiles:

$$\begin{array}{llll} P: & 3 \times \{a, b\} & 3 \times \{b, c\} & 1 \times \{a, c, d\} \\ P': & 3 \times \{a, a', b\} & 3 \times \{b, c\} & 1 \times \{a, a', c, d\} \end{array}$$

Let $f \in \{\mathrm{FT}, \mathrm{MS}\}$. We have $f(P) = \{\lhd^1, \lhd^2\}$ with $\lhd^1 = abcd$ and $\lhd^2 = dabc$. Indeed, $\mathsf{cost}_{\mathrm{FT}}(P, \lhd^i) = 2$ and $\mathsf{cost}_{\mathrm{MS}}(P, \lhd^i) = 1$ for $i \in \{1, 2\}$. These are the only axes on which both $\{a, b\}$ and $\{b, c\}$ are intervals – in other words, the cost of any other axis will be at least 3. We now focus on P', in which we have added a candidate a', clone of a. Under independence of clones, there should be an axis $\triangleleft \in f(P')$ such that $\triangleleft_{-a} = \triangleleft^1$. Among all possible axes generalizing $\triangleleft^1$, the best one for MF and FT (up to the permutation of a and a') is $\triangleleft = aa'bcd$, with a cost of 4 for FT and 2 for MS. However, axis $\triangleleft^* = daa'bc$ has cost 3 for FT and 1 for MS. Hence, there is no $\triangleleft \in f(P')$ such that $\triangleleft_{-a} = \triangleleft^1$. Thus, FT and MS do not satisfy independence of clones. □

We can actually show that independence of clones and ballot monotonicity characterize VD among scoring rules. This not only distinguishes VD from the other introduced rules, but shows its normative appeal among the entire class of scoring rules.

Theorem 6.10
Let $m \geqslant 6$, and let f be a neutral scoring rule. Then f satisfies consistency with linearity, ballot monotonicity, and independence of clones if and only if it is VD.

Proof sketch. Let f be a scoring rule satisfying neutrality, consistency with linearity, independence of clones and ballot monotonicity. As shown in Lemma 6.3, f is induced by a symmetric cost function cost with $\mathsf{cost}(A, \triangleleft) = 0$ if and only if A forms an interval in $\triangleleft$. Further, cost only depends on the approval vector $x_{A,\triangleleft}$, i.e., there exists a function $g : \{0,1\}^m \to \mathbb{R}_{\geqslant 0}$ such that $\mathsf{cost}(A, \triangleleft) = g(x_{A,\triangleleft})$ for all ballots A and axis $\triangleleft$.

The steps of the proof are as follows:[3]

1. Using ballot monotonicity, we show that there is a function h such that for all A and $\triangleleft$ such that A is not an interval of $\triangleleft$, $\mathsf{cost}(A, \triangleleft) = h(m, k_{\text{app}}, k_{\text{int}})$, where m is the number of candidates, $k_{\text{app}} = |A|$ is the number of approved candidates and k_{int} is the number of interfering candidates.

2. Using independence of clones, we show that for A not interval of $\triangleleft$, $\mathsf{cost}(A, \triangleleft)$ only depends on the sum $k_{\text{app}} + k_{\text{int}}$, i.e., there is h such that $\mathsf{cost}(A, \triangleleft) = h(m, k_{\text{app}} + k_{\text{int}})$.

3. We show that for A not interval of $\triangleleft$, $\mathsf{cost}(A, \triangleleft)$ can only take two values: $\mathsf{cost}(A, \triangleleft) = h^*_m$ if $k_{\text{app}} + k_{\text{int}} = m$ and $\mathsf{cost}(A, \triangleleft) = h_m$ otherwise.

4. Finally, we show that $h^*_m = h_m$ and that the rule is thus VD. □

Regarding the independence of the axioms in this characterization among neutral scoring rules, note that the trivial rule TRIV returning all axes satisfies every axiom but consistency with linearity, the rule that minimizes the total number of contiguous holes (and which is the scoring rule based on the function $\mathsf{cost}_G(A, \triangleleft) = |\{(x, y) \in A : \exists z, x \triangleleft z \triangleleft y \text{ and } \forall z \text{ s.t. } x \triangleleft z \triangleleft y, z \notin A\}|$) only fails ballot monotonicity, and BC only violates independence of clones. We do not have an example showing that neutrality is necessary, but this axiom can be dropped if we allow an infinitely large ground set of candidates, because then independence of clones and consistency with linearity imply neutrality for scoring rules, using standard arguments (Brandl et al., 2016, Lemma 1).

Finally, observe that the two clone axioms are quite strong: each excludes all but one of our rules. Actually, we can show that if a scoring rule satisfies neutrality and consistency with linearity, then clone-proximity and independence of clones are incompatible.

Theorem 6.11
No neutral scoring rule satisfies independence of clones, clone-proximity, and consistency with linearity.

[3] The full proof is mainly due to my coauthor Chris Dong and can be found in the original paper (Delemazure et al., 2025a).

Proof. Let f be a scoring rule satisfying all four axioms, and cost_f its cost function. As proven in Lemma 6.3, by neutrality there is a function $g_f : \{0,1\}^m \to \mathbb{R}_{\geqslant 0}$ such that $\mathsf{cost}_f(A, \lhd) = g_f(x_{A,\lhd}) = g_f(x_{A,\overleftarrow{\lhd}})$, where $x_{A,\lhd}$ is the approval vector of A and $x_{A,\overleftarrow{\lhd}}$ is the reversed vector.

Let y be the minimal cost over all approval vectors that are not intervals. In particular, $y \leqslant g_f((1,0,1,0)) = g_f((0,1,0,1))$ and $y \leqslant g_f((1,0,0,1))$. By Lemma 6.3 (using consistency with linearity), $y > 0$. Moreover, let $y' = g_f((1,0,1,1)) = g_f((1,1,0,1))$. Let $q \in \mathbb{N}$ with $q > y'/y$ and consider the following profiles:

$$\begin{array}{llll} P: & q\times\{b,c\} & q\times\{c,d\} & 1\times\{a,b,d\} \\ P': & q\times\{b,b',c\} & q\times\{c,d\} & 1\times\{a,b,b',d\} \\ P'': & q\times\{b,c\} & q\times\{c,d\} & 1\times\{a,a',b,d\} \end{array}$$

In P, for $\lhd \in \{\lhd^1, \lhd^2\}$ with $\lhd^1 = abcd$ and $\lhd^2 = bcda$, we have $\mathsf{cost}_f(\{a,b,d\}, \lhd) = y'$. All other axes break one of the pairs $\{b,c\}$, $\{c,d\}$, thus ensuring a cost of at least $q \cdot y > y'$. Therefore, $\lhd_1, \lhd_2 \in f(P)$.

Since b and b' are clones in P', by clone-proximity they should be next to each other on every axis $\lhd \in f(P')$. By independence of clones, there exists $\lhd^3$ (resp. $\lhd^4$) in $f(P')$ extending $\lhd^1$ (resp. $\lhd^2$). Combining this with neutrality, $f(P')$ contains $\lhd^3 = abb'cd$ and $\lhd^4 = bb'cda$, which thus must have the same cost. Since the ballots $\{b,b',c\}$ and $\{c,d\}$ are intervals of both of these axes and the rule is consistent with linearity, they contribute a cost of 0 and thus the cost difference of the two axes only depends on the remaining ballot $\{a,b,b',d\}$. This implies $\mathsf{cost}_f(\{a,b,b',d\}, \lhd^3) = \mathsf{cost}_f(\{a,b,b',d\}, \lhd^4)$, i.e., $g_f((1,1,1,0,1)) = g_f((1,1,0,1,1))$.

Now, consider the profile P'' which is a copy of P but with a clone a' of a. Using the same arguments as in the case of P' yields two optimal axes $\lhd^5 = aa'bcd$ and $\lhd^6 = bcdaa'$. However, let us now compare $\lhd^5$ to $\lhd^7 = abcda'$. The ballots $\{b,c\}$ and $\{c,d\}$ are intervals of both axes, and the cost of $\{a,a',b,d\}$ is the same on both, as we already showed that $g_f((1,1,1,0,1)) = g_f((1,1,0,1,1))$. Thus, $\lhd^7$ is also an optimal axis for P'', which is in contradiction with clone-proximity, since a and a' are not next to each other. □

Heredity and Partition Consistency

Independence of clones can be strengthened to *heredity*, as defined by Tydrichová (2023) for axis rules based on rankings. This axiom is equivalent to the independence of irrelevant alternatives axiom from voting theory (see Section 2.4.4), and states that if we remove *any* candidate from the profile (not just a clone), the rule should return the original axes with that candidate omitted.

Heredity
An axis rule f satisfies *heredity* if for every profile P and every subset of candidates $C' \subseteq C$, we have that for each axis $\lhd \in f(P)$, there exists $\lhd^* \in f(P_{C'})$ such that $\lhd_{C'} = \lhd^*$.

However, no reasonable axis rule can satisfy this axiom, which is not too surprising as in most contexts, independence of irrelevant alternatives is considered too strong and is not satisfiable together with very natural properties (the canonical example being Arrow's theorem).

Proposition 6.12
No axis rule satisfies heredity and consistency with linearity.

Proof. Let f be an axis rule satisfying heredity and let $P = (\{a,b\}, \{a,c\}, \{a,d\})$. Let $\lhd \in f(P)$.

In $\lhd$, there must be at least two candidates on the same side of a (as there are two sides and three candidates b, c, and d), without loss of generality b and c. By heredity, if we remove d, in $f(P_{-d})$ there must be an axis where a is in an extreme position. However by consistency with linearity, $f(P_{-d}) = \{bac, cab\}$, a contradiction. □

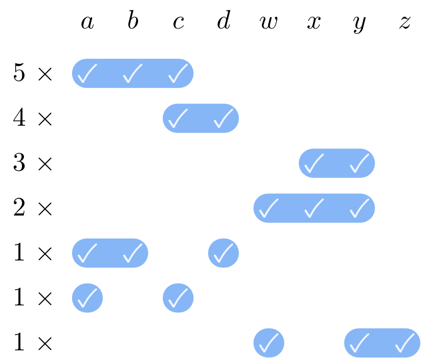


Figure 6.11: Profile P with the approval-path partition $C_1 = \{a, b, c, d\}$ and $C_2 = \{w, x, y, z\}$.

Because of this impossibility, we cannot construct an axis greedily by successively adding candidates. However, if a profile can be decomposed into subprofiles, we can expect the optimal axes to be the various concatenations of the optimal axes for the subprofiles. Specifically, given a profile P, consider the co-approval equivalence relation $\sim$ on C, with $x \sim y$ whenever there is some ballot $A \in P$ with $x, y \in A$. Moreover, take its transitive closure $\sim^*$ such that $x \sim^* y$ whenever there is a sequence $x = x_1, x_2, \ldots, x_k = y$ with $x_i \sim x_{i+1}$ for all i. We call the equivalence classes $C_1, \ldots, C_k$ of $\sim^*$ the *approval-path partition* of P. Consider for instance the profile given in Figure 6.11. The approval-path partition of this profile consists of $C_1 = \{a, b, c, d\}$ and $C_2 = \{w, x, y, z\}$, since every ballot forms a subset of one of these two classes. We can partition P into two subprofiles P_1 and P_2 in which only candidates of respectively C_1 and C_2 are approved. Note that even if they are never co-approved together, x and z are in the same equivalence class, as they are both co-approved with y. We now define a property called *partition consistency*, which in this example says that if the optimal axis in P_1 is $abcd$ and in P_2 is $wxyz$, then the optimal axes for the complete profile $P = P_1 + P_2$ should be $abcdwxyz$, $abcdzyxw$, $dcbawxyz$ and $dcbazyxw$ (up to reversal).

Partition consistency
An axis rule f satisfies *partition consistency* if for every profile P with approval-path partition $C_1, \ldots, C_k$, we have that $\lhd \in f(P)$ if and only if for each $j \in [1, k]$, the class C_j is an interval of $\lhd$ and the axis $\lhd_{C_j} \in f(P_{C_j})$.

If a rule satisfies the partition consistency axiom, we can reduce the computation time, as it becomes possible to divide the task into several sub-profiles with smaller numbers of candidates (formally, computing an optimal axis can be done in time that is fixed-parameter tractable with respect to the size of the largest set C_j in the partition). Moreover, partition consistency implies clearance, as a never-approved candidate x forms a singleton equivalence class. Thus, since VD and MF fail clearance, they also fail partition consistency.

Proposition 6.13
Partition consistency is satisfied by BC, MS and FT, but not by VD and MF.

Proof. Let $f \in \{\mathrm{BC}, \mathrm{MS}, \mathrm{FT}\}$. We show that f satisfies partition consistency. Let P be a profile with co-approval partition $C_1, \ldots, C_k$. Let $\lhd$ be an axis on which we have $a \lhd b \lhd a'$ for some candidates $a, a' \in C_j$ and $b \notin C_j$. We will show that $\lhd \notin f(P)$. Let us denote by C_j^l and C_j^r the sets of candidates of C_j on the left and right of b respectively. We have $C_j^l \neq \emptyset$ (it contains a) and $C_j^r \neq \emptyset$ (it contains a'). If there is no ballot A containing at least one candidate of C_j^l and one candidate of C_j^r, then the two sets would not be part of the same equivalence class, a contradiction.

	VD	MF	BC	MS	FT
Consistency with linearity	✓[1]	✓	✓	✓	✓
Independence of clones	✓[1]	✗	✗	✗	✗
Stability	✓	✗	✗	✗	✗
Ballot monotonicity	✓[1]	✗	✓	✗	✗
Clearance	✗	✗	✓	✓	✓
Partition consistency	✗	✗	✓	✓	✓
Veto-winner centrism	✗	✗	✗	✓	✓
Clone-proximity	✗	✗	✗	✗	✓

Table 6.1: Properties of the axis rules. The superscripts [1] indicate the characterization of VD.

Thus, there exists a ballot A^* that approves some $a \in C_j^l$ and some $a' \in C_j^r$. Because $b \notin C_j$, b is an interfering candidate for A^*.

Let $\lhd' = \lhd_{C_1} \dots \lhd_{C_k}$, where $\lhd_{C_j}$ is the restriction of $\lhd$ to candidates from C_j. Since every ballot contains only candidate from one of the C_j and since the relative order of the candidates in each subaxis is preserved, we only removed interfering candidates for each ballot by moving from $\lhd$ to $\lhd'$. It follows that $\mathsf{cost}_f(A, \lhd) \geqslant \mathsf{cost}_f(A, \lhd')$ for all ballots. As we already showed, there is some ballot $A^* \in P$ containing a and a', but not b, which is thus an interfering candidate for A^*. By moving from $\lhd$ to $\lhd'$, the candidate b is not interfering A^* anymore, so we strictly reduce the number of interfering candidates. Fom the definition of f, we deduce that $\mathsf{cost}_f(A^*, \lhd) > \mathsf{cost}_f(A^*, \lhd')$. This proves that $\lhd \notin f(P)$.

Thus, in all optimal axes $\lhd \in f(P)$, all the sets C_j form intervals, and we can write each such axis as $\lhd = \lhd_{\sigma_1} \dots \lhd_{\sigma_k}$ for some permutation $(\sigma_1, \dots, \sigma_k)$ of $[1, k]$, where $\lhd_j$ orders the candidates in C_j. For each j, let P_{C_j} be the profile obtained from P by restricting to C_j. From the definition of f, for each ballot $A \in P_{C_j}$, we have $\mathsf{cost}_f(A, \lhd) = \mathsf{cost}_f(A, \lhd_j)$. Thus $\mathsf{cost}_f(P, \lhd) = \sum_{j=1}^{k} \mathsf{cost}_f(P_{C_j}, \lhd_j)$. From this we directly deduce that $\lhd \in f(P)$ if and only if $\lhd_j \in f(P_{C_j})$ for all $j \in [1, k]$. This proves that f satisfies partition consistency.

For $f \in \{\text{VD}, \text{MF}\}$ we can use the examples from the proof of Proposition 6.6 with $C_1 = \{a, b, c, d\}$ and $C_2 = \{e\}$, $P_1 = P$ and $P_2 = \emptyset$ to show that f does not satisfy partition consistency. □

Summary of the results

In Table 6.1[4], we summarize the axiomatic results of this section. We can see that depending on the amount of information they use, the rules satisfy very different properties. Rules that use low amounts of information like VD are more robust to changes, while rules that use a lot of information like FT return more precise axes, with popular candidates being more central on the axes, and clones close to each other.

6.4 Experimental Analysis

We now evaluate the different axis rules with an experimental analysis, based on synthetic and real datasets. While the rules are in general hard to compute, for m up to about 12, we can find

[4]Note that this table could be interpreted as a binary matrix, and we could use our rules to identify an axis of the axis rules. In this case, all the rules return the same axis MF $\lhd$ VD $\lhd$ BC $\lhd$ MS $\lhd$ FT, as the profile is actually linear.

the best axes in reasonable time, using pruning and heuristics, and/or ILP encodings.

In addition to the axis rules introduced in this chapter and which are based on approval preferences, we will also consider axis rules based on rankings, that we introduced in Section 2.3.3. In particular, we will consider two rules from the literature (Faliszewski et al., 2014; Tydrichová, 2023). The first one is the equivalent of Voter Deletion for ordinal preferences, and it returns the axis $\lhd$ that minimizes the number of rankings to remove for the profile to be single-peaked. We will refer to this rule as VD-rank. The second rule is the equivalent of Forbidden Triples for ordinal preferences, and it returns the axis $\lhd$ that minimizes the number of triples of candidates that break single-peakedness. This corresponds to triples of candidates (x, y, z) such that x is ranked first in the ranking $\succ_i$, and $x \lhd y \lhd z$ or $z \lhd y \lhd x$, but $x \succ_i z \succ_i y$. We will refer to this rule as FT-rank.

We do not include the global swaps and local swaps rules (which are somewhat similar to MF and MS, respectively) in our analysis, since these are quite expensive to compute (Erdélyi et al., 2017, Theorem 6.21).

In the remainder of this section, we first compare the different rules on synthetic data, with probabilistic models with which we sample a linear profile based on a 'correct' ground truth axis and add random noise to the ballots. Then, we test our rules on real data. In particular, we use political datasets, so that we can compare the resulting axes to existing *endogenous* left-right axes. In particular, we focus on the *Voter Autrement* datasets, and votes of the Supreme Court justices of the United States of America. Finally, we also test our rules on tierlist datasets covering various non-political topics.

Probabilistic Models

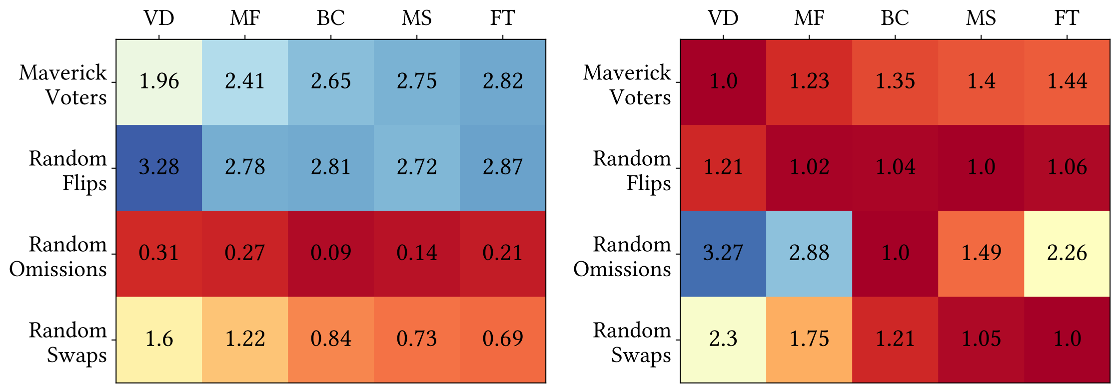
We first ran our rules on very simple probabilistic noise models. For each of our models, we first draw uniformly at random a *ground truth axis*, then we sample approval ballots i.i.d., based on this axis. As we mentioned in Section 6.2, for such probabilistic model, there necessarily exists a scoring rule that is a MLE for this model. Therefore, the performance of the rules on the different models are likely to reflect simply how similar they are to the MLE of these models. However, these experiments can give an idea of how well the rules can generalize to different models.

We first consider four models, each inspired by one of our rules, though only the *Maverick Voters* model actually is the probabilistic model of which VD is the MLE. For the other rules (MF, BC, and MS), the precise models for which they are MLEs are less natural than the intuitively similar models that we decided to use. Below, we define these models.

- *Maverick Voters:* For each voter, we randomly decide if they are a "maverick voter". With probability $p \in [0, 1/2)$, they are a maverick and we sample an approval ballot at random (whether or not it is an interval of the axis), otherwise we sample an approval ballot that is an interval of $\lhd$ uniformly at random.

- *Random Flips:* First, we sample for each voter an approval ballot that is an interval of the axis $\lhd$ uniformly at random (among all interval ballots). Then, for each candidate, we switch its status in the ballot (from approved to not approved, or conversely) independently with probability $p \in [0, 1/2)$.

- *Random Omissions:* First, we sample for each voter an approval ballot that is an interval of the axis $\lhd$ uniformly at random (among all interval ballots). Then, for each *approved* candidate, we switch its status (from approved to not approved) with probability $p \in [0, 1/2)$.



(a) Not normalized. (b) Normalized.

Figure 6.12: Average Kendall-tau distance to the ground truth axes for different rules and probabilistic models, averaged over 1 000 profiles.

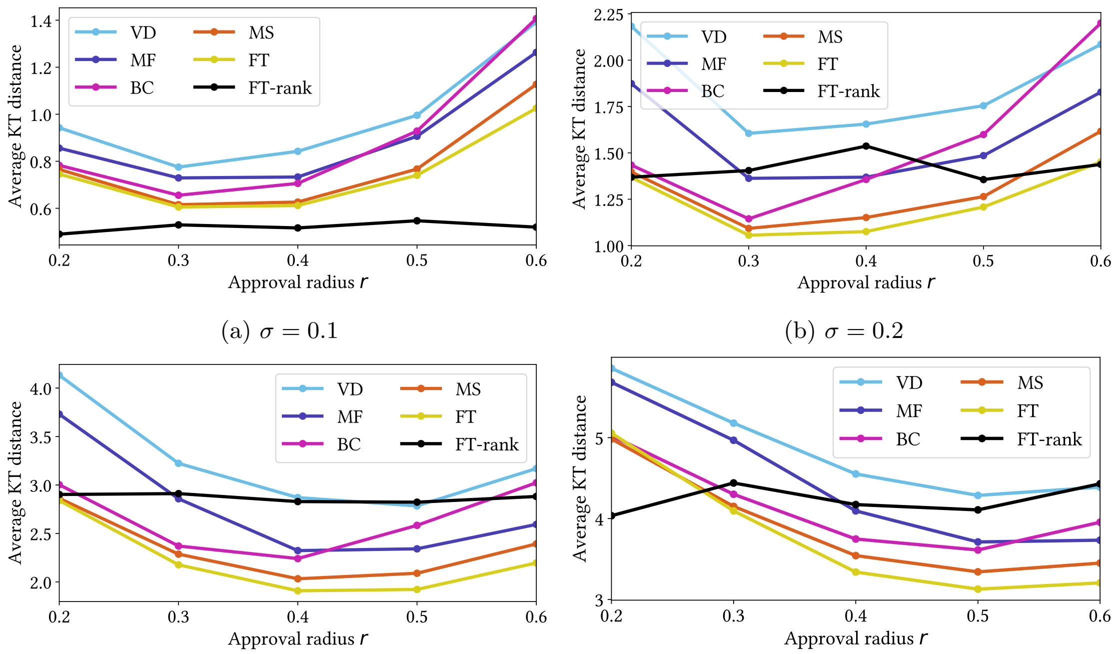
- *Random Swaps:* For each voter i, we sample an axis $\lhd_i$ using a Mallows model with center $\lhd$ and dispersion parameter $\phi \in [0, 1]$. As a reminder, the probability of $\lhd_i$ in this model is proportional to $\phi^{\mathsf{KT}(\lhd, \lhd_i)}$, where KT is the Kendall-tau distance (defined in Section 2.3.1). Once $\lhd_i$ is sampled, we sample uniformly at random an approval ballot which is an interval of $\lhd_i$.

For each model, we sampled 1 000 random profiles with $m = 7$ candidates and $n = 100$ voters. We then ran the different rules on each profile and computed the Kendall-tau (KT) distance between the axes returned by the rules and the ground truth axis. In case of a tie, we take the average KT distance over all returned axes. As a reminder, the Kendall-tau distance is defined as the number of pairwise disagreements between two rankings ($\mathsf{KT}(\lhd, \lhd') = |\{(x, y) \in C^2 : x \lhd y \text{ and } y \lhd' x\}|$). We report here the average KT distance over all 1 000 profiles. We used the following parameters for the models: $p = 0.2$ for Maverick Voters, $p = 0.3$ for Random Flips, $p = 0.45$ for Random Omissions, and $\phi = 0.5$ for Random Swaps. These parameters were chosen to obtain roughly similar KT distances across models. Figure 6.12 (a) presents those average KT distances. Despite our choice of parameters, some models clearly yield more difficult profiles than others. In particular, even with $p = 0.45$, the Random Omissions model generates profiles that are very close to the original one. In Figure 6.12 (b), we show the same results normalized such that the minimal value is 1 for each particular model. The main conclusion is that no rule really generalizes well to all models, but VD is particularly bad at generalizing beyond the Maverick Voters model. It is also interesting to note that MF and MS are not the rules that minimize the KT distance in the models they are respectively associated with, Random Flips and Random Swaps.

Euclidean Model

We now consider the 1-dimensional Euclidean model, in which voters and candidates have positions on a 1-dimensional Euclidean space (as in Section 2.5.1), but voters have noisy observations of the positions of the candidates, and their preferences depend on their distance to the *observed* positions of the candidates. This model is inspired by random utility models such as the Thurstone–Mosteller model (Thurstone, 1927; Mosteller, 1951).

More formally, each candidate and voter $x \in C \cup V$ is associated with a position $\mathsf{pos}(x) \in \mathbb{R}$ on the line. The positions $\mathsf{pos}(c)$ of the candidates describe a ground truth axis $\lhd = c_1 c_2 \dots c_m$



(a) $\sigma = 0.1$ (b) $\sigma = 0.2$

(c) $\sigma = 0.3$ (d) $\sigma = 0.4$

Figure 6.13: Evolution of the average KT distance between the axes returned by the rules and the ground truth axes, for $r \in [0.2, 0.6]$ and different values of σ, averaged over 1 000 random samples.

such that $\mathsf{pos}(c_1) \leqslant \mathsf{pos}(c_2) \leqslant \ldots \leqslant \mathsf{pos}(c_m)$. Each voter i estimates the position of each candidate c under independent normal noise: $p_i(c) = \mathsf{pos}(c) + \varepsilon_i(c)$ with $\varepsilon_i(c) \sim \mathcal{N}(0, \sigma)$, where σ is a parameter of the model. Voters approve and rank candidates based on their estimations and not on the actual positions of the candidates. More precisely, the approval set of voter i contains all candidates such that $|\,\mathsf{pos}(i) - p_i(c)| \leqslant r$, where the *approval radius* r is a parameter of the model. The ranking of voter i is given by increasing distances between $p(i)$ and $p_i(c)$: the closer the better. An interesting feature of this model is that we can compare our axis rules based on approval preferences to ranking-based ones.

We conducted experiments on 1 000 random profiles with $m = 7$ candidates and $n = 100$ voters for each set of parameters with $\sigma \in \{0.1, 0.2, 0.3, 0.4\}$ and $r \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$. Figure 6.13 shows the average Kendall-tau distance between the axes returned by the axis rules and the ground truth axes. VD-rank is not included in the charts as it always returns axes that are very far from the ground truth, with KT distances greater than 7 on average.

We find that for most values of σ and r, rules using approval preferences actually perform better than FT-rank, returning axes with a lower average KT distance to the ground truth. This is surprising, as intuitively rankings provide more information than approvals. We note however that FT-rank is better than the approval-based methods when r is either very small or very large. These are the cases in which many approval sets are of size either 0 or 1 (for small r) or of size m (for large r), and thus provide no information on candidates' proximity, giving a strong advantage to ranking-based methods. Note that none of these two extremes is particularly realistic: in the majority of real-world approval datasets, voters approve on average between 2 and 3 candidates. FT-rank is also slightly better when σ is small, but in this case all approval-based rules also perform quite well. Finally, we observe that for all values of the parameters, the axes returned by rules using

Institute	Axis ◁										
BVA	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP	R
Opinionway	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP	R
IFOP	LO	NPA	LFI	PS	EM	R	LR	DLF	FN	UPR	SP
IPSOS	LO	NPA	LFI	PS	EM	R	LR	DLF	FN	SP	UPR
Harris Interactive	LO	NPA	LFI	PS	EM	R	LR	DLF	SP	UPR	FN
Odoxa	LO	NPA	LFI	PS	EM	R	LR	DLF	UPR	SP	FN
Elabe	NPA	LO	LFI	PS	EM	R	LR	UPR	DLF	FN	SP

Table 6.2: Axes used by polling institutes during the 2017 French presidential election.

more information (e.g., FT) are on average closer to the ground truth axes than those returned by rules using less information (e.g., VD).

Voter Autrement

We also ran our rules on datasets from the *Voter Autrement* collection, which we introduced in Section 2.5.4. We used the datasets collected during the French presidential election from 2007 to 2022, for which the approval rule was tested. We did not consider datasets from the 2002 election, as the running time of our rules would be too long due to the high number of candidates (16). The number of candidates of the other elections were 12 in 2007, 10 in 2012, 11 in 2017 and 12 in 2022. We recall that the votes are weighted so that the share of total weight assigned to voters who voted for each candidate matches the national vote share of this candidate.

Baseline Axes

For these elections, there are no official "exact" axes of the candidates. Thus, to evaluate our rules, we will compare the axes they return to the ones obtained using exogenous information (e.g., based on expert or voter surveys) or to the axes used in practice by polling institutes. Indeed, polling institutes frequently order the candidates of the presidential elections, for instance in the questions they ask in their surveys (*"For which candidate would you vote if the election was next Sunday?"*), or when presenting the results of their surveys.

Consider for instance the case of the 2017 election. The axes used by the polling institutes are reported in Table 6.2. We can see that they mostly agree on the relative positions of the main parties, and that most of their disagreements are about some of the "small" candidates, for which it is actually hard to say if they are right-wing or left-wing. In 2017, this was the case for *Jean Lassalle* (R), *François Asselineau* (UPR), and *Jacques Cheminade* (SP). Even though they might be more right-wing than left-wing, they were not the most far-right candidates of the election: if they are often placed at the extremes, this is more because they are not fitting the axis. The other main kind of disagreements between polling institutes is for pairs of adjacent parties, for instance between the trotskyist parties NPA (New Anticapitalist Party) and LO (Worker Struggle) in 2017, or the center-left parties EELV (Green Party) and PS (Socialist Party) in 2022[5]. We similarly collected all the axes used by the polling institutes for the other French presidential elections, and we provide them in Appendix A.

Let us now turn to existing methods for building axes of candidates or parties. The Manifesto Project (Lehmann et al., 2024) uses information extracted from the parties' manifestos to assign

[5]This particular couple of parties appears very ideologically similar to the voters. In the 2012 CSES survey (The Comparative Study of Electoral Systems, 2023), 37.7% of respondents ordered EELV on the left of PS, 36.9% had the reverse opinion, and 25.4% placed them on the exact same position.

them a position on a left-right scale, from which we can derive an ordering. However, it is often criticized for being unreliable. For instance, in their most recent dataset (2018), they place the communist party PCF on the right of the center-right party MoDem.

Another popular method is to decide the positions of the candidates and parties based on opinions of experts or representative samples of voters. Opinions are collected in surveys, that generally ask respondents to place parties (or candidates) on a scale from 0 to 10 based on their left-right ideology. We can aggregate the answers by taking the median or average position, and then order the parties. The Chapel Hill Expert Survey (Jolly et al., 2022) is maybe the most important expert survey asking for the political position of parties. The experts have to indicate the position of parties on different scales, in particular a "left-right ideology" scale, and a "left-right economic" scale. On the "left-right ideology" scale, the aggregated answers of the 2019 survey gives the following order (note that other parties were not proposed by the survey):

$$\text{PCF} \lhd \text{LFI} \lhd \text{EELV} \lhd \text{PS} \lhd \text{MoDem} \lhd \text{EM} \lhd \text{LR} \lhd \text{DLF} \lhd \text{FN}$$

Note however, that on the "left-right economic" scale, the order is slightly different:

$$\text{LFI} \lhd \text{PCF} \lhd \text{EELV} \lhd \text{PS} \lhd \text{MoDem} \lhd \text{EM} \lhd \text{FN} \lhd \text{DLF} \lhd \text{LR}$$

indicating that the problem of ordering the candidates is not perceived as unidimensional by experts. If we now look at a voter survey, such as The Comparative Study of Electoral Systems (2023), we obtain the order LFI $\lhd$ PS $\lhd$ EM $\lhd$ LR $\lhd$ FN, which is consistent with the axes used by polling institutes, but only covers a small subset of the parties (other parties were not proposed by the survey). We also computed axes for the 2007 and 2012 elections based on these two surveys using the same methodology, and we provide them in Appendix A.

Qualitative Analysis

We now look at the axes returned by the axis rules we introduced in this chapter. We will continue to focus on the 2017 election, but the axes obtained for the datasets of the other elections are provided in Appendix A. Table 6.3 shows the axes returned by our axis rules for the datasets of the 2017 election. We can see that, like polling institutes, all the rules mostly agree on the position of the main parties, and disagree only on the positions of the small "hard to place" parties R, UPR and SP. Because our rules tend to push the less popular candidates towards the extremes (as shown by our axiomatic analysis), these small candidates are almost always placed at one of the extremes, generally on the right of the axes, as they seem to share more supporters with the candidates from the right. The second important source of disagreement is the relative order of similar parties, such as the center-left parties PS and LFI or the far-right parties FN and DLF. In particular, the rules that use more information such as MS and FT tend to put the most popular candidate of each pair closer to the center, which can cause some inaccuracies (for instance, FT places LFI closer to the center than PS in some datasets). Overall, the axes returned by our rules are consistent with the axes used by polling institutes and with the opinion-based ones.

For the online dataset, we also ran the ranking-based axis rules, using the rankings submitted by the participants for the IRV method, and restricting to the 5 755 participants that ranked all candidates. The results are displayed in the last rows of Table 6.3. The VD-rank axis is clearly less convincing than any of the axes returned by the approval-based rules. This corroborates other observations in the literature. For instance, Sui et al. (2013) ran experiments on 2002 Irish General Election data and found that the VD-rank axis only fits 0.4%–2.9% of voters. Escoffier et al. (2021)

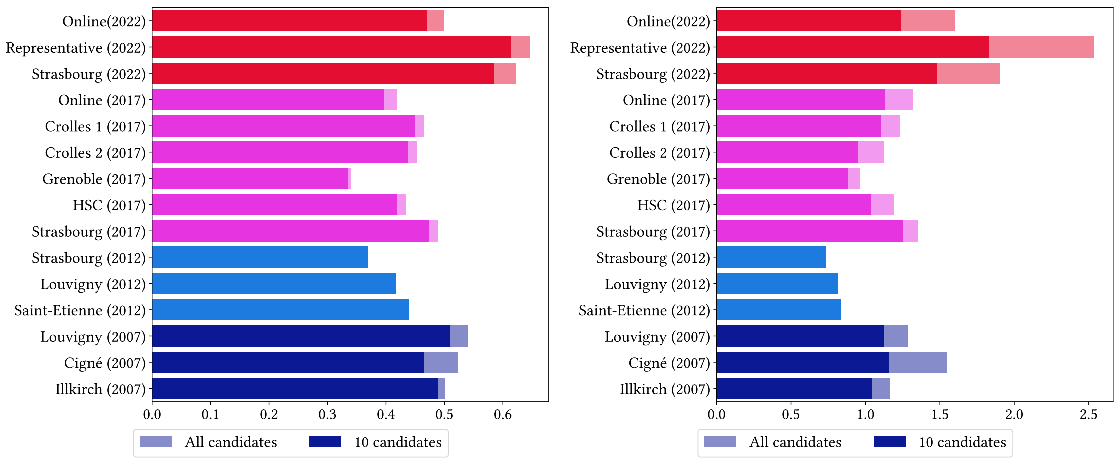
Dataset	Rule	Axis ◁										
Strasbourg	VD, MF	R	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP
	BC, MF	R	LO	NPA	PS	LFI	EM	LR	FN	DLF	UPR	SP
	FT	R	LO	NPA	PS	LFI	EM	LR	DLF	FN	UPR	SP
HSC	VD, MF	SP	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R
	BC	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	SP
	MS, FT	SP	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R
Grenoble	VD, MF	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	R	SP
	BC, MS, FT	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
Crolles 1	VD, MF	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	SP	R
	BC, MS	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
	FT	LO	NPA	PS	LFI	EM	LR	FN	DLF	R	UPR	SP
Crolles 2	VD	R	LO	NPA	LFI	PS	EM	LR	FN	DLF	SP	UPR
	MF	R	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	SP
	BC	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
Online	VD	R	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	SP
	MF	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	R	SP
	BC, MS	LO	NPA	LFI	PS	EM	LR	DLF	RN	R	UPR	SP
	FT	LO	NPA	PS	LFI	EM	R	LR	DLF	RN	UPR	SP
Online	VD-rank	FN	DLF	R	LO	NPA	LFI	PS	EM	SP	UPR	LR
	FT-rank	LO	NPA	R	LFI	PS	EM	LR	DLF	FN	UPR	SP

Table 6.3: Axes returned by axis rules on the datasets of the 2017 French presidential election. In case of ties, we report all the axes returned by the rule.

ran experiments on a similar French presidential election dataset and also observed that the optimal axis found using VD-rank was very different from the orderings discussed in French media. In our case, the optimal VD-rank axis only covers less than 4% of voters. For comparison, the approval version of VD returns an axis covering more than 60% of voters for the same dataset. The FT-rank axis is however much closer to the ones returned by approval-based rules and the ones used by the polling institutes, the main disagreement being again the position of *Jean Lassalle* (R).

The Optimal Costs

In Figure 6.14, we show the costs of the optimal axes for all the datasets we considered, using VD and MS. Note that for each dataset, we divided the costs by the number of voters who approved at least two candidates (more precisely by the sum of their weights). We can see that the costs of the optimal axes for VD are quite similar across datasets, ranging between 0.3 and 0.5 per voter, meaning that in general, between 50% and 70% of all voters approving at least two candidates approve an interval of the optimal axis. With MS, we observe that different datasets of the same election have similar costs for their optimal axes, but costs differ from one election to another. For instance, the cost of the optimal axes are lower for the 2012 election than for the 2022 one. This remains true if we reduce the set of candidates in the preference profiles such that each dataset has $m' = 10$ candidates. One way to interpret this is to say that the political landscape was more unidimensional in 2012 than in 2022. However, the optimal costs heavily depend on which candidates are running in the election. In particular, we suspected that some "small" candidates, that are hard to place on the axis, even for a political experts, were contributing a lot to the cost of the optimal axes, relatively to their importance. Figure 6.15 highlights this phenomenon. Indeed, while for most candidates, there exist candidates with whom they are very often co-approved,



(a) Voter Deletion.

(b) Minimal Swaps.

Figure 6.14: Costs of the optimal axes when considering all candidates, and when restricting the profile to the 10 candidates with the most votes at the actual election, for datasets of the *Voter Autrement* collection, using VD and MS. All the costs are divided by the total weight of voters who approved two or more candidates.

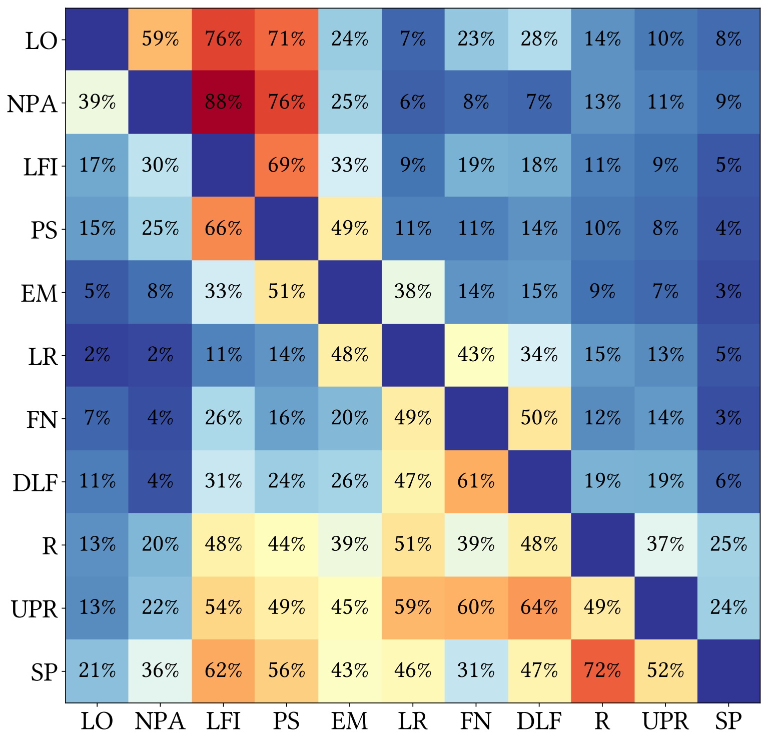
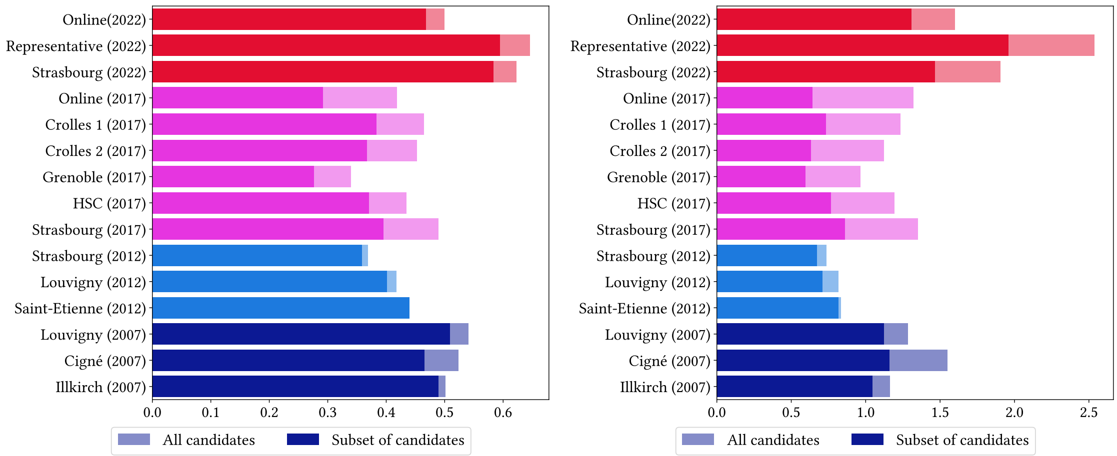


Figure 6.15: Co-approval of pairs of candidates for the *Voter Autrement* dataset of the 2017 election, collected in Grenoble. The number in each cell indicates the percentage of time the column candidate was approved by a voter who approved the row candidate.



(a) Voter Deletion. (b) Minimal Swaps.

Figure 6.16: Costs of the optimal axes when considering all candidates and when excluding some candidates from the *Voter Autrement* datasets, using VD and MS. Costs are divided by the total weight of voters with two or more approved candidates. The new number of candidates are respectively 10, 9, 8 and 11 for the 2007, 2012, 2017 and 2022 elections.

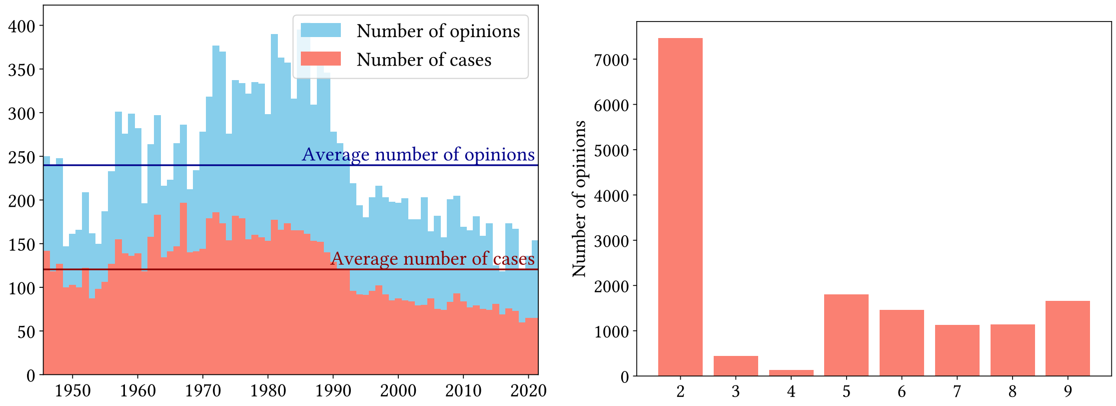
and candidates with whom they are almost never co-approved, for some other candidates, their relationship to other candidates is unclear. For instance, we can see on Figure 6.15 that in the dataset collected in Grenoble in 2017, *Jacques Cheminade* (SP), *François Asselineau* (UPR) and *Jean Lassalle* (R) have high co-approval values with *almost all other candidates*, indicating that they do not have clear “neighbours” or clear “opponents” on the axis.

Thus, we additionally computed the cost of the optimal axis excluding from the profile the “small” candidates for which it is unclear where they should be placed on the axis (note that these are not necessarily the least popular candidates). This corresponds to four candidates: *Frédéric Nihous* CPNT (2007), *Jacques Cheminade* SP (2012 and 2017), François Asselineau UPR (2017) and Jean Lassalle R (2017 and 2022). We also removed Gérard Schivardi from the 2007 datasets since he was not approved by enough voters, leading to inaccuracies. This leaves 10 candidates in 2007, 9 in 2012, 8 in 2017 and 11 in 2022. The axes we obtain with our axis rules using these reduced profiles are the same as the ones in Table 6.3 when restricted to the new subset of candidates (thus satisfying the heredity property), but with *much lower costs*, as we can see in Figure 6.16. In particular, for the 2017 datasets, we almost halved the MS cost by removing only three “small” candidates from the election.

Finally, we observed in these datasets that in the majority of the cases, there is no violation of the heredity property when adding one candidate (i.e., adding a candidate did not change the relative order of the other candidates). Still, all our rules violate this property in some datasets, but when it is the case, the changes are marginal, generally a “small” candidate switching from one extreme to another, or two similar candidates being swapped.

Supreme Court

We now present our experiments based on the data from the Supreme Court of the United States. We used our rules to obtain an ordering of the 9 justices of the Supreme Court. The dataset is based on the opinions authored and joined by the justices, derived from the Supreme Court



(a) Number of cases and opinions.

(b) Distribution of the number of justices.

Figure 6.17: Statistics about the SCOTUS dataset. In particular, the number of cases and opinions per year, and the distribution of the number of justices who joined an opinion. Note that the number of cases corresponds to the number of majority opinions.

Database[6] (Spaeth et al., 2023),which contains data for Supreme Court decisions since 1946.

The Dataset

The Court consists of 9 justices who *vote* on each case about which of the two parties to the case wins. The Court then publishes a *majority opinion* explaining the Court's reasoning. Justices can also submit *concurring opinions* and *dissenting opinions*, and *join* any of the opinions submitted by others. Concurring opinions explain additional or alternative reasons, written by justices who voted with the majority. Dissenting opinions explain why a justice did not vote with the majority. Note that there might be several dissenting and concurring opinions. Each opinion, concurrence, or dissent becomes a ballot "approving" the justices that joined in it (so in that sense, the "candidates" are the justices, and the "voters" are the opinions). The intuition is that justices joining the same opinion share an ideology and should be placed close together.

In our experiments, we discarded all terms with more than 9 justices (e.g., if one justice is replaced mid-term), giving us 65 terms between 1946 and 2021, and thus 65 approval profiles. We also removed all opinions joined by only one justice, so that all "approval ballots" contain at least two justices. In Figure 6.17, we show some statistics about these datasets. The average number of opinions per year is $n = 240$ (for 121 cases per year on average). Moreover, around 48% of the opinions are joined by only two justices.

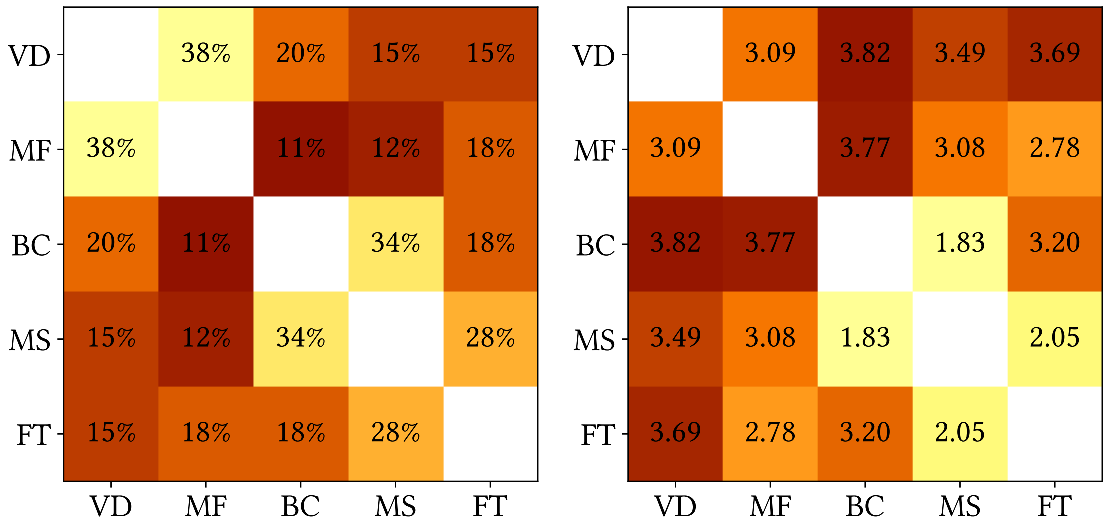
The Martin-Quinn Method

The problem of ordering the justices has been extensively studied; the standard method used by political analysts is the *Martin-Quinn* (MQ) method, which uses a dynamic item response theory model (Martin and Quinn, 2002). The Martin-Quinn method for deriving an axis of justices uses only the binary vote data (i.e., whether a justice voted for or against the winning opinion), and its underlying model assumes that a decision divides the axis of justices in the middle, with all justices to one side of the cutoff voting the same way. One issue with this approach is that justices may vote for (or against) the same opinion but have different reasons for it. It could be that the

[6] http://scdb.wustl.edu/

Rule	Avg KT	Same median	Same axis
VD	4.94	53.8 %	1.54 %
MF	4.22	58.5 %	3.08 %
BC	3.68	56.9 %	3.08 %
MS	3.55	64.6 %	1.54 %
FT	**3.43**	**66.2 %**	**7.69 %**

Table 6.4: Average Kendall-tau distance between the axes returned by the rules and the MQ axes, percentage of the time the axes have the same median justice than the MQ ones, and percentage of the time the rules return the MQ axis.



(a) Exact matching.

(b) Kendall-tau distance.

Figure 6.18: Percentage of exact match and average Kendall-tau distance between the axes returned by the different rules.

most progressive and most conservative justices vote the same way, while the centrist justices vote the other way, for example due to procedural reasons. This is not well-captured by the model. In addition, the model does not use some relevant information. For example, if two justices very frequently join each other in their concurring or dissenting opinions, this suggests that these justices should be placed near each other on the axis.

Axis Rules Comparison

We computed the axes returned by our rules for these 65 terms and compared them to the axes obtained by the established Martin-Quinn method, by computing the KT distance between the axes. Table 6.4 shows the average KT distance between the axes returned by our rules and the Martin-Quinn axes. We see that these distances are on average quite low (noting that the worst possible KT distance for $m = 9$ is 18), even if the returned axes rarely match exactly with the Martin-Quinn ones. Moreover, we observe that FT comes closest, while VD is relatively far away. We also checked how often the axes computed by our rules agreed with the Martin-Quinn axis on which justice is placed in the median position. This is of particular interest since the median justice tends to be pivotal. All rules agreed with MQ on who was the median justice for more than half of the terms. FT agrees most frequently, choosing the same median justice in 66% of the terms. We also compared the axes returned by our rules to each other. Figure 6.18 shows the percentage of terms for which the axes returned by the different rules are the same, and the average KT distance between them. We see that the axes returned by the rules are in general quite similar, especially between BC and MS, with an average KT distance of 1.83, and 34% of exact matches.

Figure 6.19 shows the evolution of the positions of the justices on the axes for the terms between 2010 and 2021, according to the axes produced by the Martin-Quinn method and by our rules. It is very clear that the Martin-Quinn method is smoother over time, which is by the rule's design, since it takes the justices' positions of the last term as a prior to compute their positions in the next term. Our rules are less stable, but they clearly separate the more conservative from the more liberal justices. From these figures, the rules that use more information, such as MS and FT, seem to be more stable over time than rules that use less information, such as VD and BC. For future work, we see potential in adapting our rules to obtain methods that are smoother over time, and perhaps more interesting than the Martin-Quinn method (as they will satisfy axiomatic properties).

Evolution of the Optimal Costs

The axes returned by the rules are not the only interesting information that we can obtain, and as for the *Voter Autrement* datasets, we were also interested in the costs of the optimal axes for each rule. We also wanted to compute the costs of the Martin-Quinn axes with our rules and compare them to the optimal costs. Since each term has a different number of opinions n, we divided the costs by the number of opinions joined by at least two justices. The results are shown on Figure 6.20. The first observation is that for most terms, the cost of the Martin-Quinn axes are very close to the cost of the optimal axes, especially for BC, MS and FT. The second observation is that the costs of the optimal axes and of the Martin-Quinn axes seem to slightly decrease over time, regardless of the rule. One way to interpret this is through the lens of the global *polarization* of American politics, that can also be observed in Congress and among partisan voters during the last decades (McCarty et al., 2006). One consequence of this polarization is a global ideological alignment of the opinions of each party's supporters on a wide range of topics (i.e., if they agree on one topic, they will agree on most of the others), and which might be one reason why the optimal axes are becoming more accurate over time. Finally, Figure 6.20 (f) shows again that the optimal costs of some rules are close: this is the case for VD and MF, and for BC and MS. In our dataset, the average cost per opinion of the optimal axes is 0.44 for VD (i.e., on average 56% of the opinions are intervals of the optimal axis), 0.52 for MF, 0.89 for BC, 1.09 for MS and 2.72 for FT.

Tierlists

Finally, we conducted experiments on tierlist datasets, that we introduced in Section 2.5.3 and used in the experiments of Chapter 3. As a reminder, these datasets were collected from the website *TierMaker.com*, on which users can create tierlists by ranking sets of items in different categories. Formally, a tierlist contains a set of labels (e.g., $\{S, A, B, C, D, F\}$), and the voter associates each candidate to some label. Since there exists a relative order between the labels (e.g., $S \succ A \succ \cdots \succ F$), this gives us a weak order of the candidates for each voter. The approval ballots are the set of candidates that received the best possible label for each voter.

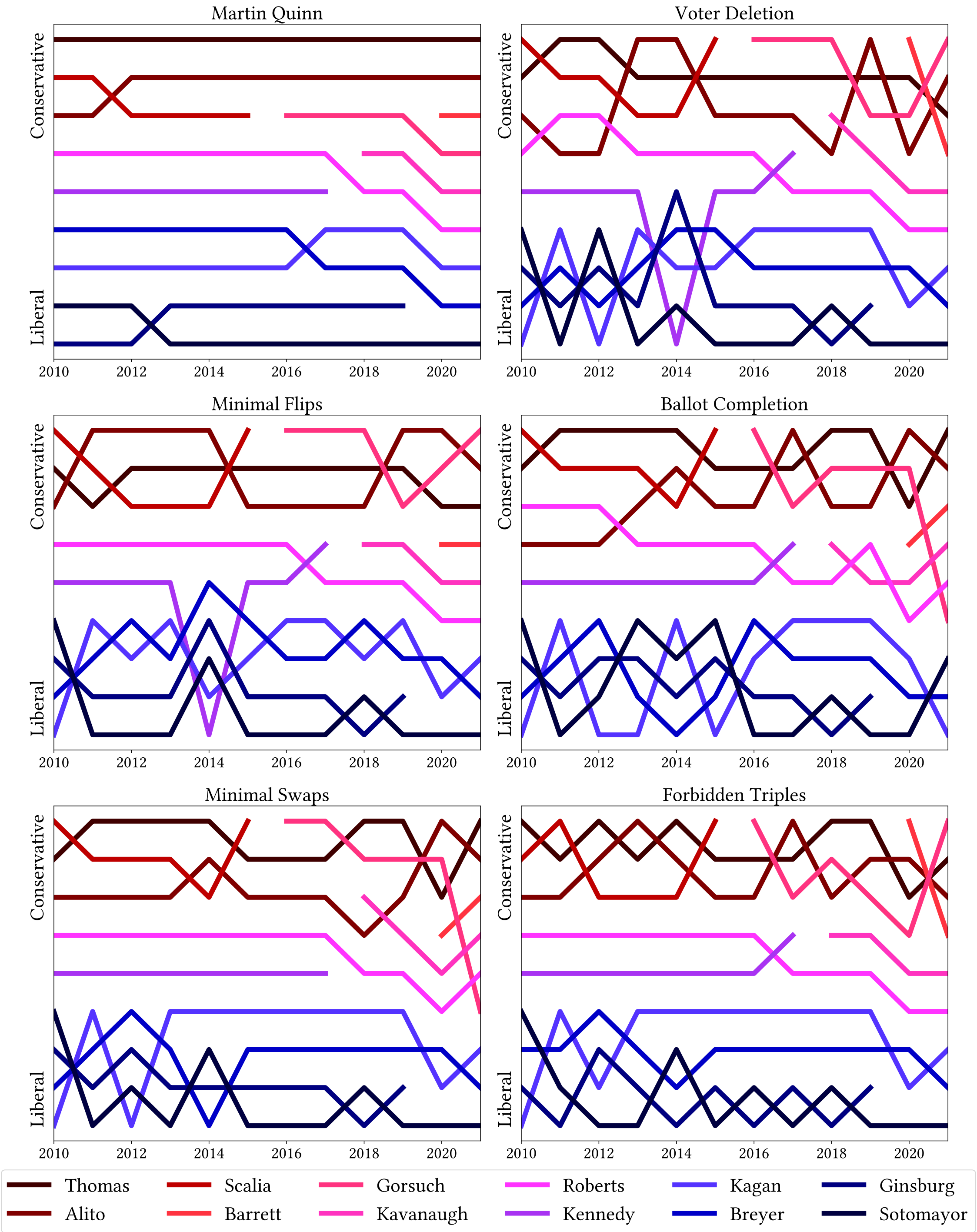
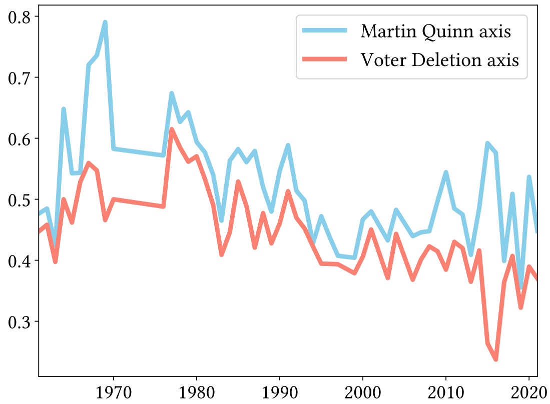
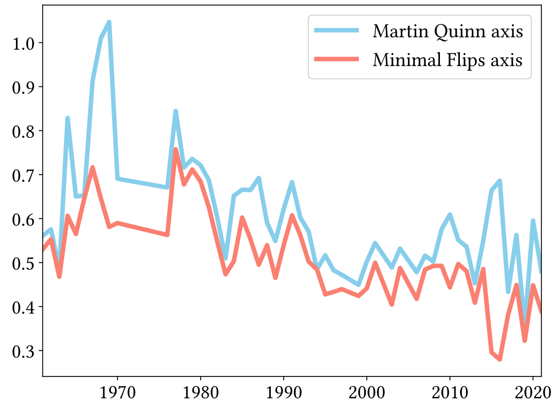


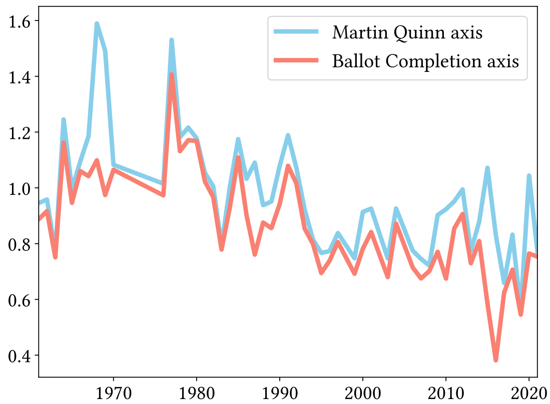
Figure 6.19: Axes of the justices of the Supreme Court of the United States between 2010 and 2021, using the Martin-Quinn method and using our axis rules.



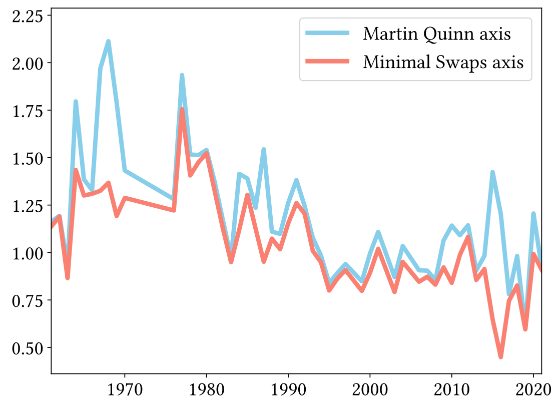
(a) Voter Deletion costs.



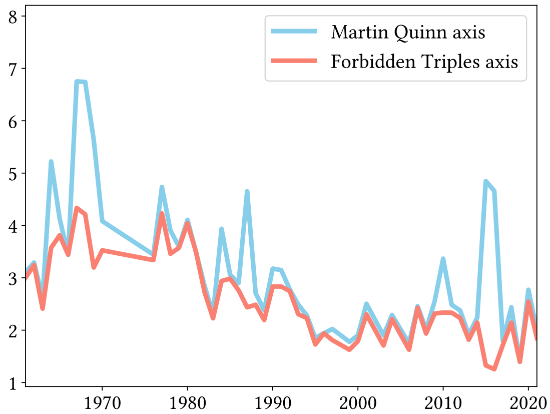
(b) Minimal Flips costs.



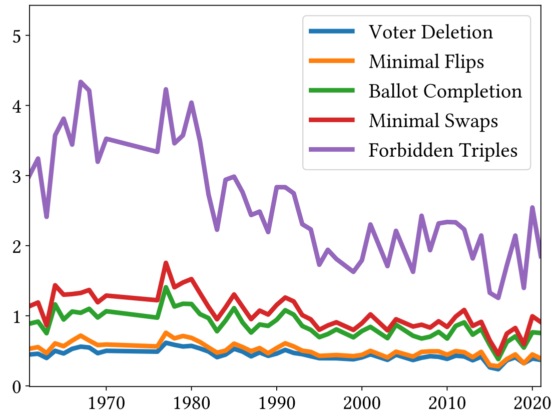
(c) Ballot Completion costs.



(d) Minimal Swaps costs.



(e) Forbidden Triples costs.



(f) Optimal costs.

Figure 6.20: Figures (a) – (e) show, for each term, the costs of the optimal axes and of the Martin-Quinn axis, divided by the number of opinions joined by at least two justices. In Figure (f), we compare the optimal costs (divided by the number of opinions) of the different rules.

Dataset	n	m	m^*	#app.
Planets	58	9	–	2.64
Numbers	50	10	–	2.56
Months	244	12	–	2.80
Harry Potter Movies	324	8	–	2.66
Star Wars Movies	4 002	11	–	2.71
Spider Man Movies	346	10	–	2.80
Taylor Swift Albums	169	12	–	3.24
Colors	618	11	–	3.08
School Courses	214	11	15	2.67
European Countries	624	10	51	3.94

Table 6.5: Datasets of tierlists used in the experiments. n is the number of voters, m the number of candidates used for the experiments, m^* the original number of candidates, and #app. the average number of approved candidates per voter.

Dataset	Months	Star Wars	Spider Man	School Courses	European Countries
Axis $\triangleleft$	March	Solo	Venom	Chemistry	Italy
	Apr.	Ep. II	Amazing SM 2	Physics	France
	May	Ep. I	Amazing SM 1	Math	UK
	Aug.	Rogue One	SM 3	Technology	Germany
	June	Ep. III	SM 1	Music	Switzerland
	July	Ep. V	SM 2	Art	Sweden
	Dec.	Ep. IV	Spiderverse	PE	Norway
	Oct.	Ep. VI	No way home	History	Denmark
	Nov.	Ep. VIII	Homecoming	Social studies	Iceland
	Sep.	Ep. VII	Far from home	Foreign lang.	Finland
	Jan.	EP. IX		Literature	
	Feb.				

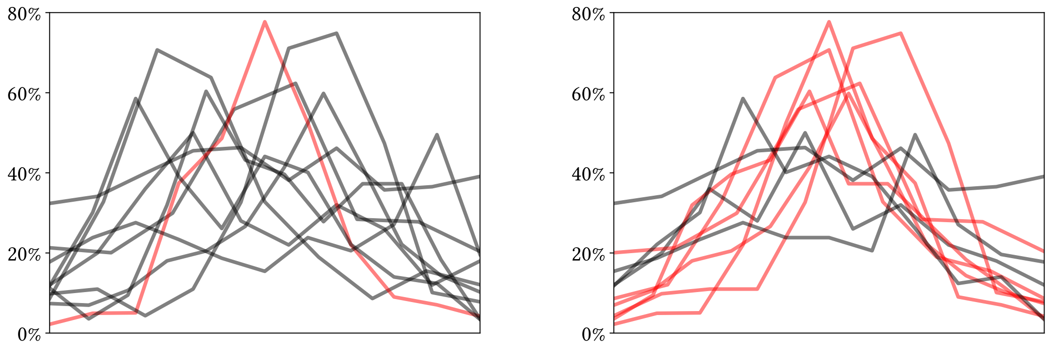
Table 6.6: Axes returned by FT for some tierlist datasets. The colors indicate groups of candidates (e.g., seasons of the year or trilogies of movies).

Optimal Axes

We list in Table 6.5 the datasets used in our first experiment.[7] We selected datasets on topics for which there exists a (more or less) "natural" axis. For instance, the order of the months in the year, or the chronological order of the movies of some series. However, we do not expect that axis rules will return these axes: there is not necessarily a good reason for them to be correlated with voters' preferences. For some datasets, we reduced the number of candidates in order to keep the running time reasonable. In that case, we removed the candidates with the least approvals. Moreover, we removed voters approving zero or one candidate from all the datasets, as their votes are intervals of any axis.

We show in Table 6.6 the axes returned by the Forbidden Triples rule for some of these datasets. The complete results for all rules and all datasets can be found in Appendix A. Most of the resulting axes are hard to interpret, and not particularly close to the *natural* axis. However, our rules seem to have correctly identified *clusters* of similar candidates. For instance, the movies of each trilogy are grouped together on the axes of the *Star Wars* dataset, and similarly for the *Spider Man*

[7] Recall that these datasets contain preferences of the voters, even if it seems unnatural to have preferences on some of these topics (e.g., months of the year, planets, or numbers).



(a) Voter Deletion. (b) Forbidden Triples.

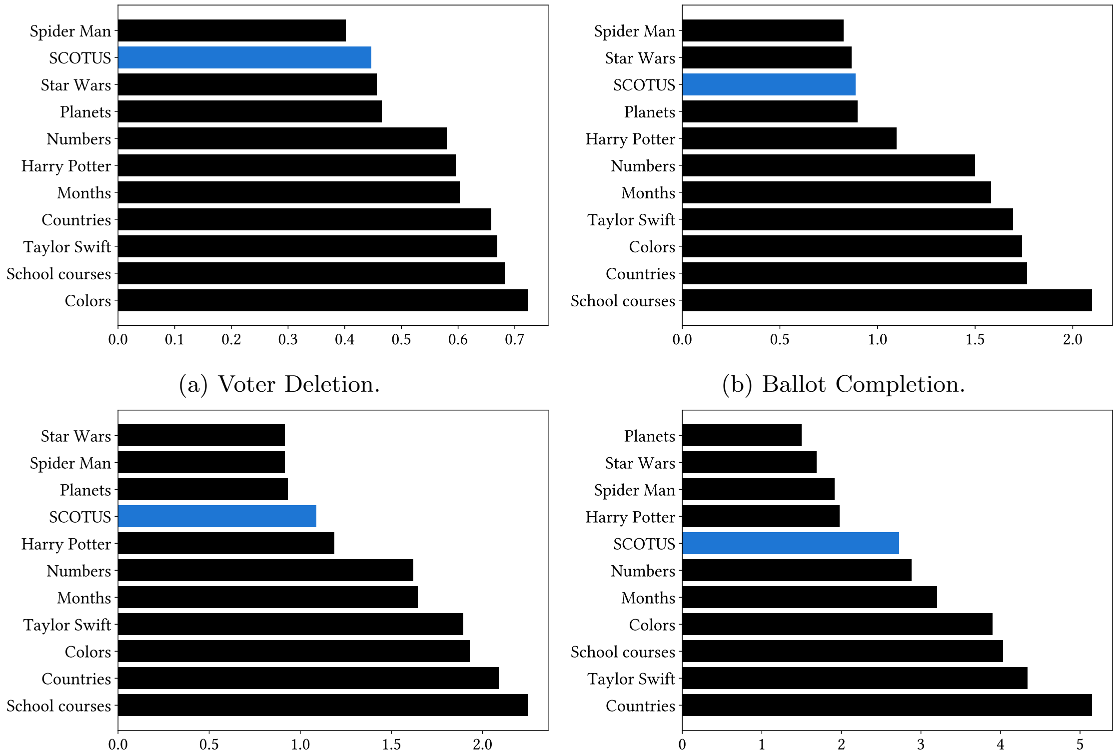
Figure 6.21: Distribution of approval scores based on the positions of the candidates on the axis. Red lines indicate single-peaked distributions.

dataset. On the *European countries* dataset, the nordic countries are grouped together, and on the *School courses* dataset, the hard science courses are grouped together, as well as the humanities courses. Finally, the best example is probably the *Months* dataset, for which the axes returned by our rules group the months by season, and even order the seasons correctly.

What is striking on these axes is that the popular candidates are generally close to the center, and the less popular ones are near the extremes. This is especially true for the rules that use more information, like FT. To illustrate this phenomenon, we show in Figure 6.21 the distribution of approval scores along the axes returned by VD and FT for all the tierlist datasets. While it is not that obvious for VD, we clearly see a "centralization effect" of the popular candidates for FT. Moreover, for 6 of the 10 axes returned by FT, the distributions of approval scores along the axes are single-peaked: when going from left to right on the axis, the approval scores of candidates are increasing until we reach a peak (the approval winner), then the scores are decreasing. This leads to some surprising "bugs" in the axes, in which a popular candidate is placed in the middle of a group to which it does not belong. This is for instance the case of *December* in the *Months* dataset, which is the most popular month, and appears between the summer and autumn months on the axes (see in Table 6.6). Note that we did not observe a centralization effect this strong for the SCOTUS dataset, in which the distribution of approval scores along the axis was single-peaked for only 5% of the terms when we used the FT rule. This is likely due to the fact that in the SCOTUS dataset, all the justices have similar "approval scores" (i.e., number of opinions joined).

Optimal Costs

Finally, Figure 6.22 shows the costs of the optimal axes for the different datasets. All the costs are divided by the number of voters with two or more approved candidates. We added to the figure the average costs of the axes obtained with the SCOTUS dataset, for comparison. We see that for most datasets, the optimal costs are quite high. In particular, for half of these datasets, we have more than 60% of non-interval voters for the optimal VD axes. However, for some datasets, the optimal axes have small costs, sometimes smaller than for the SCOTUS dataset. In particular, with the *Spider Man* and *Star Wars* datasets, our rules returned axes with low costs. This could be explained by the fact that in these datasets, there are clear groups of candidates (e.g., different trilogies), while this is not always the case for other datasets.



(a) Voter Deletion.

(b) Ballot Completion.

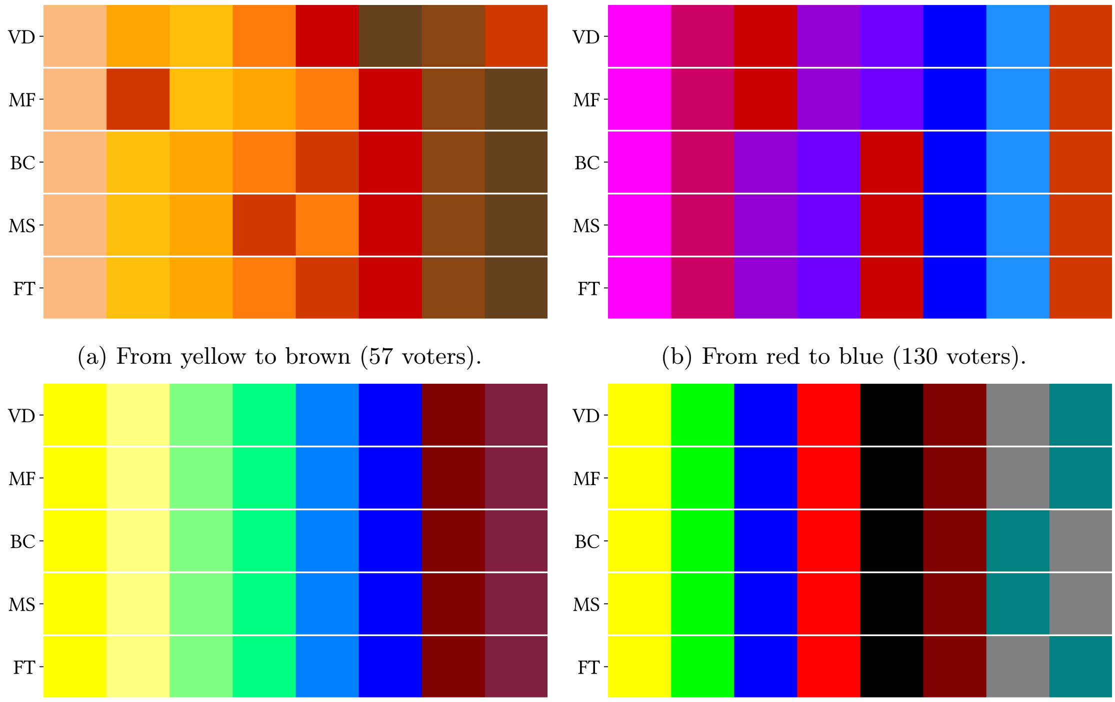
(c) Minimal Swaps.

(d) Forbidden Triples.

Figure 6.22: Costs of the optimal axes for the different rules on the tierlists datasets, divided by the number of voters with two or more approved candidates. The blue bars indicate the average cost of the optimal axes for the SCOTUS dataset.

Color Axes

Finally, we conducted experiments on two additional tierlist datasets of preferences over *colors* (different from the dataset mentioned in Table 6.5). They each have 33 candidates (which are different colors), and respectively 268 and 283 voters. Because of the high number of candidates, the palette is more nuanced and several colors are very similar, which makes the analysis more interesting. Indeed, we can select subsets of colors that contain interesting nuances, or groups of similar colors. Figure 6.23 shows the axes returned by our rules on four selected subsets, and we provide figures for more color palettes in Appendix A. We observe again that our rules are good at clustering similar candidates (here, colors), as it is the case in Figure 6.23 (c), in which we selected four pairs of *quasi*-clones, and in Figure 6.23 (d), in which we selected the four most flashy and the four most dull colors of the palette. In both cases, the similar colors are correctly grouped together. However, it seems that the axes returned are not always "satisfying" in the sense that they do not show the colors in the correct nuance order. This is for instance the case in Figure 6.23 (b) in which we would have expected to have red and blue at the extremities and the more purple/rose colors in the center. A reason why it is not the case here is probably the same as we discussed earlier: more popular candidates tend to go to the center of the axis, and less popular candidates are pushed towards the extremes. In some cases though, the returned axes are quite satisfying, like the one returned by FT in Figure 6.23 (a).



(a) From yellow to brown (57 voters).

(b) From red to blue (130 voters).

(c) Clones of candidates (104 voters).

(d) Flashy and dull (101 voters).

Figure 6.23: Axes of colors for some subsets of eight colors, obtained using the different axis rules. We provide the number of voters with at least two approved candidates for each subset.

6.5 Discussion

In this chapter, we introduced a new model for deriving orderings of candidates solely based on the approval ballots expressed by the voters. In particular, we introduced a family of scoring rules and analyzed five natural rules from this family.

Which Rule to Use?

First of all, our axiomatic and experimental analyses showed that we can order these rules on a spectrum from the least informational (Voter Deletion) to the most informational (Forbidden Triples). We saw that two of these rules satisfy a monotonicity property, and that we could characterize Voter Deletion based on this property and an independence of clones property. Furthermore, we showed an incompatibility between two natural clones-related axioms.

Our experiments showed that all of the rules we introduced were able to correctly identify clusters of similar candidates, and to return axes that are close to the "natural" (for real data) or "correct" (for synthetic data) axes. Based on these experiments, it also seems that the more informational rules, like Forbidden Triples or Minimal Swaps, generally perform better and return axes that are closer to the correct ones. Finally, our analysis showed that our axis rules tend to push popular candidates towards the center of the axis, and less popular ones towards the extremes.

When it comes down to recommending one of the methods, it seems that the rules that use more information have more advantages in terms of axis quality, and the rules that use less information are easier to interpret. However, the choice of the rule should depend on the context of the application. For instance, in a context of ordering parties based on *their votes* on a set of bills, approvals and disapprovals have symmetrical roles, and it would make sense to use a rule like Minimal Flips, which treats approvals and disapprovals the same.

Applications of the Problem

This study gives a new perspective on the problem of building a left-right axis of political parties or leaders, that is based on concrete data and explainable methods, and not on the perception of experts or voters. Moreover, with the introduced rules, one can interpret the cost of the optimal axis as how "unidimensional" the political spectrum actually is, given a set of candidates. As we already explained, we can build an axis that is based on the preferences of voters over the political parties, but we can also use our rules to build an axis based on the votes of the parties at the parliament, as we did with the opinions of the justices of the US Supreme Court.

If we mainly focused on the political applications of the model in this chapter, our methods can be used in a variety of fields, as long as we can represent the problem as a binary matrix. As already discussed in the introduction, this is for instance the case of the relative dating problem from geology and archaeology, in which we respectively want to date strata based on the fossils they contain, and items or tombs based on the designs appearing on them. However, in these contexts, and in many others (including political ones), the computational hardness of the rules can be a curb on the applicability of the model. In particular, our algorithms can take several days for datasets of more than 15 candidates. If we want to order all the items found in an archaeological site, all the politicians of a parliament, or all the projects of a participatory budgeting election, we need to find faster methods. We also need faster methods if we want to order small sets of candidates, but want to have *immediate* feedback. Consider for instance an interactive website that displays an axis of candidates or parties based on the votes of the visitors, and votes are constantly being added. For these reasons, it would be useful to design polynomial-time computable rules that produce acceptable outputs. Greedy versions of our rules are a natural starting point, but maybe better techniques exist.

Extensions of the Model

It would also be interesting to study extensions of this model that tackle some of the issues we mentioned in this chapter. One of them is the fact that the popular candidates are pushed towards the center of the axis, which can create some inaccuracies. To tackle this, we could design rules that take into account the popularity of each candidate when building the axis (similarly to what was done by Laslier (2006) for multi-dimensional spatial representation of candidates). We could also design rules that first identify the relevant subset of candidates on which axes can be built, for instance by removing candidates that have a high co-approval value with all other candidates.

Another interesting extension tackles the *stability over time* issue that we encountered with the Supreme Court dataset. Indeed, it would be interesting to design rules that take into account the axis of the previous term when building the next one. We could also generalize our model to approval profiles with incomplete preferences (Imber et al., 2022). This is for instance interesting if we want to order the parties in a parliament and they can either vote in favor, against, or abstain.

Other interesting directions for future work include studying how we can adapt our model to other types of structures, like circular axes, in which the first and last candidates of the axis are next to each other. We could also study how to adapt this model to more than one dimension, or how to introduce metric distances between candidates on the axis.

Chapter 7

Identifying Conflicting Pairs of Candidates via Rankings

7.1 Introduction

In the previous chapter, we saw that we could apply methods and concepts from social choice theory to solve problems beyond the selection of winners. In particular, we saw how we could take advantage of expressive preferences to *learn* about the structure of the electorate, and of the set of candidates. Unlike the problems studied in the first part of this thesis, the goal was not to find a social outcome for the voters based on their preferences, but to describe the organization of the society. In this chapter, we study another problem, in which we aim at learning about the electorate. Motivated by the increasing polarization of our societies, our objective will be to identify the pairs of candidates that are *inducing the most conflict* among voters.

Since each voter has their own perspective, and their own opinions on the candidates, it is almost inevitable in a situation of collective decision that there will be conflicts. Not only it is inevitable, but the existence of conflicts and disagreements between voters is generally sane: A society in which everyone thinks the same is characteristic of a totalitarian society. On the other hand, having *a lot* of divisions and conflicts is not desirable either, and it can even be dangerous for the survival of the society. This is a major concern of our century: several recent studies showed that global polarization and social conflicts are on the rise in many countries (Boxell et al., 2024; Draca and Schwarz, 2024), and that the increase in polarization is often correlated to the erosion of the democracy, and to a lack of trust in the institutions (Somer and McCoy, 2019; McCoy et al., 2018). In that sense, one motivation for identifying the largest sources of conflict within an electorate is to use this knowledge to hopefully *reduce* this conflict. First, because identifying conflicting groups of voters can help to find communication channels to improve dialogue between those groups. Second, because by identifying the candidates, or issues, that are causing the most conflict, we can address these issues as a priority and seek compromises.

This is not the only reason why identifying conflicting pairs of candidates is interesting. Consider for instance that someone wants to organize the most engaging *sports events* involving two teams or players. Then, selecting the pairs of candidates that are inducing the most conflict among sport supporters could be an interesting idea. Similarly, a *political debate* between the candidates that are inducing the most conflict would be very engaging. For socially relevant deliberations, as well as scientific or business discussions, debating conflicting ideas is a way to boost the creativity with passionate discussions. This reasoning can be applied to any field: the more conflict is induced by two alternative choices, the more engaging will be a "clash" between them.

How to Measure Conflict?

Thus, our goal is to find pairs of candidates that are generating conflict in the electorate, using only the preferences of the voters. First, we need to specify what we mean by conflict. Intuitively, two candidates a and b are inducing the most conflict when the following two conditions are fulfilled: (1) half of the electorate prefers a to b, and the other half prefers b to a, and (2) every voter likes a or b the most and the other one the least. The first condition ensures that the electorate is perfectly divided between the two candidates, and the second condition that the voters have very strong opinions on these candidates. More generally, we want to find two candidates that are supported by two groups of voters of similar sizes, *and* that are perceived ideologically as far from each other as possible in these two groups. However, this dual nature of conflict poses a significant conceptual challenge, as we need to balance these two aspects. For instance, if one pair of candidates divides more evenly the electorate than another pair, but voters have stronger opinions on the candidates of the other pair, which one should be considered as the most conflicting?

Let us now analyze how we can measure conflict under different preference formats. Clearly, uninominal preferences are not expressive enough when there are more than $m \geqslant 3$ candidates, as we only know the favorite candidate of each voter.

The natural way to define conflict with approval preferences would be to say that the conflict induced by two candidates is linked to *the number of voters who approve one but not the other*. Thus, let us denote $n^{a \succ b}$ the number of voters who approve a but not b, and $n^{b \succ a}$ the number of voters who approve b but not a. Then, the conflict is maximal when $n^{a \succ b}$ and $n^{b \succ a}$ are as *large* as possible, but also as *close* as possible (to ensure that we have a balanced conflict). Two natural ways to find pairs of candidates inducing the most conflict would then be to select the pairs that maximize *the product* $n^{a \succ b} \times n^{b \succ a}$, or *the minimum* $\min(n^{a \succ b}, n^{b \succ a})$. Maximizing the sum is not interesting, as it might lead to selecting a pair $\{a, b\}$ such that every voter approves a and disapproves b (so everyone agrees on which candidate is the best). While no work in social choice specifically looked at this problem, the product of the sizes of different supporters groups has already been identified as a good measure of ethnic polarization and potential conflicts (Montalvo and Reynal-Querol, 2005). Unfortunately, approval preferences completely hide the *intensity* of the preferences that voters have over the candidates. For instance, if a is approved by a voter but not b, that could mean that a is his favorite candidate and b his least favorite, or that a is just above their approval threshold, and b just below.

In this chapter, we define solutions to our problem that use the ordinal preferences (rankings) of voters. Note that with ordinal preferences, we can still define $n_{a \succ b}$ as the number of voters who prefer a candidate a to another candidate b as in the approval case, but we can also use the *intensity* of the preferences each voter has for the candidates, which is maximal when one of the two candidates is ranked *first*, and the other is ranked *last*.

Multi-winner Rules

As we aim at selecting a pair of candidates using voters' preferences, our task technically falls into the framework of *multi-winner voting rules*, that we discussed in Section 2.3.2. Such rules, given the preferences of voters and a committee size, select a winning *committee* (i.e., a subset of the candidates) of the desired size that aims at meeting a certain objective. As we mentioned in

Section 2.3.2, most multi-winner voting rules follow one of three objectives: (1) *individual excellence*, focusing on selecting candidates that are individually the best ones, (2) *diversity*, focusing on representing as many voters as possible, and (3) *proportional representation*, focusing on selecting a committee that proportionally represents the voters.

Our case is different, as our objective is not to select the winners of an election, but to identify the candidates that are inducing the most conflict. Focusing on diverse candidates might appear as an interesting start to fulfil our goal: intuitively, the two most diverse candidates are more likely to also be those that divide the voters the most. However, this is not necessarily the case, for the simple reason that rules seeking diversity are still aiming to find *popular* candidates. Indeed, if half of the voters have the preferences $a \succ b \succ c$, and the other half $a \succ c \succ b$, then any rule designed for electing winners, including the ones aiming for diversity, will include a in the selected committee, while the only pair of candidates that induces conflict among the voters is $\{b, c\}$: for the other pairs, all voters agree on the ordering of the candidates.

Finally, Dong et al. (2024) pursued yet another goal, that can be seen as opposite to ours, as they aim at selecting a committee of candidates that are the most *interlacing*, in the sense that the selected candidates should have overlapping sets of supporters. In that sense, they aim at minimizing the conflict induced by the candidates in the committee. Another difference with our work is that they use approval preferences while we are using ordinal preferences.

Related Polarization Measures

In the literature, several interesting notions of conflict and polarization have already been defined, and some of them are based on ordinal preferences. However, in all of these works, the aim is to measure conflict either at the level of *single candidates* (i.e., the voters are very divided on this candidate/issue and have strong opinions on it) or at the level of *the whole profile* (i.e., the voters are very divided on most candidates/issues), while in this chapter, we aim at measuring conflict at the level of *pairs of candidates*. Further works could generalize our approach to measure conflict at the level of *subsets of candidates* of any size. To see the differences between these several notions of conflict, consider the following three profiles of ordinal preferences:

P_1 :	50% $a \succ b \succ x_1 \succ \cdots \succ x_{99}$	50% $x_1 \succ \cdots \succ x_{99} \succ a \succ b$
P_2 :	50% $a \succ x_1 \succ \cdots \succ x_{99} \succ b$	50% $b \succ x_1 \succ \cdots \succ x_{99} \succ a$
P_3 :	50% $a \succ x_1 \succ \cdots \succ x_{99} \succ b$	50% $b \succ x_{99} \succ \cdots \succ x_1 \succ a$

In P_1, both candidates a and b are very divisive candidates: half of the voters put them first, and the other half put them last. Nonetheless, they do not form a conflictual pair: they are next to each other in all rankings, and all voters agree that a is better than b. In P_2 however, the two candidates form a conflicting pair: the voters who place a in first position in their ranking also place b in last position, and conversely. Still, the profile as a whole is not very polarized: the voters only disagree on the positions of a and b. A more polarized preference profile is P_3, in which the first half of the voters rank the candidates in the reverse order of the second half.

The earliest notions of polarization based on preferences are inspired by the work of Esteban and Ray (1994), who investigated a measure of polarization for *a distribution* of values. In other words, individuals are described by a value in some set (for instance $\mathbb{N}$), representing one of their characteristics. In this paper, the authors identify three key aspects of polarization, of which two align with the aspects of conflict we mentioned earlier: "(1) there must be a high degree of *heterogeneity* between the different groups and (2) there must be a high degree of *homogeneity* within each group". Consequently , the measure they introduce relies on the size of the groups (the larger and more equally divided, the more polarized) *and* on the "ideological" distance between the groups (the more distant, the more polarized). This work was later generalized by Ozkes (2013), who adapted these measures to the case of ordinal preferences.

More recently, Can et al. (2015, 2017) investigated and characterized another family of polarization measures, that is also based on ordinal preferences. In this work too, the goal is to quantify the polarization of preference profiles *as a whole*. Interestingly, the polarization measure they focus on is defined as a sum of some *pairwise* polarization measure over all pairs of candidates. Thus, they are indirectly defining a measure of conflict for pairs of candidates, which can be linked to the conflict measure "partitioning ratio" we introduce in Section 7.2. Faliszewski et al. (2023a) also investigated polarization measures for preference profiles, though they are using a completely different approach, based on the properties of the Kemeny rule. They additionally investigated the related notions of diversity and agreement in elections with rankings. Other works study related measures for preference profiles of rankings, for instance diversity measures (Hashemi and Endriss, 2014) and cohesiveness measures (Alcalde-Unzu and Vorsatz, 2016).

Finally, some other works aim at measuring the divisiveness induced by *individual* candidates. Colley et al. (2023b) recently proposed such a divisiveness measure (see also Navarrete et al. (2023)). In their work, they are also quantifying how divisive a candidate is by doing the sum of some pairwise conflict measure over all pairs of candidates that include this candidate. This pairwise conflict measure also shares similarities with the "partitioning ratio" measure that we introduce in Section 7.2. Note however that taking the two most divisive candidates will not necessarily give a pair of candidates that induces a lot of conflict, as these candidates might be individually divisive, but not necessarily conflicting with each other (consider for example the profile P_1 at the beginning of this subsection).

Outline of the Chapter

The first contribution of this chapter is the various measures of conflict that we introduce, and the trade-off that naturally arises from these notions (Section 7.2). We then define in Section 7.3 several rules that aim at selecting the most conflicting candidates, and we analyze how they relate to our conflict measures. We proceed in Section 7.4 to axiomatically evaluate these rules. In particular, we establish new foundational axioms and show that they are totally incompatible with classical multi-winner voting rules. Defining more demanding axioms, we arrive at an impossibility result between a monotonicity and an efficiency axiom. Finally, we present the experimental evaluation of the rules and of the conflict measures in Section 7.5, using both synthetic and real-life preferences. These experiments enable us to validate our theoretical insights and to show how the rules would behave in some real-world applications.

7.2 Conflict Measures

In this chapter, we consider profiles of rankings $P = (\succ_1, \dots, \succ_n)$. By $\overleftarrow{P} = (\overleftarrow{\succ}_1, \dots, \overleftarrow{\succ}_n)$, we denote profile P with all ballots being reversed. In particular, for all $i \in V$, and all $a, b \in C$, $a \succ_i b$ if and only if $b \overleftarrow{\succ}_i a$.

Let $d_i(a, b)$ be the rank difference between two candidates a and b in a vote $\succ_i$, that is $d_i(a, b) = \sigma_i(a) - \sigma_i(b)$ (where $\sigma_i(c)$ is the rank of candidate $c \in C$). Note that this value can be negative if a is ranked below b. For instance, in the vote $a \succ_i b \succ_i c \succ_i d \succ_i e$, we have $\sigma_i(b) = 2$, $\sigma_i(e) = 5$, $d_i(b, e) = 5 - 2 = 3$ and $d_i(e, b) = -d_i(b, e) = -3$. We also denote by σ_i^{-1} the inverse of the ranking

σ_i, that is $\sigma_i^{-1}(j) = c$ if and only if c is the candidate ranked j-th in $\succ_i$. Furthermore, for all pairs of candidates $a, b \in C$, we recall that $V^{a\succ b} = \{i \in V \mid a \succ_i b\}$ is the set of voters preferring a to b in P. We have $V^{a\succ b} \cup V^{b\succ a} = V$ and $V^{a\succ b} \cap V^{b\succ a} = \emptyset$. Finally, we say that a pair of candidates $\{a, b\}$ is *conflicting* if their ordering is not unanimous, i.e. $V^{a\succ b} \neq \emptyset$ and $V^{b\succ a} \neq \emptyset$. In other words, a pair of candidates is conflicting if neither of them Pareto-dominates the other one in P.

We consider in this chapter the multi-winner voting model that we introduced in Section 2.3.2. As a reminder, an (irresolute) committee voting rule f is a function that takes as input a profile P and a committee size $k \geqslant 2$ and outputs a non-empty set $f(P, k)$ of committees, such that for each committee $W \in f(P, k)$, we have $|W| = k$ and $W \subseteq C$. In this chapter, we focus on the case of two candidates (i.e., $k = 2$), and thus write $f(P)$ instead of $f(P, 2)$ for simplicity.

Before defining any rule, we need to understand better what it means for a pair of candidates to induce conflict among the voters (beyond the non-Pareto domination condition). To this end, we propose in this section several measures of conflict, which are not always compatible.

Intuitively, a pair of candidates is perfectly polarizing if there are two groups of equal sizes that have conflicting preferences on this pair (i.e., they disagree on which candidate is better), and all voters have very strong opinions on the candidates. In other words, a pair $\{a, b\}$ is maximally conflicting if half of the voters rank a first and b last and the other half rank b first and a last. We formalize these two features to describe the conflict between two candidates in the notions of social *partitioning ratio* α and candidates' *discrepancy* β. However, these two measures are not sufficient to capture all the aspects of the conflict between two candidates, and we can have the same values of α and β for two pairs of candidates that are in very different conflicting situations. Thus, we introduce two other measures to complete the picture, called the *discrepancy balance* γ and the *group discrepancy imbalance* ϕ.

Partitioning Ratio

The social *partitioning ratio* α represents the "global" aspect of the conflict between two candidates. It is maximal when the voters are perfectly divided into two groups of equal sizes, one preferring a to b and the other b to a. Formally, we define the partitioning ratio as follows.

Partitioning Ratio α
The partitioning ratio of a pair of candidates $a, b \in C$ is:

$$\alpha(a, b) = \frac{2}{n} \min(|V^{a\succ b}|, |V^{b\succ a}|) \in [0, 1].$$

If all voters prefer a over b, then $|V^{b\succ a}| = 0$, and the partitioning ratio $\alpha(a, b) = 0$ is minimal. Thus, a pair of candidates $a, b \in C$ is conflicting if and only if $\alpha(a, b) > 0$. If voters are perfectly divided between a and b, then $|V^{a\succ b}| = |V^{b\succ a}| = n/2$ (assuming n is even) and the partitioning ratio $\alpha(a, b) = 1$ is maximal. We also note that the partitioning ratio α satisfies a form of *independence of irrelevant alternatives* (defined in Section 2.4.4): the value of $\alpha(a, b)$ is independent of other candidates $c \neq a, b$. A drawback of satisfying this axiom is that the measure completely ignores the intensity of the preferences of the voters, that can be inferred from the positions of the candidates in the rankings. The next measure aims at capturing this intensity.

This measure of conflict is rather natural. In particular, we can find very similar ideas used in the definitions of the measures of polarization that exist in the literature and that we mentioned in the introduction. For instance, Can et al. (2015) propose a way to measure the polarization of a preference profile as a whole, that is equal to the average partitioning ratio $\alpha(a, b)$ over all pairs of candidates $a, b \in C$. Similarly, the partitioning ratio appears in the formula used to compute the divisiveness of a single candidate in the work of Colley et al. (2023b).

Discrepancy

The "local" aspect of conflict is that the voters might individually have strong opinions on the candidates. We measure this with the candidates' *discrepancy* β:

Discrepancy β
The discrepancy of a pair of candidates $a, b \in C$ is:

$$\beta(a,b) = \frac{1}{n \cdot (m-1)} \sum_{i \in V} |d_i(a,b)| \in [0,1].$$

The discrepancy $\beta(a,b)$ is maximal when all voters rank one of a and b first and the other one last. In that case $|d_i(a,b)| = m-1$ for all $i \in V$, and $\beta(a,b) = 1$. Note that it is in particular maximal when all voters rank a first and b last, though in this case, a and b are not really "conflicting" (a Pareto-dominates b). On the other hand, it is minimal if a and b are *clones* (defined in Section 2.4.4). In that case, we have $|d_i(a,b)| = 1$ for all $i \in V$, and $\beta(a,b) = 1/(m-1)$. Note that the discrepancy is always strictly greater than 0.

Moreover, for a given profile P, the *maximal* discrepancy between two candidates is always greater than 1/3. To see that, one just needs to observe that the average discrepancy over all pairs of candidates is always 1/3. Indeed, we have:

$$\begin{aligned}\sum_{a,b \in C} |d_i(a,b)| &= \sum_{k=1}^{m-1} \sum_{j=1}^{k} |d_i(\sigma_i^{-1}(k+1), \sigma_i^{-1}(k+1-j))| = \sum_{k=1}^{m-1} \sum_{j=1}^{k} j = \sum_{k=1}^{m-1} \frac{k(k+1)}{2} \\ &= \frac{(m-1)m(2m-1)}{12} + \frac{m(m-1)}{4} = \frac{(m-1)m(m+1)}{6}\end{aligned}$$

Moreover, there are $m(m-1)/2$ pairs of candidates, giving the following average discrepancy:

$$\begin{aligned}\frac{\sum_{a,b \in C} \beta(a,b)}{m(m-1)/2} &= \frac{1}{n \cdot (m-1)} \frac{\sum_{a,b \in C} \sum_{i \in V} |d_i(a,b)|}{m(m-1)/2} \\ &= \frac{1}{n \cdot (m-1)} \frac{\sum_{i \in V} \sum_{a,b \in C} |d_i(a,b)|}{m(m-1)/2} \\ &= \frac{1}{n \cdot (m-1)} \frac{n \cdot m(m-1)(m+1)/6}{m(m-1)/2} = \frac{1}{(m-1)} \frac{m+1}{3} > 1/3\end{aligned}$$

By pigeonhole principle, there exists at least one pair of candidates which has a discrepancy greater than the average discrepancy, i.e. $\max_{a,b \in C} \beta(a,b) > 1/3$.

Discrepancy Balance

Partitioning ratio and discrepancy already highlight a trade-off between giving more importance to voters having strong opinions or to be evenly split. Yet, there is another phenomenon involved for understanding the conflict induced by candidates. Consider the following profile of two voters:

$$a \succ x \succ c \succ d \succ y \succ b$$
$$c \succ y \succ b \succ a \succ x \succ d$$

In this profile, both (a, b) and (x, y) have discrepancy $\beta = 3/5$ and partitioning ratio $\alpha = 1$. However, the discrepancy of the pair (a, b) is not equally distributed between the two voters. Indeed, the supporter of a strongly prefers a over b, while the supporter of b is close to indifferent between the two candidates. On the other hand, the discrepancy of the pair (x, y) is equally distributed between the two voters. This motivates the notion of discrepancy balance γ:

Discrepancy Balance γ
The discrepancy balance of a pair of candidates $a, b \in C$ is:

$$\gamma(a, b) = \min(\mu_{a \succ b}/\mu_{b \succ a}, \mu_{b \succ a}/\mu_{a \succ b}) \in [0, 1],$$

where $\mu_{a \succ b}$ is the average discrepancy between a and b among supporters of a:

$$\mu_{a \succ b} = \frac{\sum_{i \in V^{a \succ b}} |d_i(a, b)|}{|V^{a \succ b}|}.$$

If $V^{a \succ b} = \emptyset$ or $V^{b \succ a} = \emptyset$, we define $\gamma(a, b) = 0$. The intuition behind this measure is that we want the average discrepancy of each group of supporters to be as similar as possible, in order for the conflict to be balanced. This measure is maximal if the average discrepancy is the same in both groups of supporters, regardless of the size of each group. In particular, it can be equal to 1 even if $n - 1$ voters prefer a to b and only one voter prefers b to a. Measures α, β and γ are not independent of each other. For instance, $\beta(a, b) = 1$ and $\alpha(a, b) > 0$ implies $\gamma(a, b) = 1$ (but $\gamma(a, b) = 1$ does not imply anything for $\beta(a, b)$).

Note that α measures if the electorate is evenly divided between two candidates, and γ measures if the discrepancy is well balanced between the two groups of supporters. By combining these two ideas, we obtain our last measure, called *group discrepancy imbalance* ϕ.

Group Discrepancy Imbalance ϕ
The group discrepancy imbalance of a pair of candidates $a, b \in C$ is:

$$\phi(a, b) = \frac{|\sum_{i \in V} d_i(a, b)|}{\sum_{i \in V} |d_i(a, b)|} \in [0, 1].$$

Note that contrary to the other measures, there is more conflict between a and b when ϕ is minimal and not maximal. In particular, if $\phi(a, b) = 0$, then the sum of the individual discrepancy between a and b is the same inside both groups of supporters, i.e., $\sum_{i \in V^{a \succ b}} d_i(a, b) = \sum_{i \in V^{b \succ a}} d_i(b, a)$. Conversely, it is equal to $\phi(a, b) = 1$ when the groups are totally imbalanced (in that case the partitioning ratio is $\alpha(a, b) = 0$). Contrary to γ, the group discrepancy imbalance ϕ is sensitive to the sizes of the groups. In particular, if the rank differences between a and b are equal for every voter (i.e., $|d_i(a, b)| = \beta \cdot (m - 1)$ for all $i \in V$), then $\gamma(a, b) = 1$, while $\phi(a, b) = 1 - \alpha(a, b)$. Thus, we expect ϕ and α to be very (negatively) correlated in practice.

To get a better understanding of these measures, consider the following example:

Example 7.1
Consider the following profile P of four voters:

$x \succ_1 a \succ_1 b \succ_1 y$ $\quad$ $a \succ_3 y \succ_3 x \succ_3 b$

$x \succ_2 a \succ_2 b \succ_2 y$ $\quad$ $b \succ_4 y \succ_4 x \succ_4 a$

In this profile, the pair $\{x, y\}$ has the highest partitioning ratio $\alpha(x, y) = 1$ as half of the voters prefer x to y and the other half prefer y to x. On the other hand, for every other pair, one candidate is supported by three voters, and the other by only one, giving a partitioning ratio of 1/2. It is easy to see that the values of discrepancy for the pairs $\{a, b\}$ and $\{x, y\}$ are the same since in both cases two voters have extreme opinions on the candidates, and two voters have them next to each other in their rankings. This gives a discrepancy $\beta(a, b) = \beta(x, y) = 2/3$. Finally, we can see that the conflict is more balanced for the pair $\{a, b\}$ than for $\{x, y\}$, as $\gamma(a, b) = 5/9$ and $\phi(a, b) = 1/4$, while $\gamma(x, y) = 1/3$ and $\phi(x, y) = 1/2$ (recall that the conflict is more balanced when ϕ is lower). We can compute the values of all four measures for all the pairs of candidates in this profile, and we obtain the following table.

	$\{a, b\}$	$\{x, y\}$	$\{a, x\}$	$\{b, y\}$	$\{a, y\}$	$\{b, x\}$
α	1/2	**1**	1/2	1/2	1/2	1/2
β	**2/3**	**2/3**	5/12	5/12	7/12	7/12
γ	5/9	1/3	1/2	1/2	**5/6**	**5/6**
ϕ	1/4	1/2	**1/5**	**1/5**	3/7	3/7

Characteristic Profiles

We can also look at the values these measures take in some specific profiles. In particular, the characteristic profiles studied by Faliszewski et al. (2023a) are an interesting start.

In the *identity* (ID) profile, all voters have the same ranking $P = (\succ, \cdots, \succ)$. In this profile the partitioning ratio $\alpha(a, b)$ is equal to 0 for all pairs of candidates. The discrepancy $\beta(a, b)$ is equal to $|\sigma(a) - \sigma(b)|/(m - 1)$ where $\sigma(c)$ is the rank of c in $\succ$. Thus, it is equal to 1 for the pair $\{a, b\}$ where a is the first ranked candidate and b the last ranked candidate in $\succ$. Finally, the discrepancy balance $\gamma(a, b)$ is equal to 0 and the group discrepancy imbalance $\phi(a, b)$ is equal to 1 for all pairs of candidates. This makes sense, as the identity profile is the least conflicting profile. In particular, there are no *conflicting* pairs of candidates, in the sense that for each pair, one candidate always Pareto-dominates the other.

In the *uniform* (UN) profile, every possible ranking appears exactly once. In this profile, the partitioning ratio is maximal $\alpha(a, b) = 1$ for each pair of candidates $a, b \in C$, as there are as many rankings with $a \succ b$ than there are with $b \succ a$. The discrepancy $\beta(a, b)$ is equal to the average one for all pairs of candidates, i.e. $\beta(a, b) = (m + 1)/(3(m - 1)) \approx 1/3$. Finally, the discrepancy balance is maximal (i.e., $\gamma(a, b) = 1$) and the group discrepancy imbalance is minimal (i.e., $\phi(a, b) = 0$) for all pairs of candidates, as for every ranking with $a \succ b$, the profile also contains the ranking identical except that the positions of a and b are swapped. Thus, in the identity profile ID, the "local" conflict, represented by the discrepancy, was maximal, while the "global" conflict, represented by the partitioning ratio, was minimal. In the uniform profile UN, the opposite is true.

Let us now consider a third characteristic profile, called *antagonism* (AN), in which half of the voters submit a ranking $\succ$, and the other half submit the opposite ranking $\overleftarrow{\succ}$. In this profile, we have $\alpha(a, b) = 1$ for all pairs of candidates, and the discrepancy is equal to $\beta(a, b) = 1$ for the pair of candidates $a, b \in C$ where a is the first ranked candidate in $\succ$ and b is the last one. Finally, the discrepancy balance is also maximal $\gamma(a, b) = 1$ (and the imbalance minimal $\phi(a, b) = 0$) for all pairs of candidates by symmetry of the profile. Thus, as expected, the antagonism profile AN is the one for which all of our conflict measures are maximal. Table 7.1 summarizes the values of the different measures for these three characteristic profiles.

	Identity (ID)	Uniform (UN)	Antagonism (AN)
α	0	**1**	**1**
$\beta_{\max}$	**1**	$\frac{m+1}{3(m-1)}$	**1**
γ	0	**1**	**1**
ϕ	1	**0**	**0**

Table 7.1: Values of the different measures for the three characteristic profiles.

7.3 Conflict Rules

Let us now introduce some rules that aim to select the most conflicting pairs of candidates. One simple way to find conflicting pairs is to select the ones that maximize some function of the different measures α, β, γ and ϕ. More formally, the rule f would be endowed with a scoring function $s(\alpha, \beta, \gamma, \phi)$ from $[0,1]^4$ to $\mathbb{R}_{\geqslant 0}$, and would select the pairs of candidates maximizing the value $s(\alpha(a,b), \beta(a,b), \gamma(a,b), \phi(a,b))$. However, it is not clear what scoring function should be used, and how to balance the different measures, as this approach lacks an intuitive interpretation of the rule. Still, we define one simple family of scoring functions solely based on the partitioning ratio α and the discrepancy β, that we name *p-MaxPolarization* rules (*p-MaxPolar* in short).

p-MaxPolarization
The p-MaxPolarization rule selects the pairs of candidates that maximize the value of the scoring function $s_p(\alpha, \beta) = \alpha \cdot \beta^p$, for $p > 0$.

Clearly, the higher the value of $\alpha(a,b)$ and $\beta(a,b)$, the more conflicting the pair of candidates $\{a, b\}$ is, and the more likely it is to be selected by these rules. Moreover, the higher the value of p, the more weight is put on the discrepancy β in comparison to the partitioning ratio α.

Pairwise Conflict

In Section 7.2, we introduced several measures of conflict that each capture *one* aspect of conflict induced by a pair of candidate. Another way to see if a pair of candidates introduces a lot of conflict is to start from pairs of voters and see how strongly they disagree on their preferences between the two candidates. Intuitively, a pair of candidates $\{a, b\}$ is inducing a *conflict* between two voters if they disagree on the ordering between a and b. Moreover, the more distant a and b are in the voters' rankings, the greater the conflict. Starting with this, we define the *sum pairwise conflict*, noted conf^+, and the *Nash pairwise conflict*, noted $\text{conf}^\times$.

Pairwise conflict
For $\circ \in \{+, \times\}$, let the conflict induced by a pair of candidates $\{a, b\}$ between two voters i, j be:

$$\text{conf}^\circ_{i,j}(a,b) = \begin{cases} 0 & \text{if } d_i(a,b) \cdot d_j(a,b) > 0 \\ |d_i(a,b)| \circ |d_j(a,b)| & \text{otherwise} \end{cases}$$

For instance, the Nash pairwise conflict induced by a and b between rankings $a \succ_i b \succ_i c$ and $b \succ_j c \succ_j a$ is $\text{conf}^\times_{i,j}(a,b) = 1 \times 2 = 2$, and the sum pairwise conflict is $\text{conf}^+_{i,j}(a,b) = 1 + 2 = 3$. One could also think of using the operators $\circ = \min$ and $\circ = \max$. However, these operators are less appealing: if we only consider the maximum (or minimum) of the two values, we completely ignore the other value. The following example illustrates this issue: if we have $d_i(a,b) = 1$ and

$d_j(a,b) = 50$, and $d_i(x,y) = d_j(x,y) = 49$, then the max-pairwise conflict will say that the conflict induced by a and b between the voters i and j is higher than the conflict induced by x and y. A similar example can be made for the min-pairwise conflict.

By extension, one way to define the conflict induced by a and b in a profile is to sum the pairwise conflict over all pairs of voters. We can first define the *MaxSumConflict* rule (*MaxSum* for short), based on the sum pairwise conflict conf^+.

MaxSumConflict
The MaxSumConflict rule selects the pairs of candidates that maximize

$$\sum_{i,j\in V} \text{conf}^+_{i,j}(a,b) = |V^{b\succ a}| \sum_{i\in V^{a\succ b}} d_i(a,b) + |V^{a\succ b}| \sum_{i\in V^{b\succ a}} d_i(b,a).$$

To derive the second formula, simply observe that we are doing a sum of $|d_i(a,b)|$, and we just need to count how many times each $|d_i(a,b)|$ appears in this sum. Since every voter $i \in V^{a\succ b}$ is conflicting with every voter $j \in V^{b\succ a}$, we can deduce that $|d_i(a,b)| = d_i(a,b)$ appears $|V^{b\succ a}|$ times in the sum. Similarly, for all $j \in V^{b\succ a}$, $|d_j(a,b)| = d_j(b,a)$ appears $|V^{a\succ b}|$ times in the sum.

We similarly define the *MaxNashConflict* rule (*MaxNash* for short).

MaxNashConflict
The MaxNashConflict rule selects the pairs of candidates that maximize

$$\sum_{i,j\in V} \text{conf}^\times_{i,j}(a,b) = \left(\sum_{i\in V^{a\succ b}} d_i(a,b)\right) \times \left(\sum_{i\in V^{b\succ a}} d_i(b,a)\right).$$

The second formula is obtained through the same reasoning as for MaxSum. Interestingly, this rule can also be expressed using some of the measures introduced in Section 7.2:

Proposition 7.1
MaxNash returns the pairs of candidates which maximize $\beta^2(a,b)(1-\phi^2(a,b))$.

Proof. For two candidates $a, b \in C$, let us denote by $s(a,b) = \left(\sum_{i\in V^{a\succ b}} d_i(a,b)\right)\times\left(\sum_{i\in V^{b\succ a}} d_i(b,a)\right)$ the MaxNashConflict score of the pair $\{a,b\}$. Let us now define the sum of discrepancies in each group of supporters $D_{a\succ b} = \sum_{i\in V^{a\succ b}} d_i(a,b)$ and $D_{b\succ a} = \sum_{i\in V^{b\succ a}} d_i(b,a)$. Then define $D^{\mathsf{min}} = \min(D_{a\succ b}, D_{b\succ a})$ and $D^{\mathsf{max}} = \max(D_{a\succ b}, D_{b\succ a})$. Now observe that:

$$\begin{aligned}
\beta(a,b) &= \frac{1}{n\cdot(m-1)} \sum_{i\in V} |d_i(a,b)| \\
&= \frac{1}{n\cdot(m-1)} \left(D_{a\succ b} + D_{b\succ a}\right) \\
&= \frac{1}{n\cdot(m-1)} \left(D^{\mathsf{max}} + D^{\mathsf{min}}\right), \text{ and} \\
\beta(a,b)\phi(a,b) &= \frac{1}{n\cdot(m-1)} \left|\sum_{i\in V} d_i(a,b)\right| \\
&= \frac{1}{n\cdot(m-1)} |D_{a\succ b} - D_{b\succ a}| \\
&= \frac{1}{n\cdot(m-1)} \left(D^{\mathsf{max}} - D^{\mathsf{min}}\right)
\end{aligned}$$

From this, we can deduce:

$$D^{\max} = \frac{n \cdot (m-1)}{2} \left(\beta(a,b)(1+\phi(a,b))\right) \quad \text{and} \quad D^{\min} = \frac{n \cdot (m-1)}{2} \left(\beta(a,b)(1-\phi(a,b))\right)$$

Finally, observe that the MaxNashConflict score is equal to

$$D_{a\succ b} \times D_{b\succ a} = D^{\max} \times D^{\min} = \frac{n^2(m-1)^2}{4}\beta^2(a,b)(1-\phi^2(a,b))$$

Since $n^2(m-1)^2/4$ is a constant, we can conclude that MaxNash selects the pairs of candidates which maximize $\beta^2(a,b)(1-\phi^2(a,b))$. □

Number of Swaps

Finally, another approach to define a rule is to select, in a preference profile P, the pairs of candidates $\{a,b\}$ that maximize the distance between P and a profile in which a and b would be non-conflicting (i.e., all voters have the same ordering between a and b). This approach is conceptually similar to the distance-rationalizable rules from classical social choice theory, which we discussed in Section 2.3.3. As a reminder, these rules aim at selecting the candidate for which the distance between the current preference profile and a profile in which there is some consensus that this candidate is the winner (e.g., voters are unanimous) is minimal (Elkind et al., 2015).

In the case of conflict, we define a rule that selects the pairs of candidates $\{a,b\}$ that *maximize* the minimum number of swaps of adjacent candidates to make $\{a,b\}$ a non-conflicting pair in the profile P. The distance we use is the *swap-distance* (or Kendall-tau distance, see Section 2.3.1), as it is the most natural distance for this model. Intuitively, the more swaps we need to perform before all voters agree on the ordering between a and b, the more conflict a and b are inducing. We call the corresponding rule *MaxSwap*.

MaxSwap
The MaxSwap rule selects the pairs of candidates that maximize

$$\min\left(\sum_{i\in V^{a\succ b}} d_i(a,b), \sum_{i\in V^{b\succ a}} d_i(b,a)\right).$$

To see how we obtain this formula, observe that the minimal number of swaps we need to perform in order to have $b \succ_i a$ (resp. $a \succ_i b$) for all voters is $\sum_{i\in V^{a\succ b}} d_i(a,b)$ (resp. $\sum_{i\in V^{b\succ a}} d_i(b,a)$). Interestingly, we can also express MaxSwap using some of the measures introduced in Section 7.2.

Proposition 7.2
MaxSwap returns the pairs of candidates which maximize $\beta(a,b)(1-\phi(a,b))$.

Proof. We reuse the notation of $D_{a\succ b}$ and $D_{b\succ a}$ from the proof of Proposition 7.1, as well as $D^{\min}$ and $D^{\max}$. Then, we simply note that the MaxSwap score is equal to

$$\min\left(D_{a\succ b}, D_{b\succ a}\right) = D^{\min} = \frac{n\cdot(m-1)}{2}\beta(a,b)(1-\phi(a,b))$$

Thus, the MaxSwap rule selects the pairs of candidates $\{a,b\}$ which maximize $\beta(a,b)(1-\phi(a,b))$. □

Example 7.2
Consider the profile P from Example 7.1:

$$x \succ_1 a \succ_1 b \succ_1 y \qquad a \succ_3 y \succ_3 x \succ_3 b$$
$$x \succ_2 a \succ_2 b \succ_2 y \qquad b \succ_4 y \succ_4 x \succ_4 a$$

In Example 7.1, we computed the values of partitioning ratio, discrepancy and global discrepancy imbalance for each pair of candidates:

	$\{a, b\}$	$\{x, y\}$	$\{a, x\}$	$\{b, y\}$	$\{a, y\}$	$\{b, x\}$
α	1/2	**1**	1/2	1/2	1/2	1/2
β	**2/3**	**2/3**	5/12	5/12	7/12	7/12
ϕ	1/4	1/2	**1/5**	**1/5**	3/7	3/7

From this, we deduce that any p-MaxPolar rule chooses $\{x, y\}$, since the pair has both the highest value of α and of β. Now, let us define $D_{c \succ d} = \sum_{i \in V^{c \succ d}} |d_i(c, d)|$ for $c, d \in C$. We have 3 voters preferring a to b with $D_{a \succ b} = 1 + 1 + 3 = 5$ and 1 voter preferring b to a with $D_{b \succ a} = 3$. Thus, the scores of $\{a, b\}$ are $1 \cdot 5 + 3 \cdot 3 = 14$ for MaxSum, $5 \cdot 3 = 15$ for MaxNash and $\min(5, 3) = 3$ for MaxSwapConflict. On the other hand, we have 2 voters preferring x over y with $D_{x \succ y} = 3 + 3 = 6$ and 2 voters preferring y over x with $D_{y \succ x} = 1 + 1 = 2$. Thus, the scores of $\{x, y\}$ are $2 \cdot 2 + 2 \cdot 6 = 16$ for MaxSum, $6 \cdot 2 = 12$ for MaxNash and $\min(2, 6) = 2$ for MaxSwapConflict. We can similarly compute the scores of all pairs of candidates:

	$\{a, b\}$	$\{x, y\}$	$\{a, x\}$	$\{b, y\}$	$\{a, y\}$	$\{b, x\}$
MaxSum	14	**16**	9	9	11	11
MaxNash	**15**	12	4	4	10	10
MaxSwap	**3**	2	2	2	2	2

Thus, MaxSum chooses the pair $\{x, y\}$, while MaxNash and MaxSwap choose the pair $\{a, b\}$. We can also check the latter using Propositions 7.1 and 7.2.

Note that if we consider the case of approval preferences, which we briefly discussed in the introduction, we can easily adapt our notion of pairwise conflict. In this case, the pairwise conflict induced by a pair of candidates $\{a, b\}$ is equal to 1 if one voter approves a but not b, and the other approves b and not a, and is 0 otherwise. This implies that MaxSum and MaxNash would correspond to maximizing the product of the size of the groups (as proposed in the introduction), and MaxSwapConflict to maximizing the size of the smallest of the two groups.

Partitioning Ratio versus Discrepancy

We will now investigate how the rules we introduced behave regarding the trade-off between the partitioning ratio α and the discrepancy β. To do so, consider a preference profile P with m candidates and n voters, and in which the pairs of candidates $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are such that the first one has maximal discrepancy $\beta(a_1, b_1) = 1$, and the second one has maximal partitioning ratio $\alpha(a_2, b_2) = 1$. In other words, every voter has a strong preference between a_1 and b_1 (but voters are not necessarily equally divided between them), and voters are perfectly divided between a_2 and b_2 (but voters might have weak preferences on them). This is for instance the case in the following profile:

$$a_1 \succ_1 a_2 \succ_1 \cdots \succ_1 b_2 \succ_1 b_1 \qquad a_1 \succ_3 a_2 \succ_3 \cdots \succ_3 b_2 \succ_3 b_1$$
$$a_1 \succ_2 b_2 \succ_2 \cdots \succ_2 a_2 \succ_2 b_1 \qquad b_1 \succ_4 b_2 \succ_4 \cdots \succ_4 a_2 \succ_4 a_1$$

For clarity, we additionally assume that the discrepancy balance is maximal $\gamma(a_1, b_1) = \gamma(a_2, b_2) = 1$. This implies that the average discrepancy is the same in each group of supporters, i.e., $\mu_{a_1 \succ b_1} = \mu_{b_1 \succ a_1} = \beta(a_1, b_1)$ and $\mu_{a_2 \succ b_2} = \mu_{b_2 \succ a_2} = \beta(a_2, b_2)$. Understanding the interplay between discrepancy and partitioning ratio boils down to the question: Which of the two pairs is more conflicting in this scenario for different values of $\alpha_1 = \alpha(a_1, b_1)$ and $\beta_2 = \beta(a_2, b_2)$? A rule that more often selects $\{a_1, b_1\}$ would favor the discrepancy β, while a rule that more often selects $\{a_2, b_2\}$ would favor the partitioning ratio α.

We can first prove that under our hypothesis on the discrepancy balance, we have for all pairs of candidates $\phi(a, b) = 1 - \alpha(a, b)$. Let $n_{\mathsf{min}} = \min(|V^{a \succ b}|, |V^{b \succ a}|)$ and $n_{\mathsf{max}} = n - n_{\mathsf{min}}$. We have:

$$\begin{aligned}
\phi(a, b) &= \frac{|\sum_{i \in V} d_i(a, b)|}{\sum_{i \in V} |d_i(a, b)|} = \frac{|\sum_{i \in V^{a \succ b}} d_i(a, b) + \sum_{i \in V^{b \succ a}} d_i(a, b)|}{\sum_{i \in V^{a \succ b}} d_i(a, b) + \sum_{i \in V^{b \succ a}} d_i(b, a)} \\
&= \frac{(n_{\mathsf{max}} - n_{\mathsf{min}})\beta(a, b)}{n_{\mathsf{min}}\beta(a, b) + n_{\mathsf{max}}\beta(a, b)} = \frac{n_{\mathsf{max}} - n_{\mathsf{min}}}{n} = \frac{n - 2n_{\mathsf{min}}}{n} \\
&= 1 - \frac{2}{n} n_{\mathsf{min}} = 1 - \alpha(a, b)
\end{aligned}$$

The equality between the first and the second lines holds because the average discrepancy is the same among both sets of voters $V^{a \succ b}$ and $V^{b \succ a}$ and is equal to $\beta(a, b)$. Thus, the sum of the rank differences $|d_i(a, b)|$ is equal to $n_{\mathsf{min}}\beta(a, b)$ for the smallest set of voters, and to $n_{\mathsf{max}}\beta(a, b)$ for the largest set. We can easily derive from this and Propositions 7.1 and 7.2 that MaxNash selects the pair of candidates that maximizes $s(a, b) = \beta(a, b)^2\alpha(a, b)(2 - \alpha(a, b))$ and MaxSwapConflict a pair that maximizes $s(a, b) = \beta(a, b)\alpha(a, b)$. Thus, in this special case with $\gamma(a_1, b_1) = \gamma(a_2, b_2) = 1$, MaxSwapConflict is equivalent to 1-MaxPolar.

We can also show that under our hypothesis, MaxSum selects the pair that maximizes $s(a, b) = \beta(a, b)\alpha(a, b)(2 - \alpha(a, b))$:

$$\begin{aligned}
&|V^{b \succ a}| \sum_{i \in V^{a \succ b}} d_i(a, b) + |V^{a \succ b}| \sum_{i \in V^{b \succ a}} d_i(b, a) \\
=&|V^{b \succ a}||V^{a \succ b}|\beta(a, b) + |V^{a \succ b}||V^{b \succ a}|\beta(a, b) \\
=&2n_{\mathsf{min}}(n - n_{\mathsf{min}})\beta(a, b) \\
=&\frac{n^2}{2} \cdot \frac{2}{n} n_{\mathsf{min}} \cdot (2 - \frac{2}{n} n_{\mathsf{min}}) \cdot \beta(a, b) \\
=&\frac{n^2}{2} \alpha(a, b)(2 - \alpha(a, b))\beta(a, b)
\end{aligned}$$

Now that we have simple formulas that only depend on $\alpha(a, b)$ and $\beta(a, b)$ for all rules, we can see under which conditions each rule selects the pair $\{a_1, b_1\}$ or $\{a_2, b_2\}$ as the most conflicting. Since we assumed that $\beta(a_1, b_1) = 1$ and $\alpha(a_2, b_2) = 1$, we have that MaxSum selects the pair $\{a_1, b_1\}$ as more conflicting than $\{a_2, b_2\}$ if and only if $\alpha_1(2 - \alpha_1) > \beta_2$. On the other hand, MaxNash selects the pair $\{a_1, b_1\}$ if and only if $\alpha_1(2 - \alpha_1) > \beta_2^2$. Since $\beta_2 \geqslant \beta_2^2$, we can conclude that MaxSum gives less importance to the discrepancy $\beta(a, b)$ than MaxNash (because it more easily selects the pair with high partitioning ratio $\{a_2, b_2\}$). Similarly, MaxSwap selects the pair

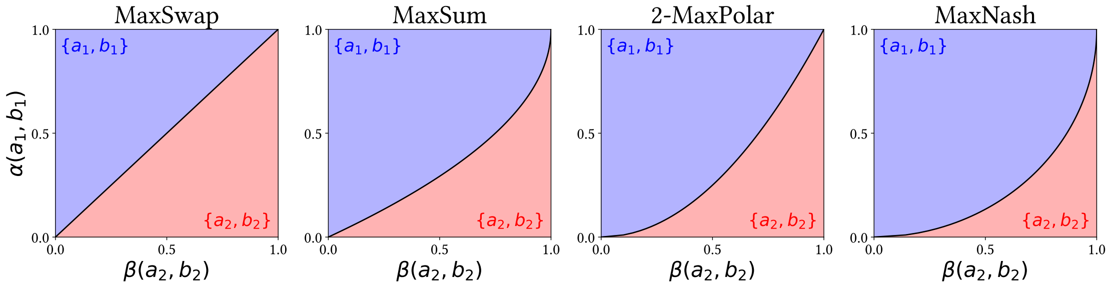


Figure 7.1: Areas in which the different rules find $\{a_1, b_1\}$ (in blue) or $\{a_2, b_2\}$ (in red) more conflicting in the (α_1, β_2) plane.

$\{a_1, b_1\}$ if and only if $\alpha_1 > \beta_2$, and since $2 - \alpha_1 > 1$, we can conclude that MaxSwap gives less importance to the discrepancy $\beta(a, b)$ than MaxSum. These results are also clear in Figure 7.1, which shows for which values of β_2 and α_1 the different rules select $\{a_1, b_1\}$ or $\{a_2, b_2\}$ as the most conflicting pair between the two: the blue area corresponds to the values for which $\{a_1, b_1\}$ (which maximizes the discrepancy) is more conflicting, and the red area to the values for which $\{a_2, b_2\}$ (which maximizes the partitioning ratio) is more conflicting. Thus, the larger the blue area, the more important is the discrepancy β for the rule, and the larger the red area, the more important is the partitioning ratio α.

7.4 Axiomatic Analysis

Now that we have defined rules and seen how they are linked to the conflict measures we defined, we proceed with axiomatic properties that we would like our rules to satisfy. Almost all the axioms that we discuss in this section are new, and have not been studied in the literature before.

Fundamental Axioms

We start by defining some fundamental axioms expected from conflict rules. First, it is clear that all the rules we introduced satisfy neutrality and anonymity, which have the same definition as in classical social choice theory (see Section 2.4.1). Let us now define axioms that are specific to our model. The first one is an efficiency axiom which ensures that unless all voters cast the same ranking, only conflicting pairs can be selected.

Conflict Consistency
A rule f is *conflict consistent* if for any profile P different from the identity profile ID, all pairs $\{a, b\} \in f(P)$ are conflicting, i.e., $\alpha(a, b) > 0$.

Note that this axiom is already incompatible with the well-studied *unanimity* axiom of multi-winner voting (see Section 2.4.2), which states that if a committee appears on top of every ranking in the profile, it should be returned by the rule. For instance, in the profile $P = \{a \succ b \succ c \succ d, a \succ b \succ d \succ c\}$, unanimity says that the rule should return the pair $\{a, b\}$. However, the only conflicting pair in this profile is $\{c, d\}$, making it impossible to satisfy both. This highlights that our model stands in stark contrast to all established objectives of multi-winner voting rules, namely, individual excellence, proportionality, and diversity, in which rules are expected to satisfy the unanimity axiom. This is the main difference with the classical paradigm of social choice: conflict rules specifically *avoid* consensual candidates.

Our second fundamental axiom for conflict rules comes from the observation that we want *"love"* and *"hate"* to have symmetrical purposes. Thus, the most polarizing pair in a profile should remain the most polarizing one if we reverse all rankings.

Reverse Stability
A rule f is *reverse stable* if for any profile P, we have $f(P) = f(\overleftarrow{P})$.

Note that this axiom could be discussed, as it is not always clear that intensities of preferences are linear in the order of the ranking. In particular, voters might have stronger opinions between their first and second choices than between their second to last and last choices. This has been observed in surveys in which voters give both their rankings and ratings of the candidates (e.g., Bronfman et al. (2015)), and this has been formalized by Segal-Halevi et al. (2020).

Proposition 7.3
MaxNash, MaxSum, MaxSwap and p-MaxPolar rules are conflict consistent and reverse stable.

Proof. All these rules give a score $s(a, b) = 0$ to non-conflicting pairs, and a score $s(a, b) > 0$ to any conflicting pair. Thus, it is clear that they are all conflict consistent.

For reverse stability, observe that $|d_i(a, b)|$ has the same value in P and $\overleftarrow{P}$ for all voters $i \in V$. Thus, the pairwise conflict induced by any pair of candidates between two voters is the same in both profiles. This directly implies that MaxSum and MaxNash are reverse stable. By additionally observing that the sets $V^{a \succ b}$ and $V^{b \succ a}$ are swapped in $\overleftarrow{P}$, we can deduce that MaxSwap is reverse stable. Finally, these two observations also imply that $\alpha(a, b)$ and $\beta(a, b)$ are the same in both profiles, and thus p-MaxPolar is also reverse stable. □

Efficiency

We now turn to axioms that sound desirable but are unfortunately not always possible to satisfy. The next axiom is inspired by efficiency axioms from classical social choice (Section 2.4.2). In particular, it is inspired by Pareto-domination, as it relies on a notion of domination between pairs of candidates. Given profile P and two conflicting pairs of candidates $\{a, b\}$ and $\{x, y\}$, we say that $\{a, b\}$ is *matching-dominating* $\{x, y\}$ if there exists a bijective function $g : V \to V$ such that

(1) $g(V^{a \succ b}) = V^{x \succ y}$ and $g(V^{b \succ a}) = V^{y \succ x}$,

(2) for all voters $i \in V$, we have $|d_i(a, b)| \geqslant |d_{g(i)}(x, y)|$, and

(3) there exists a voter $i \in V$ such that $|d_i(a, b)| > |d_{g(i)}(x, y)|$.

Note that the first condition of matching-domination imposes that $|V^{a \succ b}| = |V^{x \succ y}|$ and $|V^{b \succ a}| = |V^{y \succ x}|$, meaning that the two pairs of candidates have the same partitioning ratio $\alpha(a, b) = \alpha(x, y)$.

Example 7.3
Consider the following profile P of four voters:

$a \succ_1 x \succ_1 y \succ_1 b$ $\qquad$ $b \succ_3 x \succ_3 a \succ_3 y$

$a \succ_2 y \succ_2 x \succ_2 b$ $\qquad$ $b \succ_4 y \succ_4 a \succ_4 x$

We will focus on the pairs of candidates $\{a, b\}$ and $\{x, y\}$. We have $V^{a \succ b} = \{1, 2\}$, $V^{b \succ a} = \{3, 4\}$, $V^{x \succ y} = \{1, 3\}$ and $V^{y \succ x} = \{2, 4\}$. Consider the matching $g = \{1 \to 1, 2 \to 3, 3 \to 2, 4 \to 4\}$. We verify that it meets all the conditions:

(1) Clearly, $g(V^{a \succ b}) = V^{x \succ y}$ and $g(V^{b \succ a}) = V^{y \succ x}$.

(2) It is easy to verify that for all $i \in V$, $|d_i(a, b)| \geqslant |d_{g(i)}(x, y)|$, since $\min_i |d_i(a, b)| = 2$ and $\max_i |d_i(x, y)| = 2$.

(3) We have $g(1) = 1$ and $|d_1(a, b)| = 3 > 2 = |d_1(x, y)|$.

Hence, the pair $\{a, b\}$ dominates the pair $\{x, y\}$.

Now, the axiom we introduce says that a dominated pair should not be selected.

Matching Domination

A rule f satisfies *matching domination* if for any profile P and any two conflicting pairs of candidates $\{a, b\}$ and $\{x, y\}$, if $\{a, b\}$ is matching-dominating $\{x, y\}$, then $\{x, y\} \notin f(P)$.

We can now show the following result.

Proposition 7.4

MaxSum, MaxNash and p-MaxPolar rules satisfy matching domination, but not MaxSwap.

Proof. Consider a profile P and two conflicting pairs of candidates $\{a, b\}$ and $\{x, y\}$ such that $\{a, b\}$ is matching-dominating $\{x, y\}$. Let g be the matching function that satisfies the conditions of matching-domination. We will show that $\{x, y\}$ is not selected by MaxSum, MaxNash and p-MaxPolar rules.

For MaxSum and MaxNash, observe that two voters i and j agree on the ordering between a and b if and only if $g(i)$ and $g(j)$ agree on the ordering between x and y, because $g(V^{a \succ b}) = V^{x \succ y}$ and $g(V^{b \succ a}) = V^{y \succ x}$. If they do not agree, then we have

$$\mathrm{conf}^{\circ}_{g(i),g(j)}(x, y) = |d_{g(i)}(x, y)| \circ |d_{g(j)(x,y)}| \leqslant |d_i(a, b)| \circ |d_j(a, b)| = \mathrm{conf}^{\circ}(i, j)(a, b)$$

for $\circ \in \{+, \times\}$ because of the second condition of matching-domination. Moreover, there exists $i \in V$ with $|d_{g(i)}(x, y)| < |d_i(a, b)|$, so if we take $j \in V$ that does not agree with i on the ordering between a and b, we have $\mathrm{conf}^{\circ}_{g(i),g(j)}(x, y) < \mathrm{conf}^{\circ}(i, j)(a, b)$. Because g is bijective and MaxSum and MaxNash sum the pairwise conflict over all pairs of voters, the score of $\{a, b\}$ will necessarily be higher than that of $\{x, y\}$, which will thus not be selected.

For p-MaxPolar, we clearly have $\beta(a, b) > \beta(x, y)$ because of the second and third conditions of matching-domination, and $\alpha(a, b) = \alpha(x, y)$ because of the first one. Thus, the score of $\{a, b\}$ will be higher than that of $\{x, y\}$, which will thus not be selected.

Let us now show that MaxSwap fails this axiom. Consider the following profile P:

$$10 : a \succ x \succ y \succ b \qquad 1 : y \succ b \succ x \succ a \qquad 1 : b \succ y \succ a \succ x$$

We can show that $\{x, y\}$ is matching-dominated by $\{a, b\}$. Indeed, let g be the identity function. We have $g(V^{a \succ b}) = V^{x \succ y}$ and $g(V^{b \succ a}) = V^{y \succ x}$. Moreover, we can easily check that $|d_i(a, b)| \geqslant |d_i(x, y)|$ for all $i \in V$, and that for the first 10 voters, the inequality is strict. Thus, the three conditions are satisfied and $\{x, y\}$ is dominated. However, we need 1 swap to make $\{a, x\}$ and

$\{b, y\}$ non conflicting, and 4 swaps for $\{a, y\}$, $\{b, x\}$, $\{x, y\}$ and $\{a, b\}$. Thus MaxSwap(P) = $\{\{a, y\}, \{b, x\}, \{x, y\}, \{a, b\}\}$ and MaxSwap does not satisfy matching domination. □

Note that in the example from this proof, MaxSwap returns both $\{x, y\}$ and the pair that dominates it, $\{a, b\}$. More generally, this rule satisfies a weaker version of the axiom which asks that if a dominated pair is included in the outcome, the dominating pair should also be included.

Monotonicity

The next axiom is inspired by the monotonicity notions from the literature of classical social choice (see Section 2.4.5). The idea is the following: if one pair of candidates is the most conflicting in a given profile, adding more conflict between the two candidates should not make another pair the most conflicting, and thus be selected instead of the original one.

More formally, we say that we increased the conflict between a and b in some ranking $\succ_i$ if we increased the absolute rank difference $|d_i(a, b)|$ by swapping either a or b with one of its neighbouring candidates in the ranking. For instance, we can increase the conflict between a and b in the ranking $a \succ_i c \succ_i b \succ_i d$ by swapping b and d: $a \succ_i c \succ_i d \succ_i b$. Similarly, we say that we increased the conflict between a and b in a profile P if we increased it in one ranking $\succ_i$ and all other rankings remained the same.

Conflict Monotonicity
A rule f is *conflict monotonic* if for each profile P and each selected pair $\{a, b\} \in f(P)$, it holds that if we increase the conflict between a and b in P to obtain profile P', then $\{a, b\} \in f(P')$.

Unfortunately, we can show that it is not possible to satisfy this axiom together with conflict consistency and matching domination.

Theorem 7.5
No rule can satisfy conflict consistency, matching domination and conflict monotonicity.

Proof. Let f be a rule satisfying these three axioms, and consider the following profile P:

$$a \succ_1 b \succ_1 c \succ_1 d \qquad\qquad b \succ_2 a \succ_2 d \succ_2 c$$

By conflict consistency, only $\{a, b\}$ and $\{c, d\}$ can be selected. Assume without loss of generality that $\{a, b\} \in f(P)$. Now, consider the profile P' obtained by increasing the conflict between a and b in the second ranking twice:

$$a \succ_1 b \succ_1 c \succ_1 d \qquad\qquad b \succ_2 d \succ_2 c \succ_2 a$$

By conflict monotonicity, $\{a, b\} \in f(P')$. However, with the matching $g = \{1 \to 2, 2 \to 1\}$, we can check that $\{a, d\}$ dominates the pair $\{a, b\}$ in P'. By matching domination, we have $\{a, b\} \notin f(P')$, which is a contradiction. A similar reasoning proves the result if $\{c, d\}$ is selected instead of $\{a, b\}$. □

A direct consequence of this result is that MaxNash, MaxSum and p-MaxPolar rules do not satisfy conflict monotonicity, since they already satisfy conflict consistency and matching domination. Moreover, the same profiles that are used in the proof above can be used to show that MaxSwap does not satisfy conflict monotonicity either, as it selects $\{a, b\}$ in P but $\{a, d\}$ in P'.

Proposition 7.6
MaxNash, MaxSum, MaxSwap, and p-MaxPolar rules do not satisfy monotonicity.

This example highlights why conflict monotonicity is quite hard to achieve, but an extreme (and weaker) version of it can actually be satisfied by our rules. The idea is that instead of increasing the conflict in only *one* ranking, we increase it in *every* ranking at once, and we increase it as much as we can in every ranking. More formally, we say that P^{ab} is an *antagonization* of a profile P with respect to a pair $\{a, b\}$ if, in all votes from $V^{a \succ b}$, we shift a to the first position and b to the last, and in all votes from $V^{b \succ a}$, we shift b to the first position and a to the last. The relative order of all the other candidates remains the same. Note that the sets $V^{a \succ b}$ and $V^{b \succ a}$ are the same in both profiles (so the value of $\alpha(a, b)$ does not change), and for all $i \in V$, $|d_i(a, b)| = m - 1$ in P^{ab} (i.e., $\beta(a, b) = 1$).

Antagonization Consistency
A rule f is *antagonization consistent* if for any profile P, if $\{a, b\} \in f(P)$, then $\{a, b\} \in f(P^{ab})$.

This axiom is much easier to satisfy, and we can show that all our rules satisfy it.

Proposition 7.7
MaxNash, MaxSum, MaxSwap and p-MaxPolar rules are antagonization consistent.

Proof. Let us show that MaxSum, MaxNash and MaxSwap satisfy this axiom. Let f be one of these rules. Let P be a profile and $\{a, b\} \in f(P)$, and consider the antagonization profile P^{ab}. We will show that $\{a, b\} \in f(P^{ab})$.

We first show the result for MaxSum, MaxNash and MaxSwap. It is clear that the score of $\{a, b\}$ does not decrease when we go from P to P^{ab} for any of these rules, as for all voters $i \in V$, $|d_i^{ab}(a, b)| \geqslant |d_i(a, b)|$, where d_i^{ab} is the equivalent of d_i in P^{ab}. Moreover, the sets $V^{a \succ b}$ and $V^{b \succ a}$ are the same in both profiles, so the pairwise conflict induced by $\{a, b\}$ between any two voters is greater or equal in P^{ab} than in P. Similarly, the minimal number of swaps required to make the pair non-conflicting can only increase.

Now, we show that for all candidates $x, y \in C \setminus \{a, b\}$, the scores of $\{x, y\}$ for the different rules are lower or equal in P^{ab} than in P. Since the relative order between all candidates except a and b is the same in both profiles, the only difference is that a and b can be placed between x and y in P, but they surely are not in P^{ab} as a and b are both at the extremes of every ranking. Thus, for all $i \in V$, $|d_i^{ab}(x, y)| \leqslant |d_i(x, y)|$. Moreover, it is clear that the sets $V^{x \succ y}$ and $V^{y \succ x}$ are the same in both profiles, so the pairwise conflict induced by $\{x, y\}$ between any two voters is lower or equal in P^{ab} than in P. Similarly, the number of swaps required to make the pair non-conflicting can only decrease.

Finally, we show that the MaxSum and MaxNash scores of $\{a, b\}$ in P^{ab} are higher than the scores of $\{a, x\}$ and $\{x, b\}$ for any $x \in C \setminus \{a, b\}$. We prove it here for $\{a, x\}$, the proof for $\{x, b\}$ is similar. Recall that in P^{ab}, a is ranked first and b last in all rankings of voters $i \in V^{a \succ b}$, and conversely in all rankings of voters $i \in V^{b \succ a}$. Thus, it is clear that for voters in $V^{a \succ b}$ we have $a \succ x$ (since a is ranked first), and in $V^{b \succ a}$ we have $x \succ a$ (since a is ranked last). Therefore, $V^{a \succ x} = V^{a \succ b}$ and $V^{x \succ a} = V^{b \succ a}$. Moreover, for all $i \in V$, $|d_i^{ab}(a, b)| = m - 1 > |d_i^{ab}(a, x)|$. Thus, it directly follows from the formulas of the rules that the score of $\{a, b\}$ is higher than the score of $\{a, x\}$ in P^{ab}. Similarly, the number of swaps required to make the pair non-conflicting is necessarily higher for $\{a, b\}$ than for $\{a, x\}$ in P^{ab}, showing the result for MaxSwap.

We can now conclude for MaxSum, MaxNash and MaxSwap: $\{a, b\}$ has the highest score in P, and it can only be higher in P^{ab}. On the other hand, the scores of all pairs not involving a or b are lower or equal in P^{ab} than in P. Thus, the pair $\{a,b\}$ still has a higher score than these pairs in P^{ab}. Moreover, it has a higher score in P^{ab} than all the pairs that involve either a or b and another candidate $x \neq a, b$. Thus, $\{a,b\} \in f(P^{ab})$.

The proof for p-MaxPolar rules follows the same idea. Let us denote by α (resp. α^{ab}) and β (resp. β^{ab}) the values of the partitioning ratio and the discrepancy in P (resp. P^{ab}). We clearly have $\alpha^{ab}(a,b) = \alpha(a,b)$ and $\beta^{ab}(a,b) \geqslant \beta(a,b)$. Thus the score of $\{a,b\}$ in P^{ab} is higher than in P: $s^{ab}(a,b) = (\alpha^{ab}(a,b))^p \beta^{ab}(a,b) \geqslant (\alpha(a,b))^p \beta(a,b) = s(a,b)$. Moreover, for all $x, y \in C \setminus \{a,b\}$, we have $\alpha^{ab}(x,y) = \alpha(x,y)$ and $\beta^{ab}(x,y) \leqslant \beta(x,y)$ for the same reasons as detailed in the proof of the other rules, so the score of $\{x,y\}$ in P^{ab} is lower or equal than in P: $s^{ab}(x,y) = (\alpha^{ab}(x,y))^p \beta^{ab}(x,y) \leqslant (\alpha(x,y))^p \beta(x,y) = s(x,y)$. Since $\{a,b\} \in f(P)$, we have $s(a,b) \geqslant s(x,y)$, and thus $s^{ab}(a,b) \geqslant s^{ab}(x,y)$. Finally, for all $x \in C \setminus \{a,b\}$, we have $\alpha^{ab}(a,x) = \alpha^{ab}(a,b)$ and $\beta^{ab}(a,x) < \beta^{ab}(a,b) = 1$, so the score of $\{a,x\}$ in P^{ab} is lower than that of $\{a,b\}$: $s^{ab}(a,x) < s^{ab}(a,b)$. We can similarly prove that for all $x \in C \setminus \{a,b\}$, $s^{ab}(x,b) < s^{ab}(a,b)$. Thus, $\{a,b\}$ still maximizes the p-MaxPolar score in P^{ab} and $\{a,b\} \in f(P^{ab})$. □

Balance Preference

In Section 7.2, we highlighted the role of the discrepancy balance between the supporters of one candidate and those of the other one, motivating the definition of the measures γ and ϕ. We could argue that the conflict is more important when it is more balanced between these two groups of supporters. In this section, we define an axiom that also captures this idea, based on another dominance relation between pairs of candidates.

Formally, we say that a pair of candidates $\{a,b\}$ *balance-dominates* another pair $\{x,y\}$ in a profile P if the following conditions are true:

(1) There exists a bijective function $g : V \to V$ such that for all $i \in V$, $|d_i(a,b)| = |d_{g(i)}(x,y)|$,

(2) we have $|\sum_{i \in V} d_i(a,b)| < |\sum_{i \in V} d_i(x,y)|$.

Note that these two conditions directly imply that $\beta(a,b) = \beta(x,y)$ and $\phi(a,b) < \phi(x,y)$:

$$\beta(a,b) = \frac{1}{m} \sum_{i \in V} |d_i(a,b)| = \frac{1}{m} \sum_{i \in V} |d_{g(i)}(x,y)| = \beta(x,y)$$

$$\phi(a,b) = \frac{\left|\sum_{i \in V} d_i(a,b)\right|}{\sum_{i \in V} |d_i(a,b)|} < \frac{\left|\sum_{i \in V} d_i(x,y)\right|}{\sum_{i \in V} |d_{g(i)}(x,y)|} = \phi(x,y).$$

Thus, rules that take into account the discrepancy balance will likely prefer $\{a,b\}$ over $\{x,y\}$. We can now define the balance preference axiom.

Balance Preference
A rule f satisfies *balance preference* if for any profile P, if $\{a,b\}$ balance-dominates $\{x,y\}$, then $\{x,y\} \notin f(P)$.

The intuition behind this axiom is that if two pairs of candidates induce the same *total* amount of conflict (measured by the discrepancy) among the voters, the pair for which this amount is better divided between the supporters of the two candidates should be selected. The following example illustrates this idea.

	MaxSum	MaxNash	MaxSwap	MaxPolar
Reverse Stability	✓	✓	✓	✓
Conflict Consistency	✓	✓	✓	✓
Conflict Monotonicity	✗	✗	✗	✗
Antagonization Consistency	✓	✓	✓	✓
Matching Domination	✓	✓	✗	✓
Balance Preference	✗	✓	✓	✗

Table 7.2: Summary of the axiomatic results.

Example 7.4
Consider again the profile P introduced in Example 7.1:

$$x \succ_1 a \succ_1 b \succ_1 y \qquad a \succ_3 y \succ_3 x \succ_3 b$$
$$x \succ_2 a \succ_2 b \succ_2 y \qquad b \succ_4 y \succ_4 x \succ_4 a$$

In this profile, voters are evenly divided between x and y, but x supporters are extreme while all y supporters are more indifferent. On the other hand, a is preferred to b by 3 voters, and b is preferred to a by 1 voter, but they each have at least one extreme supporter. In this scenario, $\{x, y\}$ is more balanced in terms of group size, but $\{a, b\}$ is more balanced in terms of group discrepancy. More precisely, $\alpha(x, y) = 1 > 1/2 = \alpha(a, b)$, and $\phi(x, y) = (6-2)/(6+2) = 1/2 > 1/4 = (5-3)/(5+3) = \phi(a, b)$. Moreover, using the matching $g = \{1 \to 3, 2 \to 4, 3 \to 1, 4 \to 2\}$, we can check that $\{a, b\}$ balance-dominates $\{x, y\}$, as for all i we have $|d_i(a, b)| = |d_{g(i)}(x, y)|$ and $|\sum_i d_i(x, y)| = 6 - 2 = 4 > 2 = 5 - 3 = |\sum_i d_i(a, b)|$. Thus, a rule satisfying balance preference does not select $\{x, y\}$ in this profile.

We can show the following result.

Proposition 7.8
MaxNash and MaxSwap satisfy balance preference, but not MaxSum and p-MaxPolar rules.

Proof. Using the profile of Examples 7.2 and 7.4, we already showed that p-MaxPolar rules and MaxSum select the pair $\{x, y\}$, which is balance-dominated by the pair $\{a, b\}$. Thus, these rules do not satisfy balance preference.

Let us now prove that MaxNash and MaxSwap satisfy balance preference. For this, we can simply use the formulas of the rules obtained in Propositions 7.1 and 7.2. Since the conditions of the axiom imply that $\beta(a, b) = \beta(x, y)$ and $\phi(a, b) < \phi(x, y)$, we can directly conclude that $\{x, y\} \notin f(P)$. □

Summary

We conclude this section with a summary of the axiomatic results, presented in Table 7.2. As we can see, the only axioms that are discriminating between the rules are matching domination and balance preference. Overall, it appears that MaxNash is the one that satisfies the most axioms among the ones we introduced.

7.5 Experimental Analysis

We now compare the behavior of the rules we introduced with experiments based on synthetic and real data. We also compare them to some classical multi-winner voting rules, in order to highlight the natural incompatibility between these traditional rules and our conflictual setting.

We ran experiments on both synthetic and real data. For synthetic data, we use two different families of models: 2-dimensional Euclidean models, and mixtures of Mallows. For real data, we focus on three datasets, each intuitively representing a different level of conflict.

The 2-dimensional Euclidean model

We first consider the 2-dimensional Euclidean framework, which we introduced in Section 2.5.1. As a reminder, voters and candidates are associated with positions in a 2-dimensional Euclidean space, and the preferences of voters are based on their distances to candidates (Anshelevich et al., 2018). For a voter $i \in V$, we have $a \succ_i b$ if i is closer to a than to b. This 2-dimensional Euclidean framework has been extensively studied in the social choice literature, as it enables us to easily visualize the behavior of the rules (Enelow and Hinich, 1984). In particular, it has been used by Elkind et al. (2017a) to visualize the differences between multi-winner voting rules that have different objectives: excellence, diversity, and proportional representation (a part of their experiments is reproduced in Section 2.5.1). In this analysis, we sample profiles of $n = 100$ voters and $m = 10$ candidates, and consider two distributions of positions in the space:

(1) The uniform distribution on $[0, 1]^2$.

(2) The Gaussian distribution centered in $(0.5, 0.5)$ with standard deviation $\sigma = 0.15$ (in order to have most of the points in $[0, 1]^2$).

Visualizing the Conflict

This model is particularly interesting because it gives an intuition of what the different rules are doing by visualizing the positions of the selected candidates. In Figure 7.2, we show the positions of the selected pairs of candidates for 10 000 profiles sampled from the uniform or Gaussian distributions (in this case, positions of voters and candidates follow the same distribution). For three random instances, we marked the positions of the selected candidates by different colors. This figure shows the outcomes of three rules: MaxNashConflict, Chamberlin-Courant and Borda. For the definitions of Chamberlin-Courant and Borda in the multi-winner setting, we refer to Section 2.3.2. The goal is to highlight the fundamental difference between our rules that aim at maximizing the conflict, and classical voting rules, that have different objectives. In particular, an *excellence* objective for Borda, and a *diversity* objective for Chamberlin-Courant, which appears intuitively closer to our objective. Note that we do not present other conflict rules, as the positions of the selected candidates look very similar to the ones obtained with MaxNashConflict.

We clearly see on Figure 7.2 that while both Chamberlin-Courant and Borda select candidates that are close to the center of the space (especially Borda), MaxNashConflict selects candidates that are more on the periphery of the distribution. In particular, it clearly avoids the center of the space. Moreover, if we look at the colored points for the three random profiles, we observe that the candidates selected by MaxNashConflict are opposed to each other in the space, while the candidates selected by Borda are closer to each other. The candidates selected by Chamberlin-Courant are in between, but it is still more similar to Borda than to MaxNashConflict.

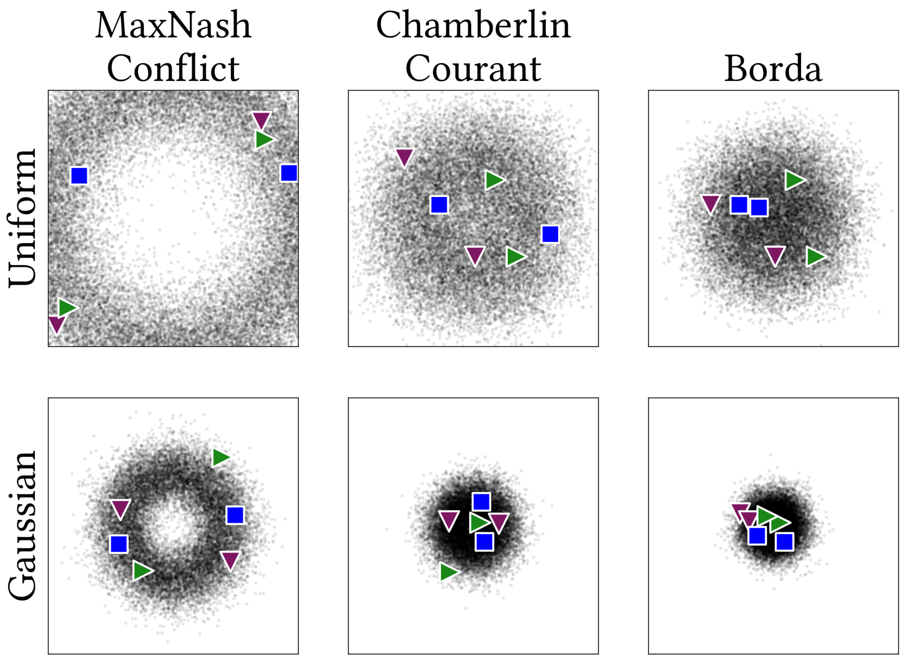


Figure 7.2: Distributions of the positions of the candidates selected by different rules and distributions of positions. Each pair of colored points corresponds to selected candidates in a single election.

Measuring the Conflict

We now compare the conflict rules to each other. In particular, we focus on MaxSum, MaxNash, MaxSwap and 2-MaxPolar. As we said, we cannot really see differences between the behavior of the different rules by simply visualizing the distribution of the positions of the selected candidates in the plane. Thus, we will not look at their positions, but compute the values of the conflict measures introduced in Section 7.2 for the pairs of candidates selected by each rule. Then, we will plot the results in a scatter plot, where each point corresponds to a pair of candidates selected in one profile, with the x-axis corresponding to the value of the partitioning ratio α (maximal when the voters are evenly divided) and the y-axis to the value of the discrepancy β (maximal when voters have strong opinions on the candidates). Moreover, the size of the points will be proportional to the value of the discrepancy balance γ and their color will depend on the group discrepancy imbalance ϕ: the redder the dot, the lower the imbalance (and thus the more balanced the conflict is). An example is given in Figure 7.3. In this figure, A has both high α and β, indicating that it represents a highly conflicting pair of candidates. B also has a high α, but an average β, indicating that voters are quite divided between the candidates of the pair, but do not necessarily have strong opinions on them. Finally, C has a low α but a very high β, indicating that most voters agree on the ordering of the two candidates, but they have strong opinions on them.

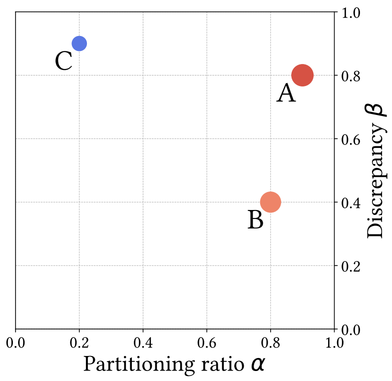


Figure 7.3: Example of a figure showing the values of the conflict measures.

In this experiment, and in the similar ones we conducted with the other datasets, we sampled

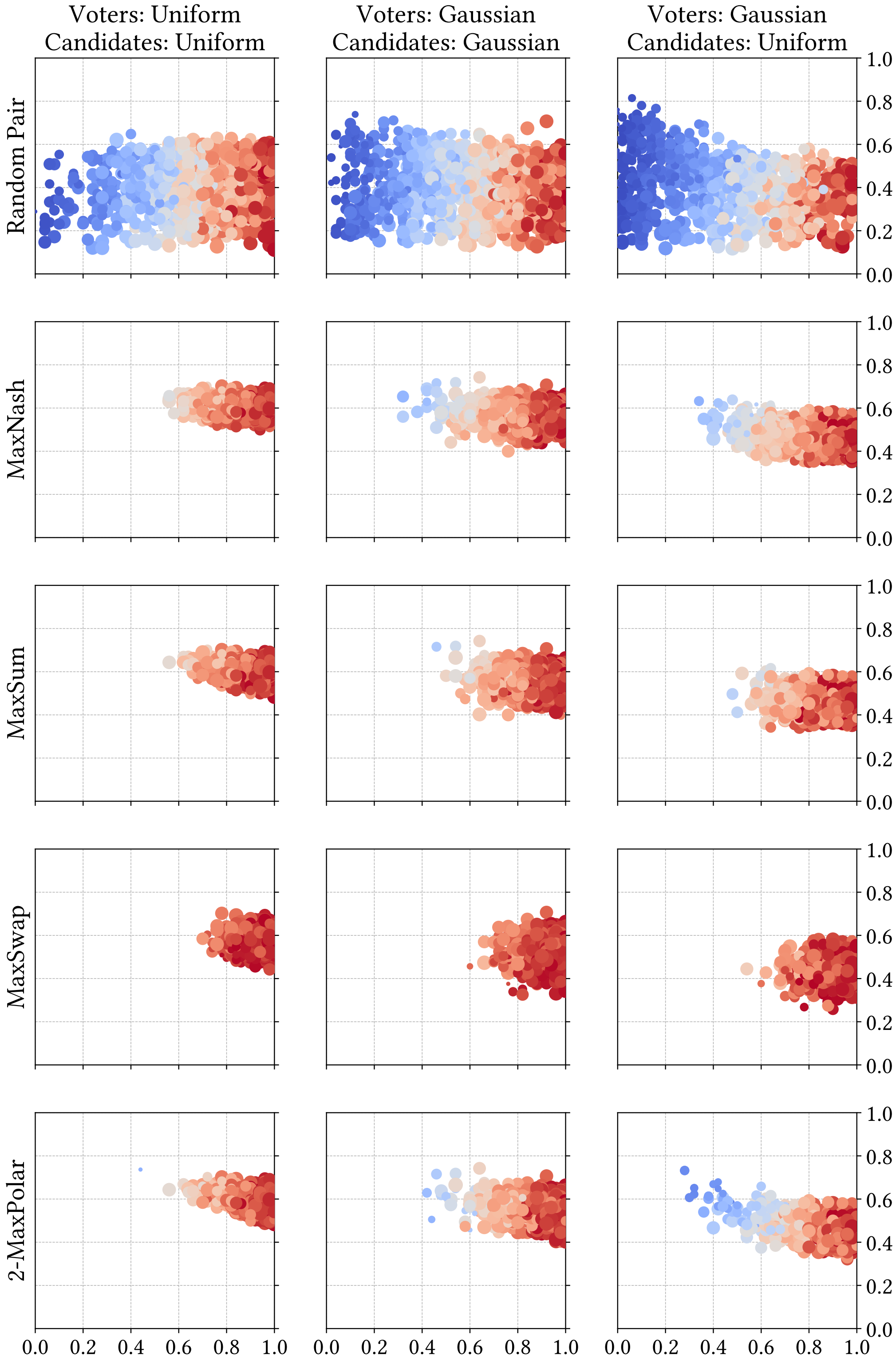


Figure 7.4: Values of the conflict measures for the pairs of candidates selected by different rules in the 2-dimensional Euclidean model. The x-axis corresponds to the partitioning ratio $\alpha(a, b)$, the y-axis to the discrepancy $\beta(a, b)$, the size of the points to the discrepancy balance $\gamma(a, b)$ and the color to the group discrepancy imbalance $\phi(a, b)$ (if ϕ is lower, the conflict is more balanced and the dot is more red).

1 000 profiles of 100 voters and 10 candidates. We used three different settings for sampling the positions of voters and candidates: (i) Uniform distribution for both voters and candidates, (ii) Gaussian distribution for both and (iii) Uniform distribution for the candidates and Gaussian for the voters. Figure 7.4 shows the values of the conflict measures for the different rules in these settings. Moreover, we included in the first row the values of the conflict measures for a random selected pair from each profile, to give an idea of the distribution of the values of the conflict measures in each setting. In particular, we observe that no pair of candidates has a very high value of discrepancy β (y-axis), especially in the first two settings. Finally, in the last setting, we observe that pairs with high α have a low β and conversely, pairs with high β have a low α. We clearly see that the partitioning ratio α is very correlated with the group discrepancy imbalance ϕ, as points on the right are redder and points on the left are bluer. This corroborates our theoretical observations from Section 7.2, which already showed that α and ϕ were linked.

These results confirm the theoretical analysis of the rules and their link to the conflict measures. Clearly, MaxSwap gives more importance to the partitioning ratio α (x-axis) and the group discrepancy imbalance ϕ (color) than MaxSum, which itself gives more importance to these two values than MaxNash. Indeed, MaxSwap sometimes returns pairs of candidates with quite low discrepancy β (< 0.4) if their partitioning ratio α is high, while MaxNash sometimes returns pairs of candidates with low partitioning ratio α (< 0.4) if their discrepancy β is high. In that sense, 2-MaxPolar is closer to MaxNash, as it seems to give more importance to the discrepancy, especially in the last setting, in which it returns pairs with very low α (< 0.3) but quite high β. Remember that 2-MaxPolar is also technically "blind" to the balance measures γ and ϕ, which can explain why we observe more points with a smaller size and a bluer color.

Mixture of Mallows models

We now present our experiments on Mallows models, or more precisely on mixtures of Mallows. We introduced Mallows models and mixtures of Mallows in Section 2.5.1, but we recall here the main points. A Mallows model has two parameters: a *central ranking* $\succ$ and a parameter $\psi \in [0, 1]$.[1] The ranking $\succ$ is the "average" ranking of the voters, but they all deviate a bit from it. If $\psi = 0$, we obtain the identity profile (ID) and all voters have the same ranking $\succ$, and if $\psi = 1$, we obtain impartial culture and each ranking is equiprobable. Thus, a Mallows model with $0 < \psi < 1$ would be something that is between the identity and uniformity, and it will probably not be much conflictual, as most voters would have a ranking similar to $\succ$.

To obtain more antagonistic profiles, we can sample rankings from mixtures of *two* Mallows, sharing the same parameter ψ, but with different central rankings $\succ^1$ and $\succ^2$. Then, we sample each ranking in the profile randomly from one Mallows with probability 1/2. This way, we obtain a profile in which half of the voters have a ranking close to $\succ^1$, and the other half close to $\succ^2$.

We expect these 2-Mallows models to have a more antagonistic structure. To confirm this, we looked at the values of conflict measures in profiles sampled from 1-Mallows and 2-Mallows models, using different values for the parameter ψ. In Figure 7.5, we show the average value of each measure over all pairs in the profile, except for the discrepancy β, for which we show the maximal value (since the average is a constant by definition). These values are averaged over 50 profiles. We observe that pairs of candidates are on average more conflictual in 2-Mallows models with respect to the partitioning ratio α, the discrepancy balance γ and the group discrepancy imbalance ϕ, especially for low values of the parameter ψ. This makes sense, as in 1-Mallows

[1]Note that we denoted this parameter ϕ in Section 2.5.1 and in the other chapters of this thesis, but in this chapter ϕ already denotes the group discrepancy imbalance.

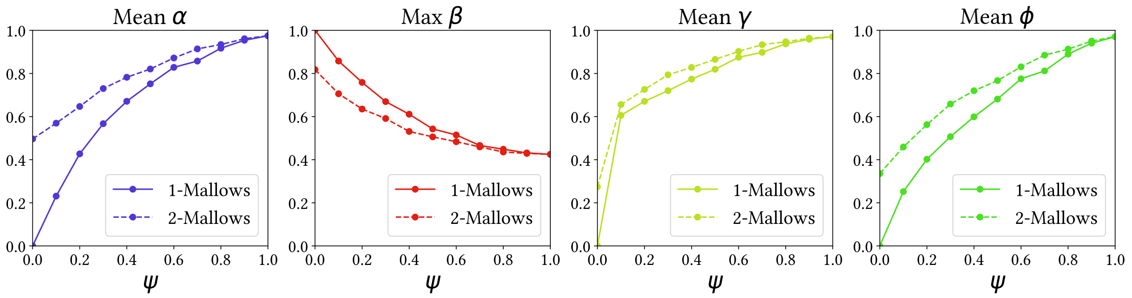


Figure 7.5: Mean/max values of different measures in profiles sampled from 1-Mallows and 2-Mallows models, averaged over 50 profiles.

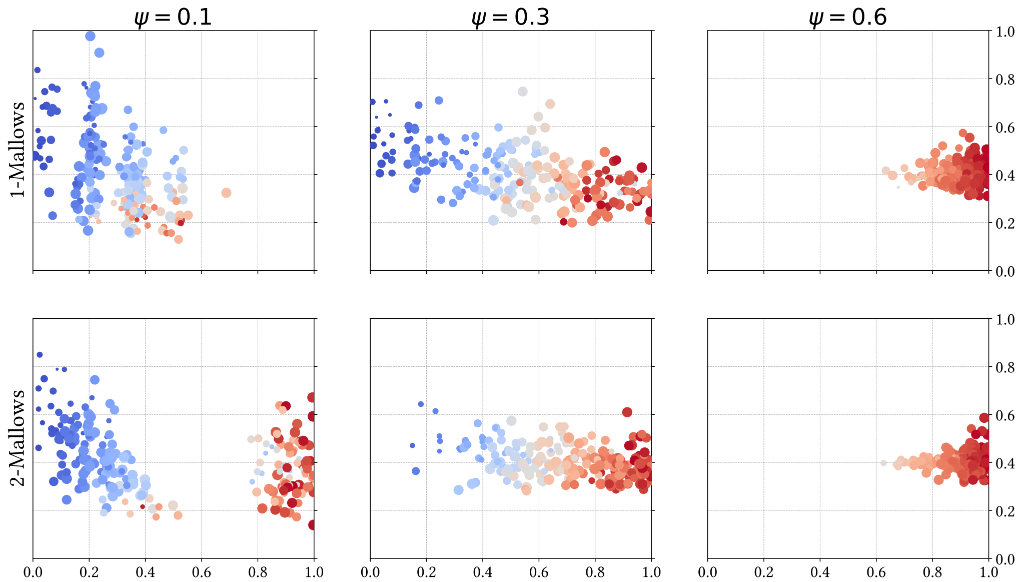


Figure 7.6: Values of conflict measures for pairs of candidates in profiles sampled from 1-Mallows and 2-Mallows models. The x-axis corresponds to $\alpha(a, b)$, the y-axis to $\beta(a, b)$, the size of the points to $\gamma(a, b)$ and the color to $\phi(a, b)$ (if ϕ is lower, the conflict is more balanced and the dot is more red).

models with low ψ, voters have almost the same ranking, and thus pairs of candidates are not highly conflictual. As ψ increases, the profiles get closer to uniformity, in which the conflict is by definition balanced (but not necessarily high), and thus these three measures tend to 1 for both 1-Mallows and 2-Mallows. However, the maximal discrepancy β is higher for 1-Mallows models than for 2-Mallows models for all values of ψ. To see why, observe that with 1-Mallows models, all rankings are similar to the central ranking $\succ$, thus the first candidate of $\succ$ is often ranked among the first candidates in voters' rankings, and the last candidate of $\succ$ is often ranked among the last ones, ensuring a high discrepancy for that pair (though almost all voters rank them in the same order). With 2-Mallows, this effect is reduced due to having two different central rankings that might cancel out each other. Finally, we deduce from Figure 7.5 that the "sweet spot" for obtaining conflicting pairs of candidates with 2-Mallows is between $\psi = 0.1$ and 0.4.

Another way to see all of this is to look at the plot of the values of the conflict measures of the different pairs of candidates from the profiles sampled using 1-Mallows and 2-Mallows, as we

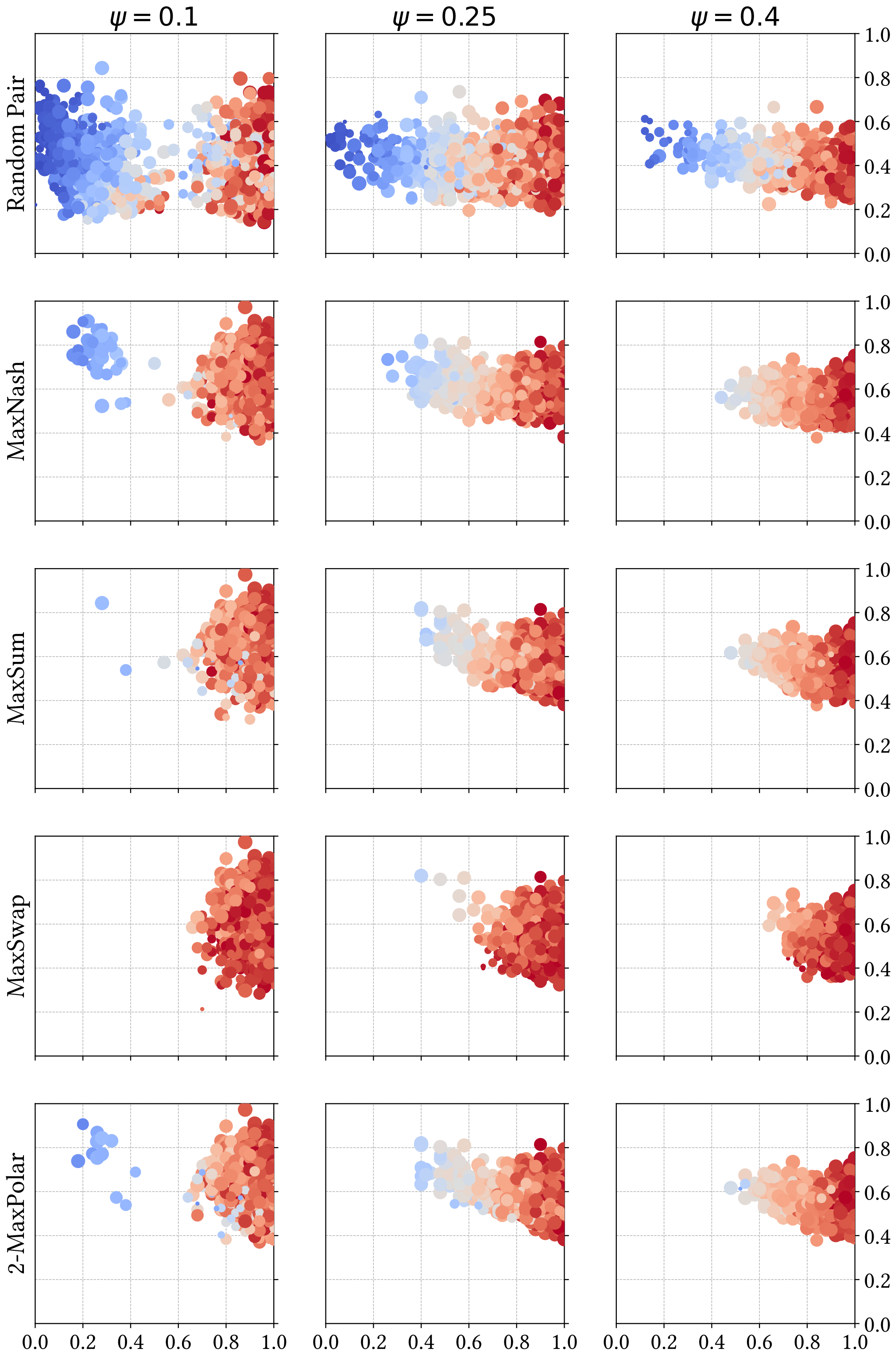


Figure 7.7: Values of the conflict measures of the pairs of candidates selected by the different rules with 2-Mallows models and three different values of ψ. The x-axis corresponds to the partitioning ratio $\alpha(a, b)$, the y-axis to the discrepancy $\beta(a, b)$, the size of the points to the discrepancy balance $\gamma(a, b)$ and the color to the group discrepancy imbalance $\phi(a, b)$ (if ϕ is lower, the conflict is more balanced and the dot is more red).

do in Figure 7.6. We see again that for low values of ψ (here, $\psi = 0.1$), 1-Mallows models fail to generate pairs with high partitioning ratio α, as all voters have a ranking similar to $\succ$, while 2-Mallows models generate pairs with either a high value or a low value of the partitioning ratio α, but no pair with an average value of α. The pairs with a high α are the ones which are ordered differently in the two central rankings $\succ^1$ and $\succ^2$: around half of the voters rank one candidate above the other, and the other half have the reverse order. The pairs with low α are the ones which are ordered the same in the two central rankings. Note that the pairs with high α do not necessarily have a high discrepancy β. In order to have a pair with both high partitioning ratio and high discrepancy, we would need for instance that $\succ^2 = \overleftarrow{\succ}^1$. Finally, we observe that when ψ increases, the two models become much more similar, with only pairs having a high partitioning ratio α and an average discrepancy β (characteristic of the uniform profile UN).

We show in Figure 7.7 the values of the conflict measures for the pairs of candidates selected by our rules, as we did for the Euclidean setting. Moreover, we also show these values for random pairs from each profile in the first row. As in the Euclidean setting, we sampled 1 000 profiles of 100 voters and 10 candidates. We observe the same differences between conflict rules with the Mallows models as with the Euclidean models. In particular, if we look at the setting with $\psi = 0.1$, MaxSwap always selects pairs of candidates with a high partitioning ratio α, i.e., such that the candidates are ranked in a different order in the two central rankings $\succ^1$ and $\succ^2$. However, MaxNash and 2-MaxPolar sometimes select pairs with a very low partitioning ratio, which means that the candidates are ranked in the same order in both central rankings. MaxSum mostly selects pairs with high partitioning ratios, however it seems to care less about the conflict balance than MaxSwap, as the latter returns pairs with a higher discrepancy balance γ (larger dots) and a lower group discrepancy imbalance ϕ (redder dots).

Real Data

Finally, we consider three different datasets of real preferences, each representing a different level of conflict:

(1) *Voter Autrement*: This dataset includes two preference profiles of voters over the candidates to the French presidential elections of 2017 and 2022, collected during *online* experiments. In these datasets, participants had to give a (truncated) ranking of candidates for the instant runoff voting (IRV) rule. For more details on these datasets, we refer to Section 2.5.4. We kept only the participants that ranked all the candidates. We have $n = 5\,755$ voters and $m = 11$ candidates in the 2017 profile, and $n = 412$ voters and $m = 12$ candidates in the 2022 profile. Recall that the voters are assigned weights based on their actual vote at the election such that the distribution of weights matches the actual results of the election.

(2) *Sushi*: This dataset contains a single profile of $n = 5\,000$ voters giving their preferences on $m = 10$ types of sushi (Kamishima, 2003).

(3) *Figure Skating*: This dataset contains juries' rankings of contestants performances in figure skating competitions (Smith, 2000). This dataset contains 49 profiles, each having $n \in [8, 10]$ judges and $m \in [10, 25]$ contestants.

The last two datasets are part of the *Preflib* collection of datasets (Mattei and Walsh, 2013). We expect the first dataset, about the French presidential elections, to be the most conflictual, as conflict is an integral part of politics. On the other hand, we expect the last dataset, about figure

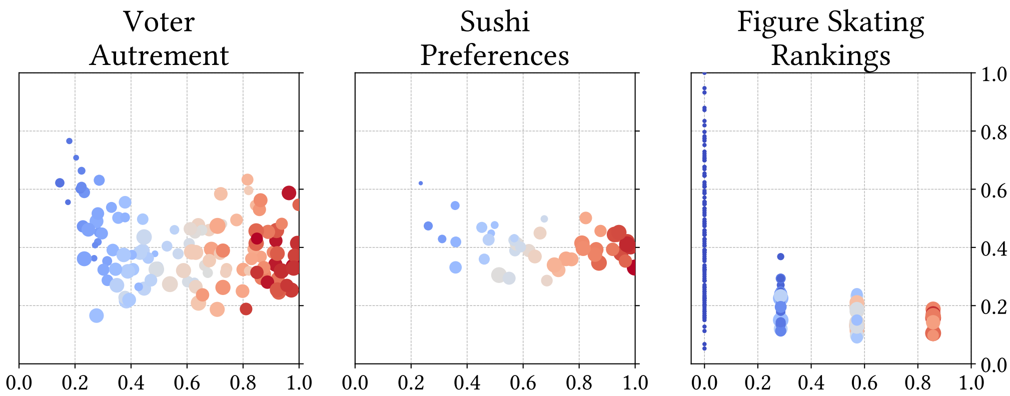


Figure 7.8: Values of the conflict measures for all pairs of candidates in the real-life datasets. The x-axis corresponds to the partitioning ratio $\alpha(a,b)$, the y-axis to the discrepancy $\beta(a,b)$, the size of the points to the discrepancy balance $\gamma(a,b)$ and the color to the group discrepancy imbalance $\phi(a,b)$ (if ϕ is lower, the conflict is more balanced and the dot is more red).

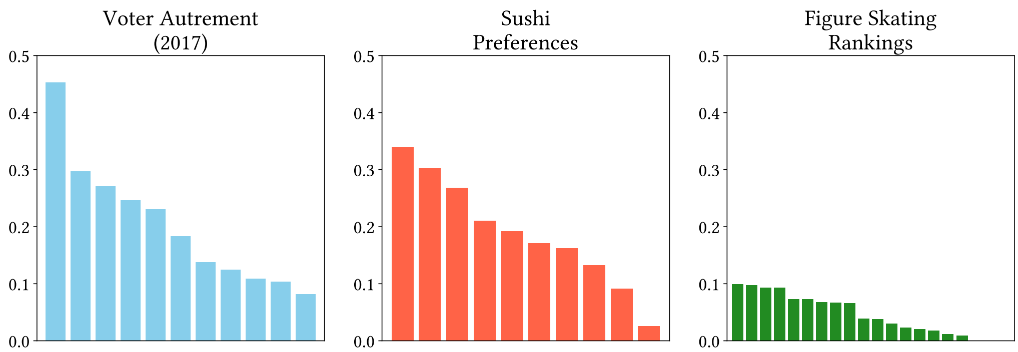


Figure 7.9: *Divisiveness* values (Colley et al., 2023b) of the different candidates of the real-life datasets, sorted in decreasing order.

skating, to be the least conflictual, as in this kind of sport competition, the performance of the contestants is usually judged on their technical skills and on the acrobatic figures presented.

These intuitions are confirmed by analyzing the values of the conflict measures of the pairs of candidates in these datasets, as shown in Figure 7.8. For the figure skating plot, we took one random dataset among the 49 available, but they all have a similar structure. At first glance, the plot for the figure skating dataset looks quite similar to what we obtained with 1-Mallows and a low value of ψ (in which all pairs have very low values of partitioning ratio α and discrepancy balance γ). However, it differs a bit as we still observe pairs of candidates with high values of α and very low discrepancy β. These are the pairs of contestants that have comparable figure skating skills, making them appear next to each other in every ranking, but for whom the judges are very divided as to which of the two is the best. On the other hand, the pairs with low α and high β correspond to pairs of contestants with a clear skill gap, such that all judges rank one very high and the other very low, but they all rank them in the same order. The plot for the *Voter Autrement* dataset looks more similar to the one of a 2-Mallows, with pairs that have high partitioning ratio and other with low partitioning ratio, but very few pairs with an average partitioning ratio. Finally, the sushi dataset has a structure similar to a 1-Mallows with an intermediate value of ψ, with not many pairs having both a high partitioning ratio and a high discrepancy.

	2017	2022
MaxNash	PS ↔ FN	LFI ↔ RN
MaxSum	PS ↔ FN	NPA ↔ RN
MaxSwap	NPA ↔ FN	NPA ↔ REC
2-MaxPolar	NPA ↔ FN	NPA ↔ RN
Borda	LFI ↔ EM	LFI ↔ EELV
Chamberlin-Courant	LFI ↔ LR	EELV ↔ RN

Table 7.3: Pairs of candidates selected in the *Voter Autrement* datasets by different rules.

These different level of conflict can also be observed if we compute polarization measures defined in the literature for whole profiles or for single candidates. Figure 7.9 shows the *divisiveness* values of the candidates, as defined by Colley et al. (2023b). For the *Voter Autrement* plot, we used the 2017 profile, and only one dataset was used for the figure skating plot. We observe that in the figure skating dataset, no candidate seems particularly divisive, while the two other datasets contain candidates with high divisiveness values. We obtain a similar picture if we compute the polarization of the preference profiles as a whole, using the measure proposed by Can et al. (2015), and which is equal to the average partitioning ratio over all pairs of candidates $\frac{2}{m(m-1)} \sum_{a,b \in C, a \neq b} \alpha(a,b)$. We obtain a polarization of 0.76 for the two *Voter Autrement* profiles, of 0.68 for the Sushi dataset, and between 0 and 0.20 for the figure skating profiles.

Conflict Measures

First, we performed the same experiment as the one we conducted with the synthetic models, by sampling random profiles from our datasets. To do so, we sampled 1 000 random profiles by selecting $n = 100$ random voters and $m = 10$ random candidates from the original profiles for each of them. Then, we compare the values of the conflict measures of the pairs of candidates selected by each rule, as well as the same values for a random pair of candidates in each profile. In this experiment, we only used the 2017 *Voter Autrement* profile, and for the figure skating dataset, each subprofile is sampled from one of the 49 profiles, selected at random. The results are presented in Figure 7.10, and the observations are the same as for the synthetic models. The case of the figure skating dataset is particularly interesting, as most pairs either have low partitioning ratio α or low discrepancy β. While rules like MaxNash and 2-MaxPolar, and to a lesser extent MaxSum, sometimes select pairs with very low α but a high β, MaxSwap never selects such pairs and almost always prioritizes the partitioning ratio α. Moreover, we observe that a lot of dots on the 2-MaxPolar plot for the figure skating dataset are very small and very blue, corresponding to unbalanced conflicts. This typically corresponds to pairs of candidates such that one is always ranked among the first and the other one always ranked among the last. If the notion of conflict is a bit tricky in such *consensual* datasets, it appears nevertheless that 2-MaxPolar has the worst performance, and one could argue that MaxSwap is performing best.

Analysis of the Voter Autrement Datasets

Let us now focus more specifically on the outcomes of the rules on the *Voter Autrement* datasets, that were gathered during the 2017 and 2022 French presidential elections. The results on the full datasets (so, without sampling as in the previous experiment) are shown in Table 7.3. As a reminder, the ideological axes of candidates that seem to be the most natural according to the poll institutes and to our *axis rules*, discussed in Chapter 6, are shown in Table 7.4. Note that we have

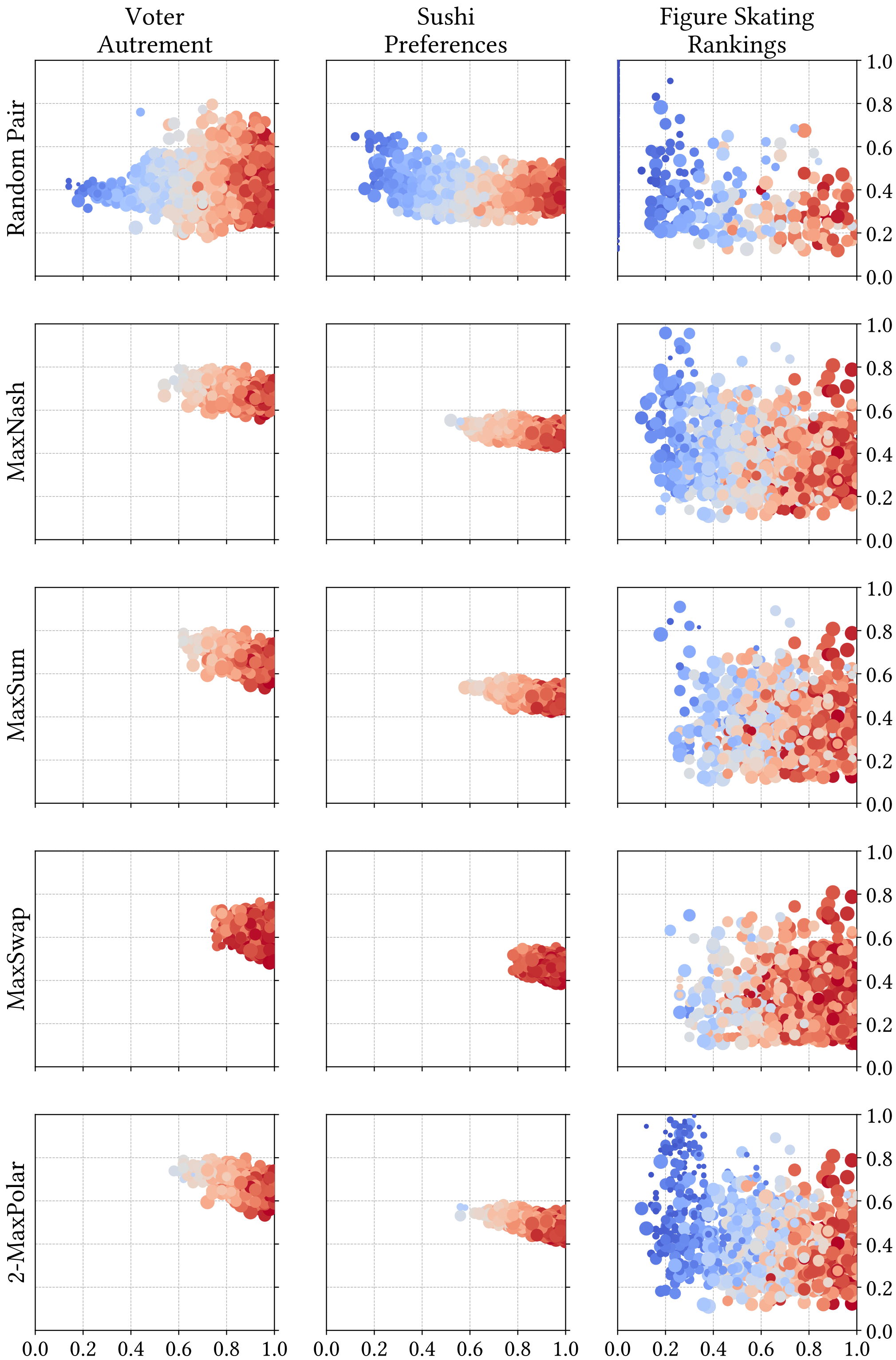


Figure 7.10: Values of the conflict measures of the pairs of candidates selected by the different rules with three different real-life datasets. The x-axis corresponds to the partitioning ratio $\alpha(a,b)$, the y-axis to the discrepancy $\beta(a,b)$, the size of the points to the discrepancy balance $\gamma(a,b)$ and the color to the group discrepancy imbalance $\phi(a,b)$ (if ϕ is lower, the conflict is more balanced and the dot is more red).

2017	LO ◁ NPA ◁ LFI ◁ PS ◁ EM ◁ LR ◁ DLF ◁ FN
2022	LO ◁ NPA ◁ LFI ∼ PCF ◁ PS ∼ EELV ◁ LREM ◁ LR ◁ DLF ◁ RN ◁ REC

Table 7.4: Natural 'ideological' order of the candidates on the left-right axis for the 2017 and 2022 French presidential elections.

intentionally removed some candidates from the axes, as they are difficult to place on a left-right spectrum (see Section 6.4 for more details), and are not relevant to our analysis.

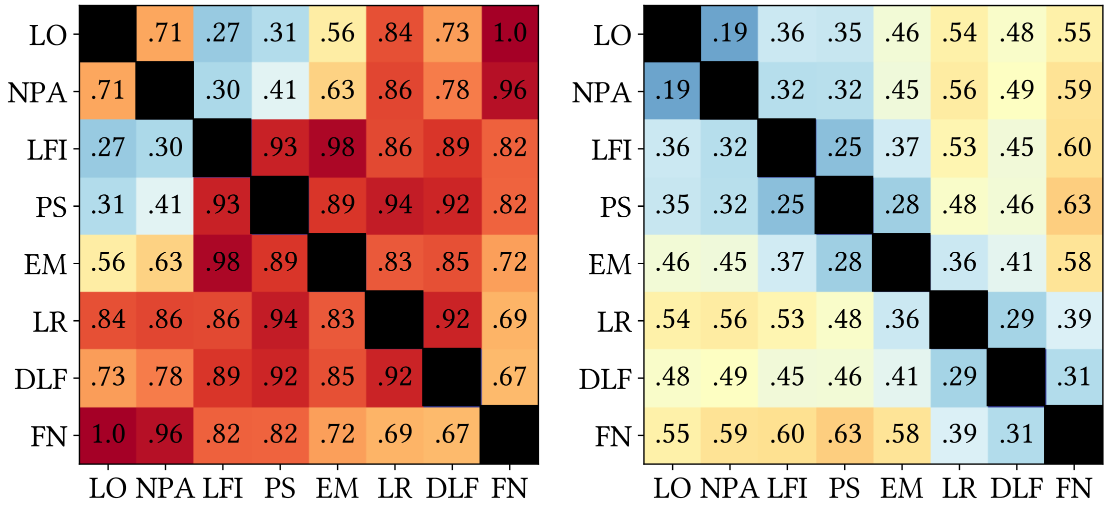
First, we clearly see a difference between our conflict rules and traditional multi-winner voting rules such as Borda or Chamberlin-Courant: conflict rules always select one candidate from the left or the far-left, and one candidate from the nationalist far-right, while the two other rules always select at least one more centrist candidate. Let us now compare the conflict rules to each other. In these datasets, a pair of candidates has a high partitioning ratio α in two cases: either the two candidates are from opposite sides of the axis, and are thus dividing the society (e.g., LFI and FN), or they are close to each other, but they have a similar level of popularity, so voters are very split between them (e.g., PS and EELV). The first case might explain why MaxSwap, which prioritizes the partitioning ratio α, selects candidates that are close to the extremes. On the other hand, rules favoring the discrepancy β like MaxNash selects candidates that are more well-known (in the sense that they have more supporters), even if they are dividing the society less evenly, as voters generally have stronger preferences on them: these candidates are ranked higher by their supporters and lower by their detractors. This is for instance the case in the 2022 dataset, for which MaxNash selects LFI (21.95% of the votes) and RN (23.15% of the votes) while MaxSwap selects the more extreme but less popular NPA (0.77% of the votes) and REC (7.07% of the votes).

We also observe that MaxSwap selected candidates with higher divisiveness values (as defined by Colley et al. (2023b)). This is not too surprising, since both MaxSwap and the divisiveness measure heavily rely on the partitioning ratio α. For the 2022 dataset, MaxSwap is actually the only rule that selects the REC candidate, which has the highest divisiveness (0.41), while the other far-right party RN ranks fifth in divisiveness with 0.30 (though there is a difference of less than 0.001 between this party and the second, third and fourth most divisive ones LO, DLF, NPA). Overall, all the conflict rules select candidates with high or medium divisiveness, which is not the case of the Borda and Chamberlin-Courant rules. For instance, they both selected EELV in the 2022 dataset, which has a very low divisiveness of 0.05.

Finally, we show in Figure 7.11 the values of α and β for pairs of candidates of the 2017 *Voter Autrement* dataset, displayed in the order of the left-right axis. Interestingly, it appears that the discrepancy β is more correlated to the relative positions of the candidates on the axis than α, despite having a much lower variance. This makes sense: voters from the left likely rank all candidates from the left above all candidates from the right, and conversely. Thus, candidates that are close to each other on the axis are also close in voters' rankings.

7.6 Discussion

In this chapter, we proposed and analyzed rules that aim at selecting the most conflicting pairs of candidates using ordinal preferences of voters, and we have shown that these rules fundamentally differ from standard multi-winner voting rules. We also proposed several measures that allow us to better understand the different forms under which conflicts appear in the preferences, as well as which of these forms are prioritized by our different rules. In particular, these measures enabled us



(a) Values of $\alpha(a, b)$.

(b) Values of $\beta(a, b)$.

Figure 7.11: Values of conflict measures of pairs of candidates in the 2017 *Voter Autrement* dataset.

to show that there is a natural trade-off between the partitioning ratio α, representing the division of the society into two groups of similar sizes, and the discrepancy β, representing the strength of voters' feeling towards the candidates.

Which Rule to Choose?

We saw through theoretical and experimental analyses that MaxSwapConflict tends to prioritize pairs of candidates having a high partitioning ratio α, while MaxNashConflict tends to prioritize the ones having a high discrepancy β. MaxSumConflict and 2-MaxPolar rules fall in between. These facts should be taken into account when selecting one of these rules for a specific application. In particular, MaxNash satisfies more axioms than the other rules. However, in our experiments, MaxSwap often selected pairs of candidates that would be considered more conflicting, especially in contexts where the conflict is not too high.

Further Work

A natural, and fairly non-trivial, follow-up direction for this work is to analyze how we could extend our rules and our axioms to a model in which we want to select subsets of more than just two candidates, such that candidates in the subset are as conflicting as possible. It is less clear how to define conflict for more than two candidates, but a possibility would be to say that it corresponds to the sum of the conflicts induced by each pair of candidates in the subset, or to the minimum conflict induced by any of the pairs. With this interpretation, we can easily adapt our rules to this new setting. Nonetheless, other approaches evaluating the whole committee at once, rather than pairs of candidates, might turn out to return better results. In particular, we could get inspiration from the measures used to quantify the polarization of preference profiles as a whole (Esteban and Ray, 1994; Ozkes, 2013; Can et al., 2015).

Concluding Remarks

In this thesis, we studied five problems motivated by real-world application of preference aggregation. In the first part of the thesis, we proposed ways to *improve upon existing voting systems* using more expressive ballots than the current ones. In the second part, we proposed *tools* that use these expressive ballots to analyze the structure of the electorate and of the candidate set, which can be particularly useful for political analyses. For each setting, we have proposed novel methods, and evaluated these methods axiomatically and experimentally. We concluded these analyses by making recommendations regarding which method(s) to use, depending on the context and on the desired normative properties. In some cases, we additionally suggested ways to implement these new methods in practice. In this short conclusion, we briefly discuss the next important step of this kind of work: the dissemination of the methods to the general public and to policy makers. We also discuss some other democratic settings in which it would be interesting to apply methods similar to the ones we used in this thesis.

Dissemination of Alternative Voting Methods

If we want the alternative voting methods we propose (in the first part of this thesis, but also in general in social choice) to be used in practice, or at least to be considered in the public debate regarding potential voting reforms, it is essential to make them known outside of the academic social choice community. There exist several channels we can use to disseminate our works. A first possibility is to inform policy makers (e.g., the members of the Parliament) about these methods, and show their benefits, while still being honest about their possible drawbacks. For instance, I had the opportunity to briefly present the methods we introduced in Chapter 5 to some policy makers, in anticipation of the debate regarding the reform of the voting system used to elect the members of the French Parliament.[2] However, to attract the attention of policy makers to alternative methods, the easiest way is probably to first convince the general public of the value of these methods. Indeed, it is essential to disseminate our works to the general public, and this is one of the goals of the *Voter Autrement* project, which allows anyone to try alternative voting methods through *in situ* and *online* experiments. More generally, online platforms such as the *Voter Autrement* surveys or the *Whale*[3] voting platform are great tools for dissemination, as they allow people to use in practice the rules that we propose. Still with this dissemination goal in mind, I sometimes discuss voting methods on my blog[4], among other data analyses. Finally, making datasets of preferences with alternative methods *open and freely accessible* is crucial to increase the interest in these methods, by allowing anyone, including other scholars, to analyze

[2] They seemed genuinely interested in our work, though they made it clear that they would not advocate for these methods *now*, because the current system is too far from them (the representatives of each constituency are currently elected with plurality with runoff) and this would make a change too radical.

[3] https://whale.imag.fr/

[4] https://theo.delemazure.fr/blog.html

and use the datasets. For instance, I frequently use *Preflib* datasets (Mattei and Walsh, 2013) in my experiments. This is also why the datasets which were collected for this thesis are freely accessible (Delemazure et al., 2024a; Delemazure and Bouveret, 2024). I have also started to clean, reformat, and publish datasets collected through past experiments, but that were not yet in open access (Baujard et al., 2025c,a).

Crucially, the methods proposed in both parts of this thesis are not only applicable to major political elections, but are also relevant for any group decision-making process involving multiple alternatives. Therefore, advocating for the adoption of these methods in smaller organizations such as local communities, associations, or companies, could be a pragmatic first step towards broader acceptance, potentially preceding large-scale political reform.

Social Choice Beyond Voting

In this thesis, we have applied the axiomatic and experimental methods to *voting* problems, in which voters can only express preferences through *ballots*. It would be interesting to find ways to conduct similar analyses in other democratic settings. For instance, we could use the axiomatic method to study the normative properties of deliberative processes, which are currently gaining attention, such as *Citizens' Assemblies* (Setälä and Smith, 2018; Dryzek et al., 2011; Setälä, 2014), or to compare elective methods and sortition methods of choosing representatives (Revel, 2023; Ebadian et al., 2022; Revel et al., 2024; Caragiannis et al., 2024). These axiomatic analyses could be complemented by experimental analyses, based on data gathered from actual deliberation or sortition processes.

Appendices

Appendix A

Details on the Experimental Analysis of Axis Rules

Poll Institutes

Institute	Axis ◁											
IFOP	PT	LO	LCR	PCF	JB	PS	LV	CPNT	UDF	UMP	MPF	FN
BVA	PT	LO	LCR	PCF	JB	PS	LV	CPNT	UDF	UMP	MPF	FN
Ipsos	PT	LO	LCR	PCF	JB	PS	LV	CPNT	UDF	UMP	MPF	FN
Sofres	LO	LCR	PT	PCF	JB	PS	LV	UDF	UMP	MPF	FN	CPNT
CSA	PT	LO	LCR	JB	PCF	LV	PS	CPNT	UDF	UMP	MPF	FN

Table A.1: Axes used by polling institutes during the 2007 French presidential election.

Institute	Axis ◁									
BVA	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	SP	FN
Harris Interactive	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	SP	FN
IFOP	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	FN	SP
Sofres	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	FN	SP
OpinionWay	SP	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	FN
LH2	SP	LO	NPA	PG	PS	EELV	MoDem	UMP	DLR	FN

Table A.2: Axes used by polling institutes during the 2012 French presidential election.

Institute	Axis ◁										
BVA	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP	R
Opinionway	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP	R
IFOP	LO	NPA	LFI	PS	EM	R	LR	DLF	FN	UPR	SP
IPSOS	LO	NPA	LFI	PS	EM	R	LR	DLF	FN	SP	UPR
Harris Interactive	LO	NPA	LFI	PS	EM	R	LR	DLF	SP	UPR	FN
Odoxa	LO	NPA	LFI	PS	EM	R	LR	DLF	UPR	SP	FN
Elabe	NPA	LO	LFI	PS	EM	R	LR	UPR	DLF	FN	SP

Table A.3: Axes used by polling institutes for the 2017 French presidential election.

Institute	Axis ◁											
BVA	LO	NPA	LFI	PCF	PS	EELV	LREM	LR	DLF	REC	RN	R
Opinionway	LO	NPA	PCF	LFI	PS	EELV	LREM	LR	R	DLF	REC	RN
IFOP	LO	NPA	PCF	LFI	PS	EELV	LREM	LR	DLF	RN	REC	R
IPSOS	NPA	LO	LFI	PCF	EELV	PS	LREM	LR	R	RN	DLF	REC
Harris Interactive	LO	NPA	PCF	LFI	PS	EELV	LREM	LR	DLF	RN	REC	R
Cluster17	LO	NPA	PCF	LFI	EELV	PS	LREM	R	LR	DLF	RN	REC
Odoxa	LO	NPA	PCF	LFI	EELV	PS	LREM	R	LR	DLF	RN	REC
Elabe	NPA	LO	PCF	LFI	PS	EELV	LREM	LR	DLF	RN	REC	R

Table A.4: Axes used by polling institutes during the 2022 French presidential election.

Surveys

Year	Scale	Axis ◁							
2014	Ideology	PG	PCF	EELV	PS	MoDem	UMP	MPF	FN
	Economic	PG	PCF	EELV	PS	FN	MoDem	UMP	MPF
2010	Ideology		PCF	EELV	PS	MoDem	UMP	MPF	FN
	Economic		PCF	PS	EELV	MoDem	UMP	MPF	FN
2006	Ideology		PCF	EELV	PS	UDF	UMP	MPF	FN
	Economic		PCF	PS	EELV	UDF	FN	UMP	MPF

Table A.5: Axes derived from the answers to the Chapel Hill Expert Survey.

Year	Axis ◁							
2017			LFI	PS		EM	LR	FN
2012			PG	PS	EELV	MoDem	UMP	FN
2007	LCR	PCF		PS	LV	UDF	UMP	FN

Table A.6: Axes derived from the answers to the CSES.

Axes Returned by our Rules

Dataset	Rule	Axis ◁											
Louvigny	VD, MF	PT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN	CPNT
	BC, MS	PT	LO	PCF	JB	LV	LCR	PS	UDF	UMP	MPF	FN	CPNT
	FT	PT	LO	PCF	JB	LCR	PS	LV	UDF	UMP	MPF	FN	CPNT
Cigné	VD,MF*	CPNT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN	PT
	MF*	PT	CPNT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN
	MF*	CPNT	PT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN
	BC	PT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN	CPNT
	MS, FT	LO	PCF	LV	JB	LCR	PS	UDF	UMP	MPF	FN	CPNT	PT
Illkirch	VD	PT	CPNT	PCF	LO	JB	LCR	LV	PS	UDF	UMP	FN	MPF
	MF*	PT	PCF	LV	JB	LO	LCR	PS	UDF	UMP	FN	MPF	CPNT
	MF*	CPNT	PCF	LV	JB	LO	LCR	PS	UDF	UMP	FN	MPF	PT
	BC,MS	PCF	LO	JB	LV	LCR	PS	UDF	UMP	FN	MPF	CPNT	PT
	FT*	PCF	JB	LO	LV	LCR	PS	UDF	UMP	FN	MPF	CPNT	PT
	FT*	PCF	LO	JB	LV	LCR	PS	UDF	UMP	FN	MPF	CPNT	PT

Table A.7: Results for the 2007 French presidential election. The '*' indicate ties between axes.

Dataset	Rule	Axis ◁									
Strasbourg	VD*, MF	LO	NPA	EELV	PG	PS	MoDem	UMP	FN	DLR	SP
	VD*, BC, MS, FT	SP	LO	NPA	EELV	PG	PS	MoDem	UMP	FN	DLR
Louvigny	VD	SP	LO	NPA	EELV	PG	PS	MoDem	UMP	FN	DLR
	MF, BC, MS, FT	LO	NPA	EELV	PG	PS	MoDem	UMP	FN	DLR	SP
Saint Etienne	VD, MF, BC, MS	SP	LO	NPA	EELV	PG	PS	MoDem	UMP	FN	DLR
	FT	SP	LO	NPA	EELV	PG	PS	MoDem	FN	UMP	DLR

Table A.8: Results for the 2012 French presidential election. The '*' indicate ties between axes.

Dataset	Rule	Axis ◁										
Strasbourg	VD, MF	R	LO	NPA	LFI	PS	EM	LR	DLF	FN	UPR	SP
	BC, MF	R	LO	NPA	PS	LFI	EM	LR	FN	DLF	UPR	SP
	FT	R	LO	NPA	PS	LFI	EM	LR	DLF	FN	UPR	SP
HSC	VD, MF	SP	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R
	BC	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	SP
	MS, FT	SP	UPR	LO	NPA	LFI	PS	EM	LR	FN	DLF	R
Grenoble	VD, MF	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	R	SP
	BC, MS, FT	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
Crolles 1	VD, MF	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	SP	R
	BC, MS	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
	FT	LO	NPA	PS	LFI	EM	LR	FN	DLF	R	UPR	SP
Crolles 2	VD	R	LO	NPA	LFI	PS	EM	LR	FN	DLF	SP	UPR
	MF	R	LO	NPA	LFI	PS	EM	LR	FN	DLF	UPR	SP
	BC	LO	NPA	LFI	PS	EM	LR	FN	DLF	R	UPR	SP
Online	VD	R	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	SP
	MF	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	R	SP
	BC, MS	LO	NPA	LFI	PS	EM	LR	DLF	RN	R	UPR	SP
	FT	LO	NPA	PS	LFI	EM	R	LR	DLF	RN	UPR	SP

Table A.9: Results for the 2017 French presidential election.

Dataset	Rule	Axis ◁											
Strasbourg	VD	PCF	LO	NPA	LFI	EELV	PS	LREM	LR	REC	RN	R	DLF
	MF	PCF	LO	NPA	LFI	EELV	PS	LREM	LR	R	RN	REC	DLF
	BC, MS	LO	NPA	PCF	PS	LFI	EELV	LREM	LR	RN	R	REC	DLF
	FT	LO	NPA	PCF	PS	LFI	EELV	LREM	LR	R	RN	REC	DLF
Representative	VD	PCF	LO	NPA	LFI	EELV	PS	LREM	LR	RN	REC	DLF	R
	MF	LR	LREM	PS	EELV	LFI	NPA	LO	PCF	R	DLF	RN	REC
	BC	LO	NPA	PCF	PS	EELV	LFI	LREM	LR	R	RN	REC	DLF
	MS	LO	PS	NPA	PCF	EELV	LFI	R	LREM	LR	RN	REC	DLF
	FT	LO	NPA	PS	EELV	PCF	LFI	R	LREM	LR	RN	REC	DLF
Online	VD	PCF	LO	NPA	LFI	EELV	PS	LREM	LR	DLF	REC	RN	R
	MF	LO	NPA	LFI	PCF	PS	EELV	LREM	LR	R	RN	REC	DLF
	BC	LO	NPA	PCF	LFI	EELV	PS	LREM	LR	R	RN	REC	DLF
	MS	LO	NPA	PCF	LFI	PS	EELV	LREM	LR	R	RN	REC	DLF
	FT	LO	NPA	LFI	PCF	PS	EELV	LREM	LR	R	RN	REC	DLF
	VD-rank	DLF	R	PCF	LO	NPA	LFI	EELV	PS	EM	LR	RN	REC
	FT-rank	LO	NPA	PCF	LFI	PS	EELV	EM	LR	R	RN	DLF	REC

Table A.10: Results for the 2022 French presidential election

Tierlists

Rule	VD	MF	BC	MS	FT
Score	0.60	0.75	1.58	1.64	3.20
	Feb.	Mar.	Mar.	Mar.	Mar.
	Mar.	Apr.	Apr.	Apr.	Apr.
	Apr.	May	May	May	May
	May	Aug.	Aug.	Aug.	Aug.
	Aug.	Jun.	Jun.	Jun.	Jun.
	Jun.	Jul.	Jul.	Jul.	Jul.
	Jul.	Dec.	Dec.	Dec.	Dec.
	Dec.	Oct.	Oct.	Oct.	Oct.
	Oct.	Nov.	Nov.	Nov.	Nov.
	Nov.	Jan.	Sep.	Sep.	Sep.
	Sep.	Feb.	Jan.	Jan.	Jan.
	Jan.	Sep.	Feb.	Feb.	Feb.

Table A.11: Ordering of the months given by the different rules. Each color corresponds to a season.

Rule	VD	MF	BC	MS	FT
Score	0.47	0.53	0.90	0.93	1.5
	Pluto	Pluto	Pluto	Mercury	Pluto
	Jupiter	Mars	Mercury	Uranus	Mercury
	Earth	Jupiter	Mars	Neptune	Jupiter
	Saturn	Earth	Earth	Saturn	Saturn
	Neptune	Saturn	Saturn	Earth	Earth
	Uranus	Neptune	Jupiter	Jupiter	Neptune
	Venus	Uranus	Neptune	Mars	Uranus
	Mars	Venus	Uranus	Venus	Mars
	Mercury	Mercury	Venus	Pluto	Venus

Table A.12: Ordering of the planets given by the different rules. Red corresponds to rocky planets and blue to gas planets.

Rule	VD	MF	BC	MS	FT
Score	0.58	0.72	1.5	1.62	2.88
	8	8	8	8	8
	6	3	6	6	6
	9	6	9	9	9
	7	9	7	3	3
	3	7	3	7	7
	1	4	4	4	4
	2	2	2	2	2
	4	1	1	1	1
	5	10	10	10	10
	10	5	5	5	5

Table A.13: Ordering of the numbers from 1 to 10 given by the different rules.

Rule	VD	MF	BC	MS	FT
Score	0.60	0.69	1.10	1.19	1.98
	Ep. 2	Ep. 2	Ep. 2	Ep. 2	Ep. 2
	Ep. 1	Ep. 1	Ep. 1	Ep. 1	Ep. 1
	Ep. 4	Ep. 5	Ep. 4	Ep. 4	Ep. 4
	Ep. 3	Ep. 7.2	Ep. 3	Ep. 3	Ep. 3
	Ep. 7.2	Ep. 3	Ep. 7.2	Ep. 7.2	Ep. 7.2
	Ep. 5	Ep. 4	Ep. 5	Ep. 5	Ep. 5
	Ep. 7.1	Ep. 6	Ep. 6	Ep. 6	Ep. 6
	Ep. 6	Ep. 7.1	Ep. 7.1	Ep. 7.1	Ep. 7.1

Table A.14: Ordering of the Harry Potter movies given by the different rules. Red corresponds to 2001–2005 movies and blue to 2007–2011 movies.

Rule	VD	MF	BC	MS	FT
Score	0.46	0.53	0.87	0.91	1.68
	Solo	Solo	Solo	Solo	Solo
	Ep. II	Ep. II	Ep. II	Ep. II	Ep. II
	Ep. I	Ep. I	Ep. I	Ep. I	Ep. I
	Ep. VI	Ep. VI	Rogue One	Rogue One	Rogue One
	Ep. IV	Ep. IV	Ep. III	Ep. III	Ep. III
	Ep. V	Ep. V	Ep. V	Ep. V	Ep. V
	Ep. III	Ep. III	Ep. IV	Ep. IV	Ep. IV
	Rogue One	Rogue One	Ep. VI	Ep. VI	Ep. VI
	Ep. VIII	Ep. VIII	Ep. VIII	Ep. VIII	Ep. VIII
	Ep. VII	Ep. VII	Ep. VII	Ep. VII	Ep. VII
	Ep. IX	Ep. IX	Ep. IX	Ep. IX	Ep. IX

Table A.15: Ordering of the Star Wars movies given by the different rules. Red corresponds to the prequels, green to the original trilogy, and blue to the sequels.

Rule	VD	MF	BC	MS	FT
Score	0.40	0.49	0.83	0.91	1.91
	Amazing SM 2	Venom	Venom	Venom	Venom
	Amazing SM	Amazing SM	Amazing SM 2	Amazing SM 2	Amazing SM 2
	Venom	Amazing SM 2	Amazing SM	Amazing SM	Amazing SM
	SM 3	No way home	SM 3	SM 3	SM 3
	SM	Spiderverse	SM	SM	SM
	SM 2	SM 2	SM 2	SM 2	SM 2
	Spiderverse	SM	Spiderverse	Spiderverse	Spiderverse
	No way home	SM 3	No way home	No way home	No way home
	Homecoming	Homecoming	Homecoming	Homecoming	Homecoming
	Far from home	Far from home	Far from home	Far from home	Far from home

Table A.16: Ordering of the Spiderman movies given by the different rules. Red corresponds to the Sam Raimi trilogy, green to the MCU movies, and blue to the Amazing Spiderman movies.

Rule	VD	MF	BC	MS	FT
Score	0.67	0.93	1.69	1.89	4.34
	Fearless (T)	Fearless	Fearless	Fearless	Fearless
	Fearless	Red	Red	Red	Red
	Red	Speak Now	Fearless (T)	Fearless (T)	Speak Now
	Speak Now	Red (T)	Speak Now	Speak Now	Red (T)
	Folklore	Evermore	Evermore	Red (T)	Evermore
	Evermore	Folklore	Folklore	Evermore	Folklore
	Red (T)	Reputation	Red (T)	Folklore	1989
	Midnights	1989	1989	1989	Reputation
	Reputation	Midnights	Reputation	Reputation	Midnights
	1989	Lover	Midnights	Midnights	Lover
	Lover	Fearless (T)	Lover	Lover	Fearless (T)
	Taylor Swift	Taylor Swift	Taylor Swift	Taylor Swift	Taylor Swift

Table A.17: Ordering of the Taylor Swift albums given by the different rules. Red corresponds to the 2008-2014 albums, blue to the 2017–2022 albums.

Rule	VD	MF	BC	MS	FT
Score	0.72	0.91	1.74	1.93	3.90
	Gray	Gray	Gray	Brown	Gray
	White	White	White	Gray	Pink
	Black	Black	Black	Green	White
	Red	Red	Red	White	Black
	Green	Green	Purple	Black	Purple
	Blue	Blue	Blue	Red	Blue
	Purple	Purple	Green	Blue	Red
	Pink	Pink	Pink	Purple	Green
	Orange	Orange	Orange	Pink	Yellow
	Yellow	Yellow	Yellow	Orange	Orange
	Brown	Brown	Brown	Yellow	Brown

Table A.18: Ordering of the colors given by the different rules.

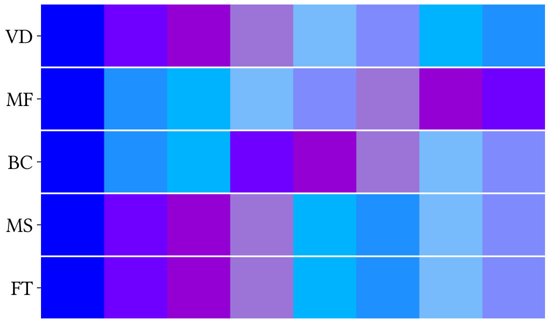
Rule	VD	MF	BC	MS	FT
Score	0.68	0.85	2.10	2.25	4.03
	Literature	Chemistry	Chemistry	Chemistry	Chemistry
	Art	Physics	Physics	Physics	Physics
	Technology	Math	Math	Math	Math
	Math	Technology	Technology	Technology	Technology
	Physics	PE	Music	Music	Music
	Chemistry	Music	Art	PE	Art
	Music	Art	PE	Art	PE
	PE	Literature	History	History	History
	Social studies	Social studies	Social studies	Social studies	Social studies
	History	History	Foreign language	Foreign language	Foreign language
	Foreign language	Foreign language	Literature	Literature	Literature

Table A.19: Ordering of the school courses given by the different rules. Red corresponds to the humanities and blue to the sciences.

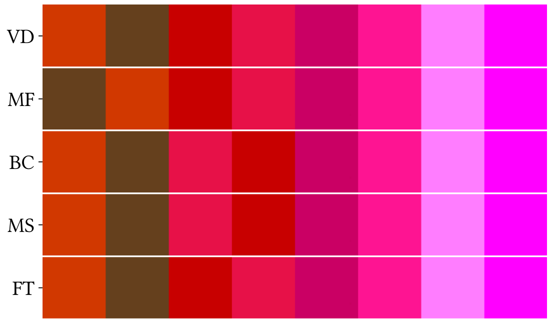
Rule	VD	MF	BC	MS	FT
Score	0.66	0.91	1.77	2.09	5.15
	Italy	Italy	Italy	Italy	Italy
	France	France	France	France	France
	UK	UK	UK	UK	UK
	Germany	Germany	Germany	Germany	Germany
	Switzerland	Sweden	Sweden	Switzerland	Switzerland
	Sweden	Switzerland	Switzerland	Sweden	Sweden
	Norway	Norway	Norway	Norway	Norway
	Denmark	Denmark	Iceland	Iceland	Denmark
	Finland	Finland	Finland	Finland	Finland
	Iceland	Iceland	Denmark	Denmark	Iceland

Table A.20: Ordering of the European countries given by the different rules. Blue indicates to the Nordic countries.

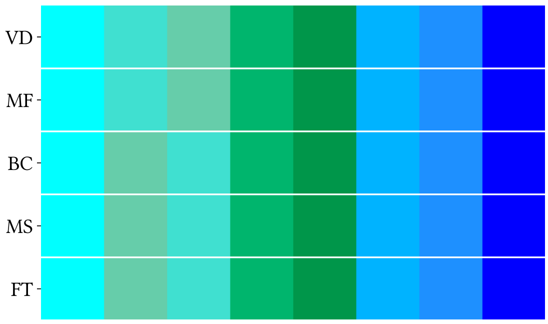
Colors Tierlists



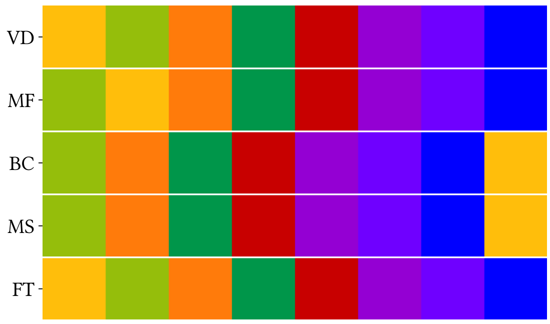
(a) Nuances of blue (138 voters).



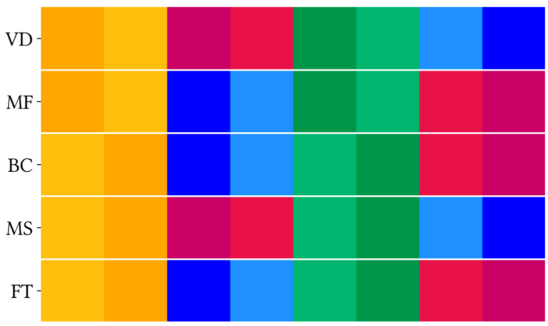
(b) Nuances of red (85 voters).



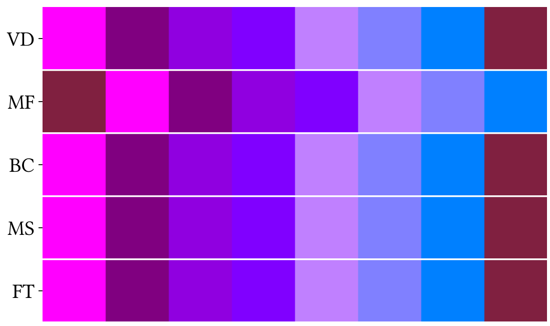
(c) From blue to green (124 voters).



(d) Colors of the rainbow (130 voters).

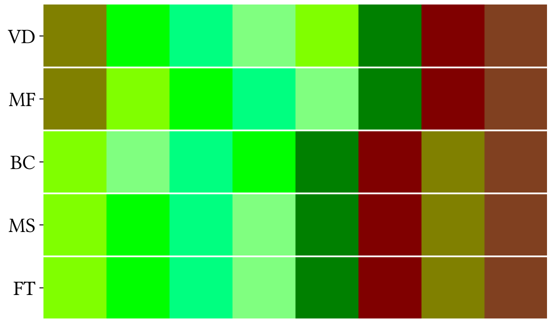


(e) Clones of candidates (119 voters).

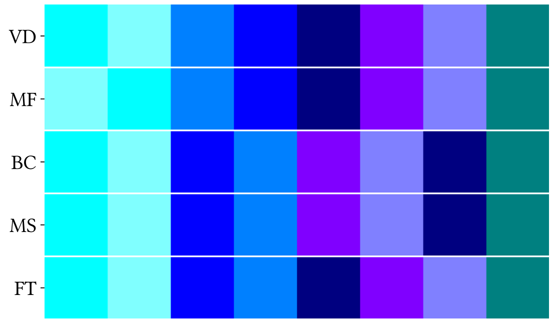


(f) From blue to purple (121 voters).

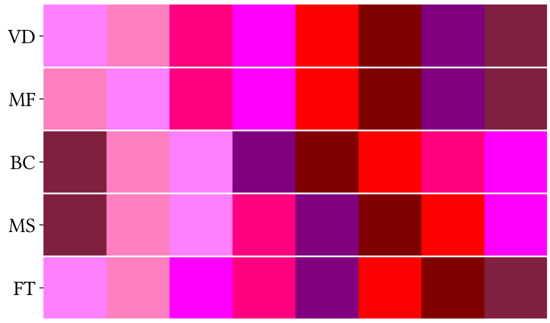
Figure A.1: Axes of colors for some subsets of eight colors, obtained using the different axis rules.



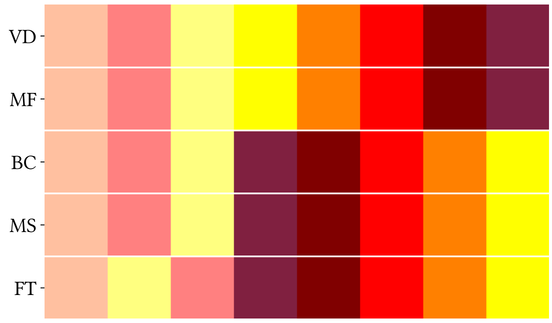
(a) Green to brown (61 voters).



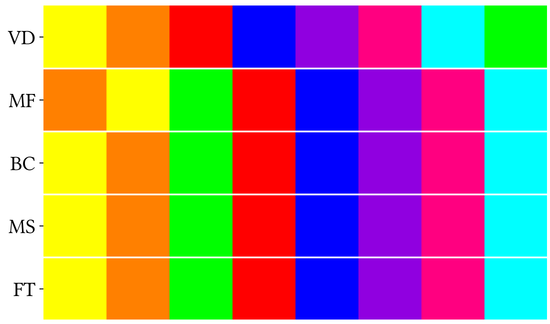
(b) Nuances of blue (113 voters).



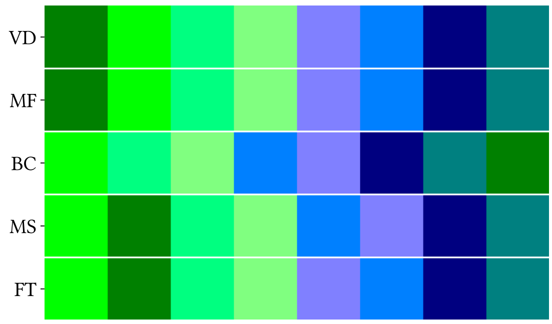
(c) From pink to purple (83 voters).



(d) From yellow to red (76 voters).



(e) Colors of the rainbow (101 voters).



(f) From green to blue (104 voters).

Figure A.2: Axes of colors for some subsets of eight colors, obtained using the different axis rules.

Appendix B

Résumé long en français

B.1 Introduction

Les systèmes de vote que nous connaissons et que nous utilisons régulièrement pour les élections politiques majeures comme les présidentielles ou les législatives souffrent de nombreux défauts, largement documentés et bien connus des électeurs : division des voix entre candidats proches, dilemme du vote « utile », ou encore le fait que les votes pour les partis sous le seuil électoral soient ignorés dans les élections parlementaires proportionnelles. Beaucoup de ces défauts découlent du format des bulletins de vote, souvent trop peu expressifs : en général, les électeurs ne peuvent choisir qu'un seul candidat. Pourquoi ne pas plutôt permettre aux électeurs de s'exprimer sur plusieurs candidats, en indiquant par exemple lesquels ils approuvent parmi la liste qui leur est proposée, ou quel est leur classement des candidats ? Dans la première partie de cette thèse, nous considérons sérieusement ces possibilités.

Une fois que l'on commence à s'intéresser aux formats de bulletins de vote expressifs comme les bulletins d'approbation ou les classements dans le cadre des systèmes de vote, on remarque que ces bulletins peuvent également être très utiles pour *décrire* le paysage politique et la structure de l'électorat. En particulier, ils permettent de facilement observer que certains candidats sont *proches*, typiquement s'ils sont appréciés par les mêmes électeurs. Partant de cette idée, dans la seconde partie de cette thèse, nous étudions des problèmes dont l'objectif est de construire des outils d'analyse politique à partir des préférences des électeurs.

Choix social et préférences expressives

Formellement, un problème de choix social est défini par un ensemble de *candidats*, un ensemble de *votants* qui expriment leurs *préférences* sur les candidats (à l'aide de *bulletins de vote*), et d'un ensemble de résultats possibles. Par exemple, dans une élection présidentielle, les résultats possibles sont les différents candidats.

Il existe plusieurs formats de bulletins de vote pour exprimer des préférences : *uninominal* (chaque votant choisit un seul candidat), *d'approbation* (les votants peuvent choisir plusieurs candidats), *par classement* (les votants classent les candidats par ordre de préférence), ou encore *cardinal* (les votants associent une note à chaque candidat). Dès lors que l'on utilise des bulletins au format plus *expressif* que le format uninominal, la littérature académique a montré que l'on peut définir de nombreuses méthodes intéressantes pour la sélection de gagnants.

Une fois que le problème est bien défini et que des méthodes d'agrégation de préférences ont été proposées, la question est alors de savoir *laquelle de ces méthodes devrait-on utiliser en pratique ?* Il n'y a généralement pas de réponse toute faite, et chaque méthode s'accompagne d'avantages et d'inconvénients. Il nous faut alors des moyens d'évaluer et de comparer ces méthodes. Différentes approches existent, et dans cette thèse, nous allons nous concentrer sur deux d'entre elles.

Une des approches les plus anciennes d'étude des méthodes d'agrégation de préférence en choix social est l'*analyse axiomatique*, qui repose sur la définition de propriétés considérées comme désirables pour ces méthodes. On vérifie alors quelles méthodes satisfont quelles propriétés. Avec la normalisation de l'informatique et des ordinateurs au cours du XXe siècle, l'*analyse expérimentale* est devenue un autre outil incontournable pour étudier les méthodes d'agrégation. L'idée était d'abord de simuler des profils de préférences à l'aide de modèles probabilistes, puis de tester les méthodes sur ces profils artificiels. Enfin, de plus en plus d'analyses expérimentales se font maintenant aussi sur des données *réelles*, puisque de nombreux jeux de données de préférences ont été collectés ces dernières années. Ces deux approches (axiomatique et expérimentale) sont complémentaires, et permettent de mieux comprendre les méthodes d'agrégation, ce qui est essentiel pour choisir laquelle de ces méthodes utiliser en pratique.

Aperçu du contenu de la thèse

Les contributions de cette thèse sont organisées en deux parties. Dans la première partie, nous étudierons comment des bulletins de vote plus expressifs peuvent servir à améliorer des systèmes de vote utilisés en pratique. Dans la seconde partie, nous verrons comment il est possible d'utiliser les bulletins de vote expressifs pour construire des outils d'analyse et de description du paysage politique.

Malgré la diversité des problèmes étudiés dans cette thèse, nous verrons dans la Section B.2 que ces problèmes ont de nombreuses similarités. En particulier, pour évaluer les méthodes proposées, nous conduisons systématiquement une analyse axiomatique et une analyse expérimentale.

Partie I : Améliorer les systèmes de vote. Cette première partie est consacrée à l'étude de trois problèmes de vote, tous motivés par des applications concrètes. Dans la Section B.3, nous étudierons *le vote par approbation à deux tours*, une amélioration du scrutin uninominal majoritaire à deux tours, qui est très utilisé pour des élections à vainqueur unique (notamment en France), mais qui souffre de nombreux défauts à la fois en théorie et en pratique. Notre proposition est de remplacer les bulletins *uninominaux* du premier tour par des bulletins d'*approbation*. Dans la Section B.4, nous étudierons *le vote alternatif*, un système de vote utilisé dans de nombreux pays (comme l'Irlande ou l'Australie), qui est basé sur des bulletins de vote par classement. Nous proposerons alors de généraliser ce mode de scrutin en permettant aux électeurs de voter avec des *préordres complets*, c'est-à-dire des classements qui peuvent contenir des égalités. Ce type de bulletin est plus expressif que les bulletins de vote par classement, et permet aux électeurs d'indiquer lorsqu'ils sont indifférents entre certains candidats. Enfin, dans la Section B.5, nous nous pencherons sur le problème des « votes perdus » dans les élections parlementaires avec seuil électoral (comme en France pour les élections Européennes, ou en Allemagne), et nous verrons comment il est possible d'améliorer la représentation des électeurs en leur permettant de voter avec des *classements (tronqués)*.

Partie II : Construire des outils d'analyse du paysage politique. Cette seconde partie est quant à elle consacrée à l'étude de deux problèmes dans lesquels l'objectif est d'agréger les préférences des électeurs pour obtenir une représentation de certains aspects du paysage politique.

Dans la Section B.6, nous verrons comment il est possible d'utiliser les préférences d'approbation des électeurs pour construire un axe *gauche-droite* (c'est-à-dire un axe unidimensionnel) des candidats, à l'aide de méthodes explicables et interprétables. Enfin, dans la Section B.7, nous proposerons des méthodes d'identification des paires de candidats induisant le plus de *conflit* au sein de l'électorat, à partir des classements des votants.

Chacun des cinq problèmes de cette thèse repose sur un article scientifique publié dans une conférence internationale (AAAI, IJCAI ou EC), bien que chaque chapitre ait été ici réécrit pour rendre la thèse plus cohérente et agréable à lire. D'autres travaux de recherche ont été réalisés durant cette thèse, mais ont été omis pour des raisons de cohérence et de longueur de la thèse. Cela inclut cinq articles publiés dans des conférences internationales, ainsi que deux articles en cours de soumission dans des revues. Ces articles sont énumérés dans la Section 1.3.

B.2 Outils d'analyse du choix social computationnel

Chaque chapitre de cette thèse est consacré à l'étude d'un problème d'agrégation de préférences. Malgré leur diversité, ces problèmes partagent cinq aspects majeurs : (1) Pour chaque problème, nous avons à disposition les *préférences* des électeurs (ou *votants*) sur des *candidats*; (2) L'objectif principal est d'agréger ces préférences pour obtenir un *résultat* d'un certain type (par exemple, choisir le meilleur candidat); (3) Pour ce faire, nous définissons des *règles* (ou *méthodes*) qui prennent en entrée les préférences des votants et retournent un résultat du type attendu. Enfin, nous comparons ces règles entre elles au travers (4) d'une analyse *axiomatique* et (5) d'une analyse *expérimentale*. Dans cette première partie, nous présentons ces différents aspects.

Les préférences

Soit un ensemble V de votants de taille $|V| = n$ et un ensemble C de candidats de taille $|C| = m$. On suppose que l'on connaît le profil $P = (X_1, \dots, X_n)$ de préférences des votants sur les candidats, exprimées dans un certain format $X_i \in \mathcal{X}$ (où $\mathcal{X}$ est le *domaine* des préférences). Le format des préférences (généralement exprimées à l'aide de bulletins de vote) joue un rôle crucial dans la manière dont un problème peut être résolu. Ci-dessous, nous passons en revue les différents formats possibles.

Préférences uninominales. Le format le plus basique (et le plus ancien) d'expression de préférence est le format *uninominal*, qui consiste pour un votant à ne donner le nom que d'un seul candidat, généralement celui qu'il préfère.

Classements. Un autre format très courant est celui des *classements* (notés $\succ_1, \dots, \succ_n$). Dans leurs classements, les votants ordonnent les candidats par ordre de préférence (on parle également de *préférences ordinales*). Formllement, un classement $\succ_i$ est un *ordre strict total.*

Bulletins d'approbation. Dans les *bulletins d'approbation* (notés $A_1, \dots, A_n$), les votants peuvent sélectionner plusieurs candidats à la fois (on dit alors que ces candidats sont *approuvés* par le votant). Formellement, $A_i \subseteq C$ est un sous-ensemble de l'ensemble des candidats, sans contrainte de taille. Ce format est très ancien, et était notamment utilisé pour les conclaves papaux entre 1294 et 1691 (Colomer and McLean, 1998).

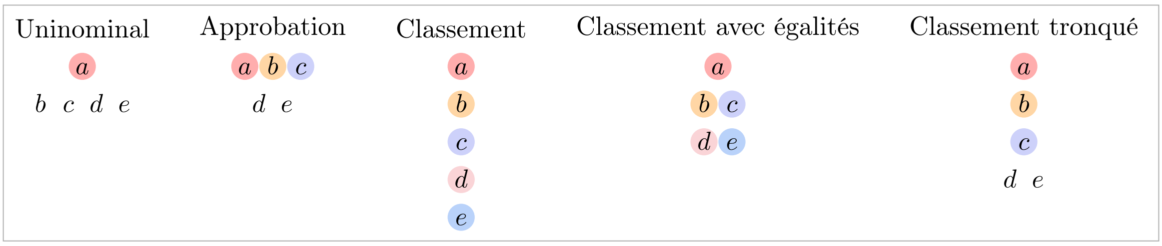


Figure B.1: Exemples de bulletins de différents formats.

Classements avec égalités. D'un côté, les classements permettent à l'électeur de s'exprimer sur tous les candidats, mais ne permettent pas d'exprimer des indifférences. À l'inverse, les bulletins d'approbation créent deux niveaux d'indifférence (les candidats approuvés et les non-approuvés), mais ne permettent pas de hiérarchiser les candidats au sein de ces deux groupes. Les *classements avec égalités* (notés $\succcurlyeq_1, \dots, \succcurlyeq_n$) sont un compromis entre les deux, et conservent les avantages des deux formats. Un classement avec égalités est un classement dans lequel plusieurs candidats peuvent être classés à la même position (ce qui permet d'exprimer des indifférences). Formellement, un classement avec égalités $\succcurlyeq_i$ est un *préordre total* (en anglais, on parle de *weak order*). Les classements et les bulletins d'approbation sont des cas particuliers des classements avec égalités, mais il y en a d'autres, comme les *classements tronqués*, où un votant ne classe qu'une partie des candidats. La Figure B.1 donne des exemples de bulletins de vote dans ces différents formats.

Évaluations. Enfin, on parle d'*évaluation* lorsqu'à partir d'un ensemble (ordonné) Z de « notes » possibles, un votant associe une note à chaque candidat. Des exemples typiques sont les avis en ligne sur de nombreux sites (avec $Z = \{0, 1, 2, 3, 4, 5\}$ pour le nombre d'étoiles) ou bien les mentions au bac (avec $Z = \{\text{Très bien}, \text{Bien}, \text{Assez Bien}, \text{Passable}, \text{Insuffisant}\}$). Il est également possible de prendre un ensemble continu, comme $Z = \mathbb{R}$, ou $Z = [0, 1]$. On parle aussi parfois d'utilité, de coût, de score, ou de préférences cardinales.

Familles de problèmes et règles de vote

Nous venons de voir différents formats de préférences (ou de bulletins de vote) possibles, intéressons-nous maintenant à ce que l'on peut souhaiter obtenir comme résultat en *agrégeant* ces préférences. Un problème typique est le choix d'un vainqueur unique d'une élection. Cependant, le résultat désiré n'est pas forcément une sélection de vainqueur, et nous considérons également dans cette thèse des problèmes dont l'objectif est d'obtenir une *description* de la structure de l'électorat.

Une fois que le type de résultat souhaité est bien déterminé, on peut définir des règles (ou méthodes) d'agrégation, prenant en entrée un profil de préférences $P = (X_1, \dots, X_n)$ et retournant un résultat $O \in \mathcal{O}$, où $\mathcal{O}$ est l'ensemble des résultats possibles. Par exemple, dans une élection à vainqueur unique, $\mathcal{O} = C$ l'ensemble des candidats. Si l'on autorise la règle à retourner plusieurs résultats possibles (par exemple des ex aequo), on dit que la règle est *irrésolue*. Si à l'inverse, la règle ne retourne toujours qu'un seul résultat, on dit qu'elle est *résolue*. Deux problèmes en particulier ont été très étudiés dans la littérature du choix social, et sont au cœur de cette thèse : les *élections à vainqueur unique* et les *élections de comités* (ou *élections à vainqueurs multiples*).

Élections à vainqueur unique.

Ce problème est sûrement le plus étudié de la littérature du choix social. L'objectif est de sélectionner un candidat $c \in C$ à partir des votes des électeurs. Lorsqu'il n'y a que deux candidats, la solution est simple et une seule règle satisfait un ensemble de propriétés désirables (May, 1952) : la majorité, qui retourne le candidat recevant le plus de voix entre les deux. Les choses deviennent plus complexes lorsque l'on a plus de deux candidats.

Bulletins uninominaux. Avec des bulletins *uninominaux*, il n'y a pas beaucoup de possibilités, la règle la plus sensée étant la *Pluralité*, qui consiste à élire le candidat recevant le plus de voix (on parle alors de *score de pluralité*). C'est par exemple le système utilisé au Royaume-Uni pour élire les membres du Parlement.

Bulletins d'approbation. Avec des bulletins d'*approbation*, on peut de la même manière utiliser la règle d'*approbation*, qui consiste à élire le candidat ayant reçu le plus de voix. Cette règle est connue pour satisfaire de nombreuses propriétés intéressantes (Brandl and Peters, 2022). Il existe cependant des variantes, comme l'*approbation divisée* (*split approval voting* en anglais), où chaque votant distribue au total un point, qu'il divise équitablement entre les candidats qu'il approuve.

Classements. Avec des *classements*, il existe cette fois de très nombreuses règles de vote :

- Parmi les plus étudiées, il y a les *règles de scores positionnelles*, qui attribuent un score à chaque candidat en fonction de sa position dans le classement de chaque votant. Le gagnant est alors le candidat avec le plus grand score total. L'exemple le plus connu est la règle de *Borda*, où chaque votant attribue les scores $(m-1, m-2, \ldots, 1, 0)$ aux candidats classés respectivement $1, 2, \ldots, m$ dans son classement.

- Une autre grande famille de règles est celle des *règles de score à éliminations successives*, où l'on utilise également des vecteurs de scores pour calculer le score de chaque candidat, mais où l'on élimine successivement les candidats ayant le plus petit score. Après chaque élimination, les classements sont modifiés pour tenir compte de l'élimination. L'exemple le plus connu est celui du *vote alternatif* (*instant runoff voting* en anglais), qui est utilisé dans de nombreux pays, comme l'Australie et l'Irlande. Une autre règle s'inscrit également dans cette idée d'élimination : le *vote uninominal majoritaire à deux tours*, qui est le système utilisé en France pour les élections présidentielles et législatives. Dans ce système, si aucun candidat n'obtient plus de 50% des voix au premier tour, les deux candidats (parfois plus) ayant obtenu le plus de voix se qualifient pour un second tour, où le candidat recevant le plus de voix est élu.

- Une autre grande famille de règles est celle des *règles de Condorcet*, qui élisent nécessairement le gagnant de Condorcet s'il existe. On dit qu'un candidat est un *gagnant de Condorcet* s'il est préféré à tous les autres candidats par la majorité des votants (dans un duel). Puisqu'il ne peut jamais y avoir une majorité de votants qui s'accorde pour choisir un autre candidat à la place d'un gagnant de Condorcet, ils sont souvent considérés comme des gagnants naturels. Cependant, il n'existe pas nécessairement de gagnant de Condorcet : il peut y avoir un cycle de domination. De nombreuses règles ont alors été proposées pour généraliser le principe de Condorcet aux cas où il n'existe pas de tel candidat. C'est le cas par exemple des règles de *Copeland* (Copeland, 1951), de *Minimax* (Young, 1977) ou de *Ranked Pairs* (Tideman, 1987).

Il existe bien sûr de nombreuses autres règles, comme la règle de *Kemeny* (Kemeny, 1959).

Classements avec égalités. Lorsque les préférences sont des *classements avec égalités* (c'est-à-dire des préordres complets), il est souvent possible de généraliser les règles définies pour les classements. Cela est particulièrement vrai si les classements en question sont des *classements tronqués*.

Évaluations. Enfin, il existe plusieurs manières d'agréger des *évaluation* pour sélectionner un candidat. Il est par exemple possible de choisir le candidat ayant la meilleure évaluation *médiane* (comme dans le *Jugement Majoritaire* de Balinski and Laraki (2011)), ou bien celui ayant la meilleure évaluation *minimale* (c'est le principe *égalitariste*). Enfin, si les évaluations sont numériques, on peut également choisir le candidat ayant la meilleure *moyenne* (c'est le principe *utilitariste*).

Élections de comités.

Regardons maintenant les différentes règles qui ont été introduites pour les élections à vainqueurs multiples, ou *élections de comités.* Dans ce genre d'élection, la taille du comité est généralement fixée à l'avance, on la note $k \geqslant 2$. Les élections de comités peuvent suivre différents principes, dont trois principaux ont été identifiés par Elkind et al. (2017b) : (1) l'*excellence*, où l'on cherche à sélectionner les candidats qui sont les meilleurs individuellement; (2) la *proportionalité*, où l'on cherche à représenter les groupes de votants proportionnellement à leur taille; et (3) la *diversité*, où l'on cherche à satisfaire le plus grand nombre de votants. De nombreuses règles d'élection de comités ont été proposées dans la littérature :

Bulletins uninominaux. Si les préférences sont *uninominales*, on peut choisir les k candidats ayant reçu le plus de voix, c'est le *vote non-transférable* (SNTV).

Bulletins d'approbation. Avec des bulletins d'*approbation*, il existe un grand nombre de règles. Tout d'abord, on peut généraliser les règles d'élection à vainqueur unique en sélectionnant k candidats au lieu d'un. On peut également utiliser les règles de *Thiele* ou leurs versions séquentielles (Thiele, 1895), pour lesquelles un votant reçoit de moins en moins de satisfaction pour chaque candidat en plus qu'il approuve dans le comité. Enfin, un très grand nombre de règles ont été proposées pour des élections proportionnelles, comme les *règles de Phragmén* (Phragmén, 1894) ou la *méthode des parts égales* (Peters and Skowron, 2020). Nous recommandons le livre de Lackner and Skowron (2023) pour une revue de la littérature sur les élections de comités par approbation.

Classements. Avec des *classements*, on peut également généraliser les règles d'élection à vainqueur unique. Par exemple, la règle de *k-Borda*, basée sur les scores de Borda, est intéressante si l'on veut satisfaire le critère d'excellence. La règle de *vote transférable* (STV), généralisation du *vote alternatif* (IRV) est quant à elle intéressante pour le critère de proportionnalité (Tideman, 1995; Tideman and Richardson, 2000). Enfin, la règle de *Chamberlin-Courant* (Chamberlin and Courant, 1983) est adaptée pour le critère de diversité (Elkind et al., 2017b).

Les règles que nous venons de présenter considèrent toutes que la taille du comité est fixée, mais d'autres règles ont également été proposées pour des élections de comités où la taille du comité n'est *pas fixée à l'avance* (Duddy et al., 2014; Lackner and Maly, 2021; Faliszewski et al., 2020;

Kilgour, 2016; Brandl and Peters, 2019). Enfin, le problème d'élection de comités est également proche de celui d'établir une distribution des sièges dans un parlement à partir de scores (problème d'*apportionment*), très étudié en théorie du choix social (Balinski and Young, 2010; Pukelsheim, 2014).

Axiomes

Une fois que différentes règles ont été proposées pour un problème donné, l'étape suivante est d'évaluer et de comparer ces règles. Les deux outils principaux que nous utilisons dans cette thèse sont l'*analyse axiomatique* et l'*analyse expérimentale*. Concentrons-nous d'abord sur l'analyse axiomatique. L'idée est de définir un ensemble de propriétés désirables, appelées *axiomes*, que l'on souhaiterait que les règles vérifient. Pour un axiome donné, il n'y a pas d'entre-deux : une règle est soit conforme à l'axiome, soit elle ne l'est pas.

En plus de montrer quelles règles satisfont quels axiomes, on trouve fréquemment des *incompatibilités* entre axiomes, c'est-à-dire que l'on prouve que pour un ensemble donné d'axiomes, il ne peut exister de règle qui les satisfont tous. Dans d'autres cas plus positifs, on peut prouver qu'il existe une unique règle de vote qui satisfait un ensemble donné d'axiomes, on parle alors de *caractérisation*. Nous allons maintenant passer rapidement en revue les familles d'axiomes que nous utiliserons dans cette thèse.

Symétrie. La *neutralité* et l'*anonymité* font partie des axiomes les moins controversés du choix social (pour les règles irrésolues, c'est-à-dire pouvant retourner des *ex-aequo*). Le premier impose que le résultat ne dépende pas de l'identité des *candidats*, et le second qu'il ne dépende pas de l'identité des *votants*.

Efficacité et Représentation. Les axiomes d'*efficacité* donnent des contraintes sur le résultat pour certains profils de préférences. Par exemple, pour les élections à vainqueur unique, le *critère de Condorcet* impose que s'il existe un gagnant de Condorcet, alors celui-ci doit être élu. Ainsi, un axiome d'efficacité peut imposer un gagnant (comme c'est le cas ici), mais il peut à l'inverse imposer un (ou des) perdant(s), comme le fait le critère du *perdant de Condorcet*, qui dit qu'un candidat perdant le duel à la majorité contre tous les autres candidats ne devrait pas être élu, ou encore le *principe de Pareto* (Pareto, 1919). Enfin, pour les élections de comités, les axiomes de *représentation* imposent que si un groupe de votants est assez nombreux et assez homogène dans ses préférences, alors ce groupe doit être représenté dans le comité (proportionnellement à sa taille) par ses candidats préférés (Dummett, 1984; Aziz et al., 2017a; Sánchez-Fernández et al., 2017; Aziz et al., 2018; Peters et al., 2021)

Renforcement. L'axiome de renforcement (Smith, 1973; Young, 1974) impose que si l'on obtient le même résultat avec deux profils de préférences P_1 et P_2, alors on doit également obtenir ce résultat si l'on combine les deux profils en un seul $P' = P_1 + P_2$. L'idée est que si un candidat gagne dans une région A et une région B, alors il doit gagner si l'on combine les deux régions.

Axiomes d'indépendance. Les axiomes d'indépendance imposent que le résultat d'une règle reste inchangé lorsque l'on ajoute ou enlève des candidats dans l'élection (et donc dans les préférences des votants). Le plus fort de ces axiomes est l'*indépendance aux perdants* (adaptée de *indépendance aux alternatives non pertinentes* introduite par Arrow (1950)), qui impose que le résultat reste inchangé lorsque l'on enlève un perdant de l'élection. Un axiome plus faible est l'*indépendance aux clones* (Tideman, 1987), qui impose que le résultat reste inchangé lorsque l'on ajoute ou enlève un candidat qui est *un clone* d'un autre candidat dans l'élection. On dit que deux candidats sont *clones* si ils sont classés l'un après l'autre dans tous les classements des votants (dans le cas où les préférences sont des classements), ou s'ils sont toujours approuvés ensemble (dans le cas des bulletins d'approbation).

Vote stratégique et monotonie. Les axiomes de *résistance au vote stratégique* impose qu'un votant ne puisse pas obtenir un résultat qu'il préfère en donnant de fausses préférences. Autrement dit, la meilleure stratégie pour le votant doit être de donner ses préférences *sincères*. Cet axiome étant trop fort dans la plupart des cas (Gibbard, 1973; Satterthwaite, 1975), on restreint généralement les manipulations autorisées pour le votant stratégique. Enfin, l'axiome de *monotonie* (Fishburn, 1982) impose quant à lui que si un candidat gagne une élection pour un profil de préférences P, alors il doit également gagner si l'on modifie les préférences de certains votants de telle sorte que le gagnant reçoivent plus de soutien qu'avant (par exemple, en améliorant sa position dans certains classements).

Expériences

L'analyse axiomatique est un outil puissant pour comparer les règles de vote, mais il est souvent souhaitable de compléter cette approche par une *analyse expérimentale*. L'idée est d'observer le comportement des règles qui ont été introduites en pratique sur des données *réelles*, ou des données *simulées*, générées à partir de modèles probabilistes.

Les objectifs de l'analyse expérimentale sont multiples. D'une part, on veut étudier le comportement des règles, et essayer de relier ces comportements aux résultats de l'analyse axiomatique. Il est également intéressant d'étudier les similarités entre les règles. D'autre part, et lorsque c'est possible, on veut pouvoir comparer la *qualité* des résultats retournés par les règles. Pour cela, il faut une mesure qui permettent de déterminer quel résultat est *objectivement* le meilleur. Une approche possible est celle de la *distorsion* (Procaccia and Rosenschein, 2006; Anshelevich et al., 2021). Enfin, l'analyse du jeu de données en lui-même est parfois très instructive, et peut permettre de mieux comprendre la structure des préférences des votants.

Modèles probabilistes. Pour générer des profils de préférences lorsque l'on ne dispose pas de suffisamment de données réelles, il est très fréquent d'avoir recours à des données synthétiques, générées à l'aide de modèles probabilistes, dont Boehmer et al. (2024) ont fait une revue. Ceux que nous utilisons dans la thèse sont aussi les plus courants dans la littérature. En particulier, dans le modèle d'*Impartial Culture* (la *culture impartiale*), toutes les préférences possibles sont équiprobables. Par exemple, pour les préférences ordinales, chaque classement a la même probabilité d'être choisi. Au contraire, dans le modèle de *Mallows* (Mallows, 1957), il existe une préférence « moyenne » (par exemple, un classement moyen), et les préférences des votants sont générées à partir de cette préférence moyenne en y ajoutant du bruit aléatoire. Enfin, un autre modèle très utilisé et très présent dans cette thèse est le modèle *euclidien* (Enelow and Hinich, 1984, 1990). Dans ce modèle, candidats et votants sont associés à des positions dans un espace multidimensionnel. Les préférences des votants sont alors basées sur leur distance aux candidats : plus ils en sont proches, plus ils les préfèrent.

Données réelles. Dans cette thèse, nous allons toujours tester nos règles sur des jeux de données réels (souvent en complément de données synthétiques). Les sources de ces données sont multiples, mais beaucoup viennent de la bibliothèque de jeux de données *Preflib* (Mattei and Walsh, 2013), ou des expérimentations réalisées dans le cadre du projet *Voter Autrement*. Ces expérimentations ont eu lieu en France, en parallèle des élections nationales (généralement, les élections présidentielles), depuis 2002. Ces expérimentations se sont faites selon plusieurs modalités. Dans les expérimentations *in situ*, des chercheurs se rendent le jour de l'élection dans certains bureaux de vote pour proposer aux électeurs de voter à une « élection parallèle », dans laquelle des modes de scrutin alternatifs (et basés sur des préférences plus expressives) sont utilisés, comme le vote par approbation, le vote alternatif, le vote de Borda, ou le Jugement majoritaire (Laslier and Van der Straeten, 2004; Baujard and Igersheim, 2009; Farvaque et al., 2009; Balinski and Laraki, 2010; Baujard et al., 2013; Bouveret et al., 2019; Kamwa et al., 2020; Baujard et al., 2025b). Dans les expérimentations *en ligne*, les électeurs sont invités à se rendre sur un site web pendant les quelques semaines précédant l'élection, pour encore une fois voter avec des modes de scrutin alternatifs (Van der Straeten et al., 2013; Laslier et al., 2012; Bouveret et al., 2018; Delemazure and Bouveret, 2024).[1] Ces expérimentations ont permis de récolter des données sur les préférences des électeurs, mais aussi sur leurs comportements de vote, et sur leurs opinions sur les différents modes de scrutin. Au total, ces expérimentations ont permis de collecter une vingtaine de jeux de données, cumulant plus de 70 000 participants.

B.3 Le vote par approbation avec second tour

Étant donné que la première partie de cette thèse est consacrée aux manières d'améliorer les systèmes de vote utilisés en pratique grâce à des bulletins de vote plus expressifs, il était naturel pour moi de commencer par le système de vote que je connais le plus, la pierre angulaire de la démocratie française : *le scrutin uninominal majoritaire à deux tours*. Ce mode de scrutin se déroule en deux tours. Au premier tour, les électeurs votent pour un candidat (et un seul). Si un candidat obtient plus de 50% des voix, il est élu. Sinon, les deux candidats ayant obtenu le plus de voix se qualifient pour le second tour, où les électeurs votent à nouveau pour l'un des deux candidats. Le candidat qui obtient le plus de voix au second tour est élu. Ce mode de scrutin, ou des variantes de celui-ci, est utilisé pour quasiment toutes les élections politiques à vainqueur unique en France, ainsi que dans de nombreux autres pays (84 pays l'utilisent pour élire leur chef d'État).

Ce mode de scrutin est cependant critiqué. Il a notamment été prouvé qu'il ne satisfait pas de nombreuses propriétés désirables, comme l'*indépendance aux clones* et la *monotonie*. La non-satisfaction de ces axiomes n'est pas seulement théorique : elle a des conséquences concrètes sur l'issue des élections. Par exemple, la non-satisfaction de l'axiome d'indépendance aux clones est causée par la division des voix entre les candidats proches, ce qui est une des causes de l'échec de la gauche à l'élection présidentielle française de 2002, où 8 candidats de gauche se sont présentés au premier tour, dispersant les voix de gauche et permettant à l'extrême-droite de se qualifier pour le second tour. Ce phénomène pousse également de nombreux électeurs à voter « utile ».

Il existe cependant une alternative qui ne souffre pas des mêmes défauts, le *vote par approbation*. Dans ce mode de scrutin, les électeurs peuvent voter pour plusieurs candidats. Le candidat qui obtient le plus de voix est élu. On peut toutefois se demander si un tel mode de scrutin serait facilement accepté dans les pays habitués aux élections à deux tours, comme la France. D'autant plus que le second tour permet de satisfaire quelques propriétés intéressantes (par exemple, il rend

[1] J'ai notamment contribué à l'édition 2022 de l'expérimentation en ligne, et je m'en suis inspiré pour en conduire une autre en 2024 dans le cadre de l'élection Européenne (voir Section B.5).

l'élection d'un perdant de Condorcet impossible). Une possibilité est alors de conserver le second tour, mais de modifier le premier tour en permettant aux électeurs de voter pour plusieurs candidats, c'est-à-dire en utilisant le vote par approbation. C'est cette proposition que nous explorons dans ce chapitre.

Dans un protocole de vote par approbation avec second tour, la question est alors de savoir de quelle manière sont choisis les deux finalistes pour le second tour. La solution la plus naturelle est de choisir les deux candidats ayant obtenu le plus de voix au premier tour. C'est par exemple ce qui est fait pour les élections municipales de *Saint-Louis* dans le *Missouri*, où ce système est en place depuis 2021. Cependant, il existe de nombreuses autres manières de choisir les finalistes. Nous en explorons quelques-unes dans ce chapitre, et montrons que ces différentes méthodes vérifient des propriétés théoriques différentes, et peuvent avoir un impact important sur l'issue de l'élection en pratique.

Règles d'approbation avec second tour

Pour calculer le gagnant d'une élection par approbation avec second tour, nous avons à la fois besoin de connaître les votes d'approbation des électeurs pour le premier tour, et leurs classements des candidats afin de départager les finalistes au second tour, quels qu'ils soient. On utilise pour cela la structure de *classement avec seuil* (*approval-ordinal ballots* en anglais) introduite par Brams and Sanver (2009).

Formellement, on a un profil de votes d'approbation $P_A = (A_1, \ldots, A_n)$, ainsi qu'un profil de votes par classement $P_\succ = (\succ_1, \ldots, \succ_n)$, où A_i est l'ensemble des candidats approuvés par l'électeur i, et $\succ_i$ est son classement des candidats. On suppose que les deux profils sont compatibles, c'est-à-dire que pour tout i, pour tout $x \in A_i$ et $y \notin A_i$, on a $x \succ_i y$. Dans la suite, on note pour $x, y \in C$, $S(x) = |\{i \in V \mid x \in A_i\}|$ le score d'approbation du candidat x, et $S(xy) = |\{i \in V \mid x \in A_i \text{ and } y \in A_i\}|$ le score d'approbation de la paire de candidats $\{x, y\}$.

La première étape consiste à sélectionner deux finalistes, et repose donc sur la famille des règles d'élection de comités par approbation (*approval based committee rules* en anglais), avec une taille de comité fixée à $k = 2$. Nous recommandons le livre de Lackner and Skowron (2023) pour une revue de la littérature sur ces règles. Il est à noter que les règles que nous étudions ici sont *irrésolues*, et peuvent donc retourner plusieurs paires de finalistes (en cas d'*ex aequo*).

La règle la plus naturelle est *l'approbation classique (AV)*, qui sélectionne comme finalistes les deux candidats ayant reçu le plus de votes. Le défaut de cette méthode est que si un groupe de votants est plus nombreux que tous les autres groupes, et que ces votants approuvent tous au moins deux candidats en commun, alors ces deux candidats seront choisis comme finalistes, laissant les autres groupes sans représentants au second tour. Pour obtenir plus de diversité, on peut utiliser *l'approbation proportionnelle (PAV)* ou bien *l'approbation de Chamberlin-Courant (CCAV)*. L'idée est qu'un votant qui approuve les deux finalistes contribue moins au score de la paire que dans le cas de l'approbation classique. Dans le cas de Chamberlin-Courant, une paire de candidats reçoit un seul point de chaque votant qui approuve au moins un des deux candidats, même si le votant approuve les deux.

Plus formellement, on a $\text{AV}(P_A) = \text{argmax}_{x \neq y \in C}\, S(x) + S(y)$, $\text{PAV}(P_A) = \text{argmax}_{x \neq y \in C}\, S(x) + S(y) - (1/2)S(xy)$ et $\text{CCAV}(P_A) = \text{argmax}_{x \neq y \in C}\, S(x) + S(y) - S(xy)$. On peut alors généraliser ces règles pour obtenir la famille des règles α-AV, qui sélectionnent la paire de candidats $\{x, y\}$ qui maximise $S(x) + S(y) - \alpha S(xy)$, pour $\alpha \in [0, 1]$. On retrouve AV pour $\alpha = 0$, PAV pour $\alpha = 1/2$, et CCAV pour $\alpha = 1$.

Plutôt que de choisir les deux finalistes en une seule fois, on peut également utiliser des règles

	AV	PAV	CCAV	S-PAV	S-CCAV	EnePhr	S-Phr	SAV	TRIV
Efficacité de Pareto	✓	✓	✗	✓	✗	✗	✓	✓	✗
Rés. stratégique	✓	✗	✓	✗	✗	✗	✗	✗	✓
Rés. stratégique forte	✗	✗	✗	✗	✗	✗	✗	✗	✓
Monotonie	✓	✗	✗	✗	✗	✗	✗	✗	✓
Ind. aux clones faible	✗	✗	✓	✗	✓	✗	✗	✗	✗
Efficacité d'approbation	✓	✗	✗	✓	✓	✓	✓	✗	✗

Table B.1: Résumé des propriétés des règles d'approbation avec second tour.

séquentielles, qui sélectionnent les finalistes l'un après l'autre. On peut par exemple définir des variantes des règles α-AV ainsi : le premier finaliste $x \in C$ est le candidat qui maximise le score d'approbation $S(x)$, et le second finaliste $y \in C \setminus \{x\}$ est le candidat qui maximise $S(y) - \alpha S(xy)$. On obtient ainsi la règle α-AV séquentielle (notée α-seqAV). On peut en particulier définir S-PAV et S-CCAV, qui sont les variantes séquentielles de PAV et CCAV. En dehors de cette famille de règles, nous étudions également la règle d'*Eneström-Phragmén (EnePhr)* et *la règle séquentielle de Phragmén (S-Phr)*.

Enfin, nous étudions la règle *d'approbation divisée (SAV)*, où un votant i contribue $1/|A_i|$ au score de chaque candidat $x \in A_i$ (de telle sorte qu'un votant qui approuve plusieurs candidats divise sa voix entre ceux-ci). Nous ajoutons également à l'analyse la règle *triviale (TRIV)* qui sélectionne toutes les paires de finalistes possibles.

On peut maintenant définir la famille des règles *d'approbation avec second tour (AVR)*. On définit pour cela $\textsc{maj}(P_\succ, \{x, y\})$ comme le gagnant du vote de majorité (i.e., du second tour) entre x et y selon le profil de classement $P_\succ$. Ainsi, si f est une règle d'élection de comité par approbation, on définit la règle (irrésolue) d'approbation avec second tour associée f^R comme suit :

$$f^R(P_A, P_\succ) = \bigcup_{\{x,y\} \in f(P_A)} \textsc{maj}(P_\succ, \{x, y\}).$$

Cela permet directement de définir les règles AV^R, PAV^R, CCAV^R, S-PAV^R, S-CCAV^R, SAV^R, EnePhr^R, S-Phr^R et TRIV^R. Des exemples illustrant ces règles sont donnés en Section 3.2.

Analyse axiomatique

Nous étudions d'abord les propriétés axiomatiques des règles que nous venons d'introduire. Les résultats sont résumés dans la Table B.1. Ci-dessous, nous détaillons ces résultats.

Efficacité de Pareto. Une règle satisfait l'*efficacité de Pareto* si elle n'élit jamais un candidat qui est dominé par un autre candidat, c'est-à-dire si elle n'élit jamais un candidat b pour lequel il existe un candidat a tel que (1) pour tout votant $i \in V$, on a $a \succ_i b$ et (2) il existe un votant $i \in V$ tel que $a \in A_i$ et $b \notin A_i$. Cette propriété est très faible et est satisfaite par la plupart des règles (Proposition 3.2). Il est à noter qu'il est possible de définir de légères variantes de CCAV et S-CCAV dans laquelle on résout les cas d'*ex aequo* en faveur de la paire avec le meilleur score d'approbation classique (AV), et qui satisfont alors l'efficacité de Pareto.

Résistance stratégique. Une règle est *résistante au vote stratégique* si un votant qui n'approuve aucun des gagnants en votant sincèrement ne peut pas changer le résultat de l'élection en votant stratégiquement, de telle sorte que tous les gagnants soient désormais des candidats qu'il approuve.

Il est intéressant de voir que AV^R et CCAV^R sont les seules règles α-AV qui satisfont cette propriété (Proposition 3.3 et Corollary 3.4). Il est possible de rendre la propriété plus forte en demandant qu'aucun des gagnants ne soit approuvé par le votant lorsque celui-ci vote stratégiquement (toujours à condition qu'il n'approuve également aucun gagnant en votant sincèrement). On dit alors que la règle est *fortement résistante au vote stratégique*. Cependant, cette propriété est trop forte pour être satisfaite, et elle est incompatible avec l'efficacité de Pareto (Theorem 3.5).

Monotonie. Une règle est *monotone* si un candidat qui est élu reste élu lorsque des votants améliorent sa position dans leurs classements, ou bien l'approuvent alors qu'ils ne l'approuvaient pas auparavant. AV^R est la seule règle que nous étudions (hormis la règle triviale) qui satisfait cette propriété (Proposition 3.7).

Indépendance aux clones. Dans ce contexte, deux candidats sont des *clones* si tous les votants qui approuvent l'un approuvent également l'autre, *et* si tous les votants classent les deux candidats l'un à la suite de l'autre. Une règle est *indépendante aux clones* si un candidat gagne dans un profil avec des clones si et seulement s'il gagne également dans un profil où un des clones a été enlevé (sauf si il s'agit du clone qui a été enlevé, dans ce cas, son clone doit gagner à sa place). On peut cependant montrer que cette propriété est trop forte pour n'importe quelle règle d'approbation avec second tour. On définit alors une version plus faible, qui restreint la propriété à certains profils de votes non pathologiques. On dit alors qu'une règle est *faiblement indépendante aux clones*. On peut montrer que CCAV^R et S-CCAV^R satisfont cette propriété (Proposition 3.11), et que cette propriété est incompatible avec la monotonie (Theorem 3.9).

Efficacité d'approbation. Enfin, une règle satisfait l'*efficacité d'approbation* si le gagnant est soit un gagnant du vote d'approbation (c'est-à-dire un candidat qui reçoit le plus de voix d'approbation), soit un candidat qui *bat* un gagnant du vote d'approbation au vote majoritaire. Autrement dit, cette propriété s'assure qu'*au moins un des deux finalistes* est un gagnant du vote d'approbation au premier tour, ce qui semble être naturel dans de nombreux contextes (il serait en effet difficile de concevoir que le candidat recevant le plus de votes au premier tour ne puisse pas participer au second). Nous montrons que cette propriété est satisfaite par toutes les règles séquentielles, incluant AV^R (Corollary 3.14).

Analyse probabiliste

À l'aide d'une analyse probabiliste dans le modèle euclidien à une dimension, nous avons étudié le lien entre le paramètre α des règles α-AV et α-seqAV et les positions des finalistes dans l'espace euclidien. Dans ce modèle, candidats et votants $x \in C \cup V$ sont chacun associés à une position $\mathsf{pos}(x) \in \mathbb{R}$, et un votant i approuve un candidat c ssi $|\mathsf{pos}(c) - \mathsf{pos}(i)| \leqslant d$, où d est un paramètre du modèle. Le classement d'un votant s'obtient quant à lui en ordonnant les candidats par ordre croissant de leur distance au votant. On suppose pour cette analyse qu'il existe une fonction continue de densité ϕ sur $\mathbb{R}$ telles qu'une proportion $\int_p^{p'} \phi(x)dx$ des votants ont une position dans l'intervalle $[p, p']$. On suppose de plus que cette fonction est symétrique en 0, et qu'elle est unimodale.

Nos résultats montrent que pour les règles α-AV, les finalistes ont des positions symétriques par rapport au centre (c'est-à-dire 0), et que leur position vérifie une équation dépendante de α et du paramètre d (Proposition 3.16). De manière générale, plus α est grand, plus les finalistes sont éloignés l'un de l'autre. Pour les règles séquentielles (α-seqAV), le premier finaliste est toujours le candidat le plus proche de 0, et la position du second finaliste dépend encore une fois des paramètres α et d (Proposition 3.17). De manière générale, plus α est grand, plus le second finaliste est éloigné du premier finaliste (et donc de 0). Nous avons étudié en particulier trois fonctions de densité (la fonction triangulaire, la Gaussienne, et la fonction exponentielle), avec plusieurs valeurs de d.

Analyse expérimentale

Enfin, nous avons implémenté les différentes règles et comparé leurs résultats sur des jeux de données réels de votes par approbation. Nous avons utilisé pour ces expériences 38 jeux de données provenant de sources diverses : 18 sont des jeux de données politiques collectés dans le cadre des expériences de vote *« Voter Autrement »*, les autres incluent notamment les votes pour les élections de la *Society for Social Choice and Welfare* (SSCW), des votes pour des posters de conférences, pour des chansons, ainsi que des jeux de données de *Tierlists* transformés en votes d'approbation.

On observe tout d'abord (Figure 3.6) que les différentes règles d'approbation retournent souvent des paires de finalistes différentes les unes des autres, et dans le cas des jeux de données politiques, les règles d'approbation avec second tour retournent souvent des paires de finalistes différentes de celles du scrutin uninominal majoritaire à deux tours, utilisé pour ces élections. Nous avons ensuite étudié quelques différences spécifiques entre les règles proposées.

Impact du paramètre α. Tout d'abord, nous avons étudié l'impact du choix du paramètre α pour les règles α-AV et α-seqAV. Les résultats des expériences confirment ceux de l'analyse probabiliste, et montrent que plus α est grand, plus les finalistes sont *éloignés idéologiquement* l'un de l'autre. Par exemple, AV^R choisira des finalistes assez proches idéologiquement (et proches du centre), tandis que CCAV^R choisira des finalistes plus éloignés idéologiquement (et plus éloignés du centre). C'est notamment le cas pour le jeu de données de l'expérimentation « Voter Autrement » de Grenoble en 2017, pour lequel AV^R choisit *Benoît Hamon* et *Emmanuel Macron*, tandis que CCAV^R choisit *Benoît Hamon* et *François Fillon* (Figure 3.7).

Versions séquentielles. Nous avons ensuite comparé les règles α-AV à leurs versions séquentielles. Notamment, nous avons observé que les règles non-séquentielles choisissent rarement une paire de finalistes qui ne contient pas le gagnant d'approbation du premier tour, et retournent donc souvent *la même paire* que leurs versions séquentielles. Il peut cependant arriver que cela ne soit pas le cas, comme pour le jeu de données de l'expérimentation « Voter Autrement » de 2007, où ni PAV^R ni CCAV^R ne choisissent comme finaliste le gagnant d'approbation du premier tour *François Bayrou*, et choisissent à la place *Ségolène Royal* et *Nicolas Sarkozy* (Figure 3.7).

L'approbation divisée. Enfin, nous nous sommes intéressés au cas de SAV^R. Nous avons notamment observé que cette règle défavorise les candidats de gauche dans la plupart de nos jeux de données, puisque ceux-ci sont souvent co-approuvés avec de nombreux autres candidats (et que plus un votant approuve de candidats, plus son poids dans l'élection est faible).

Conclusion et perspectives

Ces différentes analyses nous ont permis de montrer que le vote par approbation avec second tour n'est pas une seule règle, mais une *famille de règles*, et que les différentes règles de cette famille vérifient des propriétés axiomatiques différentes, ce qui peut avoir un impact important sur l'issue de l'élection. En se basant sur nos analyses, il semblerait que les règles les plus intéressantes pour remplacer le scrutin uninominal majoritaire à deux tours soient les règles AV^R (approbation classique) et $S\text{-}CCAV^R$. La première semble cependant la plus simple à comprendre et à mettre en place pour une élection de grande envergure. Enfin, des résultats d'un sondage mené à Saint-Louis, dans le Missouri (où cette règle est utilisée depuis 2021), montrent que la majorité des électeurs sont satisfaits du nouveau système de vote, et en particulier de la possibilité d'exprimer des préférences plus complexes qu'avec un bulletin uninominal.

B.4 Le vote alternatif avec des indifférences

Dans la section précédente, nous avons étudié une amélioration du scrutin uninominal majoritaire à deux tours, très utilisé pour des élections à vainqueur unique. Dans cette section, nous nous intéressons à une alternative populaire à ce mode de scrutin : le *vote alternatif* (*instant runoff voting (IRV)* en anglais). Ce mode de scrutin est notamment utilisé en Australie et en Irlande.

Dans ce mode de scrutin, les électeurs classent les candidats par ordre de préférence. On compte tout d'abord le nombre de voix reçues par chaque candidat, en ne prenant en compte que le premier choix de chaque votant, puis on élimine le candidat ayant reçu le moins de voix. On redistribue alors ses voix aux candidats suivants dans le classement des votants qui l'ont choisi en premier. On élimine ainsi les candidats un par un, jusqu'à ce qu'il n'en reste qu'un. Ce mode de scrutin est assez attractif pour plusieurs raisons : en plus de permettre aux votants de donner des préférences plus expressives, IRV satisfait des propriétés intéressantes comme l'axiome d'*indépendance aux clones*, et est donc moins sensible à la division des voix entre candidats proches. Cette méthode satisfait aussi le *critère de majorité*, qui impose qu'un candidat classé en première position par plus de la moitié des votants soit le vainqueur de l'élection. Cependant, IRV ne satisfait pas la *monotonie* : il est possible que le gagnant d'un profil de votes soit éliminé dans un autre profil de votes où certains votants ont amélioré la position de ce candidat dans leurs classements.

Un autre défaut de ce mode de scrutin est qu'il est parfois nécessaire de classer tous les candidats pour voter (à cause des règles de l'élection), ce qui nécessite un effort supplémentaire de la part des votants. Cependant, dans beaucoup de cas, les votants peuvent se limiter à classer seulement un sous-ensemble des candidats. Dans ce cas, le bulletin est ignoré une fois que tous les candidats qu'il classe ont été éliminés. Mais si un votant souhaite classer un candidat tout en bas de son classement, il doit alors classer tous les candidats au-dessus de celui-ci. Un vote par classement force également les votants à avoir un avis tranché sur chaque candidat, sans pouvoir exprimer d'*indifférences* (un votant pourrait trouver plusieurs candidats équivalents, et ne pas vouloir les classer l'un avant l'autre).

Une solution à ces problèmes est d'autoriser les indifférences dans les classements des votants, ce qui rendrait les bulletins de vote encore plus expressifs. Les votants donneraient ainsi des *classements avec égalités* sur les candidats, c'est-à-dire des classements pouvant contenir des égalités. En plus de permettre aux votants d'exprimer des indifférences, cela réduirait le nombre de bulletins invalides (comme ceux présentés en Figure 4.3).

Dans ce chapitre, nous étudions donc des généralisations du vote alternatif qui permettent aux votants d'exprimer des indifférences à l'aide de classements avec égalités. Une première possibilité a été discutée dans le journal *Voting Matters* (Meek, 1994; Warren, 1996; Hill, 2001). Dans cette généralisation, un votant qui classe $k \geqslant 1$ candidats à égalité en première position de son classement (parmi les candidats non éliminés) donne à chacun de ces candidats $1/k$ points, et 0 pour les autres. Le candidat avec le moins de points au total est éliminé. On répète cette procédure jusqu'à ce qu'il ne reste qu'un seul candidat. Nous allons appeler cette règle *Split-IRV* dans ce qui suit.

	Approval-IRV	Split-IRV
Indépendance aux clones	✓[1]	✗
Respect des majorités cohérentes	✓[1]	✗
Cohérence avec IRV	✓[2]	✓
Monotonie d'indifférence	✓[2]	✗

Table B.2: Résumé des propriétés axiomatiques des règles Approval-IRV et Split-IRV. Les superscripts [1] et [2] indiquent les deux caractérisations de Approval-IRV.

Dans ce chapitre, nous plaidons en faveur d'une autre généralisation de IRV, que nous appelons *Approval-IRV*, et dont le fonctionnement est le suivant. Chaque votant donne 1 point à chaque candidat qu'il classe en haut de son classement (parmi les candidats non-éliminés), et 0 points aux autres. On élimine alors le candidat ayant reçu le moins de points, et on répète cette procédure jusqu'à ce qu'il ne reste qu'un seul candidat. Nous allons montrer avec des arguments axiomatiques et expérimentaux que cette généralisation semble plus naturelle que Split-IRV.

Règles de score à élimination successives avec indifférences

Nous supposons dans cette partie que nous connaissons un profil de classements avec égalités $P = (\succcurlyeq_1, \dots, \succcurlyeq_n)$, où $\succcurlyeq_i$ est le classement avec égalités (c'est-à-dire un préordre complet) de l'électeur i. Formellement, un classement avec égalités partitionne l'ensemble des candidats en différentes classes d'indifférences $(C_1, \dots, C_k)$ telles que $C_1 \succ_i C_2 \succ_i \cdots \succ_i C_k$. On note alors $\tau = (|C_1|, \dots, |C_k|)$ le *type* du classement avec égalités. En particulier, $\tau = (1, 1, \dots, 1, 1)$ correspond à un ordre strict (c'est-à-dire un classement), et $\tau = (x, y)$ correspond à des préférences dichotomiques (et donc à un vote par approbation).

Nous allons concentrer notre étude sur une famille de règles : les *règles de score à éliminations successives* (Smith, 1973, Section 4), dont le principe se généralise aisément aux classements avec égalités. Le principe est le suivant : à chaque étape, chaque votant attribue un score à chaque candidat selon (i) le type de son classement, et (ii) la position du candidat dans son classement avec égalités. On compte alors le score de chaque candidat, et on élimine le candidat ayant reçu le moins de points. On répète cette procédure jusqu'à ce qu'il ne reste qu'un seul candidat. Plus formellement, une règle de score à éliminations successives f est associée à une fonction de score s qui associe à chaque type d'ordre τ un vecteur de score $s(\tau) = (s_1, \dots, s_k)$, où s_i est le score attribué à un candidat classé i-ème dans le classement avec égalités. On considère dans notre analyse que les règles sont irrésolues, et peuvent donc retourner plusieurs candidats *ex aequo*.

Les deux règles principales que nous étudions sont définies formellement comme suit. *Approval-IRV* est la règle de score à éliminations successives associée à la fonction de score s définie par $s(\tau) = (1, 0, \dots, 0)$ pour tout type d'ordre τ. *Split-IRV* est la règle de score à éliminations successives associée à la fonction de score s définie par $s(\tau) = (1/\tau(1), 0, \dots, 0)$ pour tout type d'ordre, où $\tau(1)$ est le nombre de candidats classés premiers à égalité dans le classement.

Analyse axiomatique

Nous étudions d'abord les propriétés axiomatiques des règles que nous venons d'introduire. Les résultats sont résumés dans la Table B.2. En particulier, nous avons prouvé deux caractérisations de Approval-IRV, qui sont détaillées ci-dessous.

Indépendance aux clones. Dans le cas des classements avec égalités, deux candidats sont des *clones* si leur relation aux autres candidats est la même dans chaque classement. Nous avons prouvé que Approval-IRV est indépendante aux clones, tandis que Split-IRV ne l'est pas (Proposition 4.1).

Respect des majorités cohérentes. Soit un profil de préférences dans lequel plus de la moitié des votants classent un même candidat c en première position. Une règle respecte les *majorités cohérentes* si le vainqueur de ce profil est c ou un candidat classé en première position avec c par un des votants de la majorité qui classent c en première position. Nous avons prouvé que Approval-IRV respecte les majorités cohérentes, tandis que Split-IRV ne le fait pas (Proposition 4.3).

Première caractérisation de Approval-IRV. Nous avons prouvé que Approval-IRV est la seule règle de score à élimination successives qui satisfait l'indépendance aux clones et le respect des majorités cohérentes (Theorem 4.4). Ce résultat est très intéressant, puisque ces deux axiomes sont des généralisations aux classements avec égalités de propriétés que IRV satisfait dans le cas des ordres stricts. En particulier, le respect des majorités cohérentes est une généralisation du critère de majorité de IRV. Ce résultat indique donc que Approval-IRV est une généralisation naturelle de IRV aux classements avec égalités, et qu'elle conserve les propriétés intéressantes de IRV.

Cohérence avec IRV. Les deux règles que l'on considère sont cohérentes avec IRV, c'est-à-dire que si on considère un profil d'ordres stricts P, alors les deux règles retournent le même candidat que IRV.

Monotonie d'indifférence. Une règle est *monotone d'indifférence* si un candidat qui est élu reste élu lorsqu'un votant pour qui ce candidat forme une classe d'indifférence à lui seul change son vote pour le classer à égalité avec le ou les candidats qu'il classait directement au-dessus de lui. Nous avons prouvé que Approval-IRV satisfait cette propriété, tandis que Split-IRV ne la satisfait pas (Proposition 4.12).

Deuxième caractérisation de Approval-IRV. Enfin, nous avons prouvé que Approval-IRV est la seule règle de score à élimination successives cohérente avec IRV et qui satisfait la monotonie d'indifférence (Theorem 4.13). Ce résultat est intéressant, puisqu'IRV est réputé pour ne pas satisfaire la monotonie, et qu'en autorisant les égalités dans les classements, on arrive à satisfaire une forme plus faible de monotonie.

Analyse expérimentale

Nous avons ensuite implémenté les deux règles et les avons comparées sur des jeux de données de votes réels, et des jeux de données synthétiques, générés par différents modèles probabilistes (préférences euclidiennes en une et deux dimensions, culture impartiale et modèles de Mallows). Nous avons également utilisé des modèles probabilistes pour *introduire des indifférences* dans les préférences des votants, puisque tous les jeux de données contiennent à l'origine des ordres stricts. Les paramètres de ces modèles permettent de faire varier le nombre d'indifférences.

Nous observons tout d'abord que les deux règles retournent souvent *les mêmes vainqueurs*, et cela est particulièrement vrai lorsqu'il y a très peu d'indifférences (lorsqu'il n'y en a pas du tout, les deux sont égales à IRV), ou quand il y en a beaucoup. Pour des valeurs intermédiaires, les règles retournent parfois des vainqueurs différents. Nous avons également comparé les vainqueurs de ces règles aux gagnants de différentes règles basées sur les ordres stricts, c'est-à-dire avant l'introduction des indifférences par notre modèle probabiliste. On observe que Approval-IRV retourne plus souvent le vainqueur de Condorcet que Split-IRV (Figure 4.16), et sélectionne des candidats avec un meilleur score de Borda (Figure 4.17). À l'inverse, Split-IRV retourne plus souvent le vainqueur de IRV que Approval-IRV (Figure 4.15). Cependant, cela ne nous permet pas de conclure que l'une des deux règles est meilleure que l'autre, puisque les préférences des votants sont modifiées par l'introduction des indifférences.

Enfin, nous avons comparé la distorsion des vainqueurs de chaque règle dans le cas des modèles euclidiens. Nous observons que Approval-IRV retourne des vainqueurs ayant en moyenne une distorsion plus faible (et donc meilleure) que les vainqueurs de Split-IRV (Figures 4.19 (a) and 4.19 (b)).

Conclusion et perspectives

Dans ce chapitre, nous avons étudié deux généralisations du vote alternatif aux classements avec égalités, et nous avons montré que Approval-IRV semble être une généralisation naturelle de IRV, qui conserve les propriétés intéressantes de IRV, et en apporte de nouvelles. Sur le plan expérimental, les résultats sur la distorsion tendent à montrer que Approval-IRV est une généralisation de IRV plus attractive que Split-IRV. Cependant, il serait intéressant de mener des expériences supplémentaires pour confirmer ces résultats. Approval-IRV est également plus facile à comprendre et à mettre en place pour des élections de grande envergure. Des travaux futurs pourraient étudier la résistance aux votes stratégiques de ces règles.

B.5 Classer les candidats pour les élections parlementaires avec seuil électoral

Dans les deux sections précédentes, nous nous sommes intéressés à des modes de scrutin pour des élections à vainqueur unique, et nous avons proposé des améliorations de ces modes de scrutin reposant sur des bulletins de vote plus expressifs. Dans cette section, nous nous intéressons à un autre type d'élection : les élections parlementaires. Dans ces élections, différents partis politiques se présentent, et plusieurs candidats d'un même parti peuvent être élus.

Comme pour les élections à vainqueur unique, il existe de nombreux modes de scrutin pour les élections parlementaires en pratique, mais beaucoup d'entre eux reposent sur l'idée de *représentation proportionnelle*, c'est-à-dire que les sièges dans le parlement soient répartis entre les partis politiques proportionnellement au nombre de voix qu'ils ont reçues. Ce genre de système est notamment très utilisé en Europe et en Amérique du Sud pour élire les parlements nationaux.

Afin d'éviter une trop grande fragmentation politique, qui peut rendre difficile la formation de coalitions de gouvernement, de nombreux pays qui utilisent la représentation proportionnelle imposent un *seuil électoral*, c'est-à-dire un pourcentage minimum de voix (en général, entre 0 et 6%) qu'un parti doit recevoir pour pouvoir obtenir des sièges au parlement. Les partis qui ne reçoivent pas ce pourcentage minimum de voix sont exclus de la répartition des sièges, et les votes pour ces partis sont alors « perdus » ou « gaspillés ». Par exemple, pour l'élection des représentants français au Parlement Européen de 2019, près de 20% des votes ont été « perdus » de la sorte. Ce phénomène pousse également les électeurs à voter stratégiquement, en votant pour un parti qui a une chance d'atteindre le seuil électoral, plutôt que pour le parti qu'ils préfèrent vraiment.

Dans ce chapitre, nous proposons des manières d'éviter ce gaspillage de votes, en permettant aux votants de *classer* les candidats. Cette idée n'est pas nouvelle, et a déjà été évoqué dans le débat public dans de nombreux pays. En Allemagne, l'idée d'un « Vote de remplacement » (*Ersatzstimme*) qui serait utilisé à la place du vote original si le parti pour lequel on a voté ne dépasse pas le seuil électoral a été proposé par certains activistes et politiciens.

Pour étudier ces propositions, nous introduisons le modèle formel des *règles de sélection de partis*, qui prennent en entrée un profil de classements tronqués et un seuil électoral, et qui retournent la liste des partis qui peuvent obtenir des sièges au parlement. On ne s'intéresse donc pas à la répartition des sièges entre les partis, mais seulement à la sélection des partis qui *peuvent* obtenir des sièges.

Règles de sélection de partis

Formellement, on a un profil de préférences $P = (\succ_1, \dots, \succ_n)$, où $\succ_i$ est le classement (tronqué) du votant i, c'est-à-dire que seul un sous-ensemble des partis $S \subseteq C$ est classé par le votant i.

Pour une sélection de partis $S \subseteq C$, on note $\mathsf{best}_S(i)$ le parti favori de l'électeur i parmi les partis de S. Si i n'a pas classé de parti de S, on note $\mathsf{best}_S(i) = \emptyset$. On note également $\mathsf{score}_S(c) = |\{i \in V \mid \mathsf{best}_S(i) = c\}|$ le *score* du parti c dans la sélection S. Étant donné un seuil électoral $0 \leqslant \tau \leqslant |V|$, une sélection de partis S est dite *valide* si chaque parti $c \in S$ est le parti favori de plus de τ électeurs, c'est-à-dire si $\mathsf{score}_S(c) > \tau$.

Une règle de sélection de partis f est alors une fonction qui prend en entrée un profil de classements tronqués P et un seuil électoral τ, et qui retourne une sélection de partis $f(P, \tau) \subseteq C$ valide. On supposera dans ce qui suit que les règles étudiées sont *résolues*, c'est-à-dire qu'elles ne retournent pas d'ex aequo (les cas d'ex aequo pouvant par exemple être résolus à l'aide d'un ordre total sur les partis). Nous définissons trois règles *procédurales*, qui semblent appropriées pour des élections politiques :

- *Gagnants Directs Seulement (Direct Winners Only, DO).* Cette première règle sélectionne uniquement les partis qui sont le parti favori de plus de τ votants, c'est-à-dire que $\mathrm{DO}(P, \tau) = \{c \in C \mid \mathsf{score}_C(c) \geqslant \tau\}$.

- *Vote Transférable (Single Transferable Vote, STV).* Cette règle est inspirée du vote alternatif. On commence avec l'ensemble de tous les partis $S_0 = C$, et, à chaque étape $k \geqslant 0$, si S_k est valide, on retourne S_k. Sinon, on élimine le parti c ayant le score le plus faible dans S_k, c'est-à-dire qui minimise $\mathsf{score}_{S_k}(c)$, et on considère la sélection de partis $S_{k+1} = S_k \setminus \{c\}$. On répète cette procédure jusqu'à ce que l'on trouve une sélection valide.

- *Pluralité Gloutonne (Greedy Plurality, GP).* On commence avec l'ensemble vide $S_0 = \emptyset$, et on considère les partis un par un par ordre décroissant de score de pluralité $\mathsf{score}_C(c)$. Pour chaque parti c, si $S_k \cup \{c\}$ est valide, on ajoute c à S_k, et on considère la sélection de partis $S_{k+1} = S_k \cup \{c\}$. Sinon, on ne rajoute pas le parti c (dans ce cas, $S_{k+1} = S_k$). On répète cette procédure jusqu'à ce que l'on ait considéré tous les partis. On retourne alors la sélection de partis finale S_n.

Il est à noter que les sélections de partis retournées par STV et GP contiennent systématiquement celles retournées par DO, et que ces trois règles sont calculables en temps polynomial.

Nous considérons également deux règles basées sur *l'optimisation de fonctions*, c'est-à-dire qui retournent une sélection de partis S qui maximise une certaine fonction de score $\phi(P, S)$. La règle *Pluralité Maximale (MaxP)* est basée sur la fonction de score $\phi(P, S) = \sum_{c \in S} \mathsf{score}_C(c)$, c'est-à-dire que MaxP retourne la sélection de partis S qui maximise le nombre total de première places reçues par les partis de S. La règle *Représentation Maximale (MaxR)* est basée sur la fonction de score

	DO	STV	GP	MaxP	MaxR
Maximalité de la sélection	✗	✗	✓	✓	✓
Inclusion des gagnants directs	✓[1]	✓[2]	✓	✗	✗
Représentation des coalitions	✗	✓	✗	✗	✗
Monotonie de seuil	✓	✓	✗	✗	✗
Indépendance aux partis éliminés	✗	✓[2]	✗	✗	✗
Indépendance aux clones	✗	✓	✗	✗	✓
Renforcement des partis gagnants	✓[1]	✗	✗	✗	✗
Monotonie	✓	✗	✗	✗	✗
Representative-strategyproof (un parti « à risque »)	✗	✗	✓	✓	✓
Share-strategyproof (parti « sûr » premier ou deuxième)	✓	✗	✗	✗	✗
Share-strategyproof (représentant classé premier)	✓	✗	✓	✗	✗

Table B.3: Propriétés satisfaites par les règles de sélection de partis. Les exposants [1] et [2] indiquent des résultats de caractérisation.

$\phi(P, S) = \sum_{c \in S} \mathsf{score}_S(c)$, c'est-à-dire que MaxR retourne la sélection de partis S qui maximise le nombre total de votants qui ont classé au moins un parti de S. Ces deux règles semblent moins adaptées à un contexte politique, et elles sont NP-difficiles à calculer (Theorem 5.1). Cependant, elles sont intéressantes à étudier et peuvent trouver des applications dans d'autres contextes (par exemple en *facility location*).

Analyse axiomatique

Encore une fois, nous allons d'abord étudier les propriétés axiomatiques des règles que nous venons d'introduire. Les résultats sont résumés dans la Table B.3, et détaillés ci-dessous.

Axiomes d'efficacité. Nous introduisons deux axiomes d'efficacité. Le premier, l'*efficacité faible*, impose que la sélection de partis S soit non-vide s'il existe une sélection valide non-vide. Le second, plus fort, est l'axiome de *maximalité de la sélection*, qui impose que si S est la sélection de partis retournée par la règle, alors il n'existe pas de sélection valide $S' \supsetneq S$. Alors que GP, MaxP et MaxR satisfont ces deux axiomes, DO et STV ne satisfont aucun des deux (Proposition 5.2).

Axiomes de représentation. Nous introduisons ensuite une série d'axiomes de représentation. Le plus faible est l'*inclusion des gagnants directs*, qui impose que si un parti c est le parti favori (c'est-à-dire classé en première position) de plus de τ électeurs, alors il doit être dans la sélection retournée par la règle. Cet axiome est satisfait par DO, STV et GP, mais pas par MaxP et MaxR (Proposition 5.3). Il est possible de renforcer cet axiome en imposant que s'il existe une coalition de plus de τ votants qui classent un sous-ensemble de partis $T \subseteq C$ en haut de leur classement (possiblement dans des ordres différents pour chaque votant), alors au moins un des partis de T doit faire partie de la sélection retournée par la règle. On dit alors que la règle satisfait l'axiome de *représentation des coalitions*. Cet axiome est satisfait par STV, mais pas par les autres règles (Proposition 5.4).

Variations du seuil. Les deux axiomes suivants s'intéressent aux effets des variations du seuil. En particulier, l'axiome de *monotonie de seuil* impose que si on augmente le seuil électoral τ, alors la sélection de partis retournée par la règle ne peut que diminuer, c'est-à-dire que $f(P, \tau) \supseteq f(P, \tau')$ pour tout $\tau' > \tau$ et pour tout profil de préférences P. Cet axiome est satisfait par DO et STV, mais pas par GP, MaxP ou MaxR (Proposition 5.8). Le second axiome, l'*indépendance aux partis éliminés*, impose que si $S = f(P, \tau)$ est la sélection de partis retournée par la règle pour un seuil τ, alors pour tout seuil $\tau' \geqslant \tau$, la règle sélectionne les mêmes partis dans le profil complet et dans le profil restreint aux partis de S, c'est-à-dire que $f(P, \tau') = f(P|_S, \tau')$. Cet axiome, combiné à l'inclusion des gagnants directs, caractérise STV (Theorem 5.9).

Indépendance aux clones. Deux partis sont des *clones* s'ils sont toujours classés l'un après l'autre dans les classements des votants. Nous avons montré que STV et MaxR sont indépendantes aux clones, tandis que DO, GP et MaxP ne le sont pas (Proposition 5.11).

Renforcement des partis gagnants. L'axiome de renforcement des partis gagnants impose que si un parti est sélectionné par une règle pour un profil P et un seuil τ, et qu'il est également sélectionné pour un profil P' et un seuil τ', alors il doit être sélectionné pour la combinaison des deux profils $P + P'$ et pour le seuil $\tau + \tau'$. Combiné avec l'inclusion des gagnants directs, cet axiome caractérise DO (Theorem 5.13).

Monotonie. L'axiome de monotonie impose que si un parti est sélectionné par une règle pour un profil P et un seuil τ, et qu'un votant change son vote pour améliorer la position de ce parti dans son classement (sans toucher à l'ordre relatif des autres partis), alors le parti doit toujours être sélectionné pour le profil P' obtenu et pour le même seuil τ. Cet axiome est satisfait par DO, mais pas par les autres règles (Proposition 5.15).

Résistance au vote stratégique. Enfin, nous introduisons plusieurs axiomes liés au vote stratégique. On considère pour cela deux axiomes : le premier, dit *representative-strategyproofness*, impose qu'un votant ne peut pas obtenir un représentant qu'il préfère s'il donne un classement insincère que lorsqu'il donne un classement sincère; le second, dit *share-strategyproofness*, impose qu'un votant ne peut pas obtenir un représentant qu'il préfère *ou* améliorer le score relatif (la part du score parmi la somme des scores) de son représentant en donnant un classement insincère. Ces deux axiomes étant trop forts pour être satisfaits par n'importe quelle règle dans le cas général, nous nous restreignons à certains cas particuliers, basés sur une classification des partis en différents groupes (Proposition 5.17 à 5.19)

Analyse expérimentale

Nous avons ensuite implémenté les règles que nous venons d'introduire, et les avons comparées sur un jeu de données de votes réels que nous avons collectées au cours d'une expérience de vote en ligne. En effet, nous avons construit un site internet dans le cadre de l'élection des représentants français au Parlement Européen de 2024, sur lequel nous informions les participants du problème des votes perdus, et leur proposions de voter pour les partis de leur choix en utilisant un classement tronqué (d'abord en ne classant que deux partis, puis en en classant autant que souhaité).

Nous avons collecté deux échantillons de participants. Le premier est l'échantillon *auto-sélectionné*, que nous avons recruté *via* des mailing-lists et sur les réseaux sociaux. Les participants rejoignaient l'expérience volontairement et n'étaient pas rémunérés. Nous avons réussi à obtenir 3 046 participants en une semaine pour cet échantillon. Cependant, l'échantillon n'est pas représentatif de la population française (en particulier, les jeunes, les personnes diplômées, et les personnes de gauche sont sur-représentées). Nous avons donc également collecté un échantillon *représentatif*, en recrutant des participants *via* un institut de sondage professionnel. Cet échantillon est représentatif de la population française et compte 1 000 participants, qui ont été rémunérés pour leur participation.

Les principaux résultats de nos expériences sont les suivants. Tout d'abord, les participants étaient beaucoup plus nombreux à classer un « petit parti » (c'est-à-dire un parti qui ne dépasse pas le seuil électoral) en première position de leur classement lorsqu'ils avaient la possibilité de classer les candidats, plutôt que de voter pour un seul parti. En particulier, de nombreux votants ont classé un plus petit parti en haut de leur classement que celui pour lequel ils ont indiqué voter à l'élection (Table 5.2).

Deuxièmement, nous constatons une diminution du nombre de votes « perdus » grâce aux modes de scrutin par classement. En effet, alors qu'il y avait au total 12.1% de votes perdus dans cette élection, nous obtenons jusqu'à 2.3% de votes perdus pour STV et GP dans l'échantillon auto-sélectionné, et 7.2% dans l'échantillon représentatif (Table 5.3). De plus, ce gain en représentativité n'entraîne pas une augmentation de la fragmentation politique, puisqu'au plus un parti supplémentaire est sélectionné par nos règles (Table 5.4).

On constate également que la plupart des votants sont représentés par un parti qu'ils ont classé dans leurs trois premières positions (Figure 5.2). Par conséquent, on obtient une importante réduction du nombre de votants non-représentés même en tronquant les classements des votants à trois ou quatre candidats (Figure 5.4), ce qui serait plus facile à mettre en place dans la pratique.

Enfin, nous avons également regardé quels seraient les résultats avec différents seuils (Figure 5.5), et si l'on observait les mêmes résultats en ajoutant du bruit dans les données (Figure 5.6).

Conclusion et perspectives

Dans cette section, nous avons étudié des règles de sélection de partis pour les élections parlementaires avec seuil électoral. Nous avons introduit un modèle formel pour ces règles, et nous avons étudié leurs propriétés axiomatiques. Nous avons également implémenté les règles et les avons comparées sur des données réelles collectées lors d'une expérience de vote en ligne. Nos résultats expérimentaux montrent que les règles que nous avons étudiées permettent de réduire le nombre de votes perdus.

Les différentes règles DO, STV et GP satisfont des propriétés axiomatiques différentes, mais il semble qu'en pratique STV et GP soient très similaires et permettent une meilleure représentation que DO. Cependant, DO a l'avantage d'être plus simple à mettre en place, par exemple en effectuant deux tours de scrutins avec des votes uninominaux.

Dans notre expérience en ligne, nous demandions aux participants d'indiquer quelle méthode ils préféraient entre celles qu'ils avaient testées et la méthode actuelle. Dans l'échantillon représentatif, ils sont 46% à préférer les alternatives, et 77% dans l'échantillon auto-sélectionné. De plus, ce pourcentage est plus élevé parmi les participants qui ont voté « utile » à l'élection (c'est-à-dire stratégiquement), ainsi que chez les électeurs de gauche.

B.6 Axes de candidats à partir de votes d'approbation

Dans les trois premières sections, nous avons vu comment des bulletins de vote plus expressifs pouvaient améliorer la façon dont nous élisons nos représentants, que cela soit pour des élections à vainqueur unique ou pour des élections parlementaires. Cependant, ces bulletins de vote expressifs présentent d'autres intérêts. En particulier, ils nous donnent des *informations* sur la structure sous-jacente des préférences des votants, en indiquant par exemple quels candidats sont proches les uns des autres (car appréciés par les mêmes votants).

C'est ce que nous allons faire dans cette section, dans laquelle nous étudions comment il est possible d'utiliser les votes d'approbation des votants pour construire un *axe unidimensionnel* des candidats. L'idée qu'il existe un axe unidimensionnel des candidats, ou des partis, est très présente en politique : c'est le fameux axe « gauche-droite ». Cependant, l'axe gauche-droite exact est souvent débattu, et il n'existe pas de consensus clair sur l'axe des partis et des candidats. Lorsque les médias, ou les instituts de sondage, présentent les candidats sur un axe gauche-droite, ils se fient généralement à leur intuition, ou à des estimations d'experts dans le meilleur des cas. Nous proposons dans cette section des méthodes systématiques, mathématiques et explicables pour construire de tels axes, que nous analysons avec les outils classiques de la théorie du choix social.

Ce problème trouve des applications dans de nombreux autres domaines que la politique. En réalité, partout où l'on souhaite ordonner des « objets » pour lesquels nous possédons des informations binaires (0 ou 1), il est possible d'utiliser les méthodes que nous allons présenter. C'est le cas par exemple en *archéologie* pour la datation relative des artefacts, ou des tombes, ou bien en *géologie* pour celle des couches terrestres.

Règles d'axes

Pour ce problème, nous connaissons un profil de votes d'approbation $P = (A_1, \dots, A_n)$, où $A_i \subseteq C$ est l'ensemble des candidats approuvés par l'électeur i. L'objectif est d'obtenir un *axe* $\lhd$ des candidats, c'est-à-dire un ordre total strict sur les candidats C. Cependant, contrairement à un classement, tout axe $\lhd$ est équivalent à son axe opposé (c'est-à-dire en inversant le sens).

On dit qu'un bulletin d'approbation A est un *intervalle* d'un axe $\lhd$ si pour chaque paire de candidats approuvés $a, b \in A$, pour tout candidat $c \in C$ tel que $a \lhd c \lhd b$, on a $c \in A$. En d'autres termes, le bulletin d'approbation forme un intervalle contigu sur l'axe $\lhd$. S'il existe un candidat $c \in C$ tel que $a \lhd c \lhd b$, mais $c \notin A$, on dit que ce candidat *interfère* avec l'intervalle A.

Ainsi, un axe $\lhd$ est totalement compatible avec un profil de votes d'approbation P si chaque bulletin d'approbation A_i est un intervalle de $\lhd$. Dans ce cas, on dit que le profil P est *linéaire* et l'axe gauche-droite que l'on cherche est $\lhd$. Cependant, il n'existe en général aucun axe totalement compatible avec les profils d'approbation, il faut donc se contenter d'axes *partiellement compatibles*, et définir des règles qui choisissent de tels axes.

Les règles d'axes que nous allons introduire sont des fonctions (irrésolues) f qui prennent en entrée un profil de votes d'approbation P et qui retournent un ensemble d'axes. Dans notre analyse, nous allons nous concentrer sur des règles dites *de score*. Une règle de score est associée à une fonction de coût cost qui associe à chaque bulletin A et axe $\lhd$ un coût $\mathsf{cost}(A, \lhd) \geqslant 0$. Intuitivement, si le bulletin d'approbation A est un intervalle de l'axe $\lhd$, alors le coût est de 0, et plus le bulletin est loin d'être un intervalle de l'axe, plus le coût est élevé. On note $\mathsf{cost}(P, \lhd) = \sum_{i=1}^{n} \mathsf{cost}(A_i, \lhd)$ le coût total de l'axe $\lhd$ pour le profil P. Une règle de score f est alors une fonction qui prend en entrée un profil de votes d'approbation P et qui retourne les axes qui minimisent le coût total, c'est-à-dire $f(P) = \mathrm{argmin}_{\lhd'} \mathsf{cost}(P, \lhd')$.

Nous définissons cinq règles de la sorte:

- *Voter Deletion* (VD), ou *suppression de votants*, est la règle qui sélectionne les axes qui minimisent le nombre de votants dont le bulletin d'approbation n'est pas un intervalle de l'axe. La fonction de coût associée est donc $\mathsf{cost}_{\mathrm{VD}}(A, \lhd) = 0$ si A est un intervalle de $\lhd$, et $\mathsf{cost}_{\mathrm{VD}}(A, \lhd) = 1$ sinon. Cette règle est très simple à comprendre, mais ne distingue pas les différents bulletins qui ne sont pas des intervalles. Par exemple, le coût est le même peu importe le nombre de candidats qui interfèrent avec A.

- *Minimum Flips* (MF), ou *minimum d'inversions*, est la règle qui sélectionne les axes qui minimisent le nombre de candidats qui doivent être ajoutés et/ou supprimés des bulletins pour que chaque bulletin d'approbation soit un intervalle de l'axe. La fonction de coût associée est :

$$\mathsf{cost}_{\mathrm{MF}}(A, \lhd) = \min_{x,y \in A : x \unlhd y} |\{z \in A : z \lhd x \text{ or } y \lhd z\}| + |\{z \notin A : x \lhd z \lhd y\}|$$

  Ici, la notation $x \unlhd y$ indique que $x \lhd y$ ou $x = y$.

- *Ballot Completion* (BC), ou *complétion de bulletin*, est la règle qui sélectionne les axes qui minimisent le nombre de candidats qui doivent être ajoutés aux bulletins d'approbation pour que chaque bulletin soit un intervalle de l'axe. La fonction de coût associée est :

$$\mathsf{cost}_{\mathrm{BC}}(A, \lhd) = |\{b \notin A : \text{ il existe } a, c \in A \text{ tels que } a \lhd b \lhd c\}|$$

  L'idée est que les votants aient pu oublier d'inclure certains candidats dans leurs bulletins (possiblement par manque d'information). Cette méthode est la seule qui a déjà été utilisée dans la littérature pour construire des axes politique à partir de votes d'approbation (Lebon et al., 2017; Baujard and Lebon, 2022).

- *Minimum Swaps* (MS), ou *minimum d'échanges*, est une règle où l'on compte des modifications de l'axe plutôt que du bulletin d'approbation. En effet, pour un bulletin de vote A et un axe $\lhd$, le coût est le nombre minimum d'échanges de candidats adjacents sur l'axe qu'il faut effectuer pour que A devienne un intervalle de l'axe. La fonction de coût associée est :

$$\mathsf{cost}_{\mathrm{MS}}(A, \lhd) = \sum_{x \notin A} \min(|\{y \in A : y \lhd x\}|, |\{y \in A : x \lhd y\}|).$$

- *Forbidden Triples* (FT) ou *triplets interdits*, est une règle qui minimise le nombre de « triplets interdits » de candidats. Un triplet de candidat (a, b, c) est dit *interdit* pour un axe $\lhd$ si on a $a \lhd b \lhd c$ et $a, c \in A$, mais $b \notin A$ (donc b interfère avec l'intervalle A). La fonction de coût associée est donc :

$$\mathsf{cost}_{\mathrm{FT}}(A, \lhd) = |\{(x, y, z) : x, z \in A, y \notin A, x \lhd y \lhd z\}|.$$

Toutes ces règles sont NP-difficiles à calculer (Theorem 6.2).

Analyse axiomatique

Nous avons tout d'abord étudié les propriétés axiomatiques des règles d'axes que nous venons d'introduire. Les résultats sont résumés dans la Table B.4, et détaillés ci-dessous.

Cohérence avec la linéarité. Cet axiome basique impose que si le profil de votes d'approbation P est linéaire, c'est-à-dire qu'il existe un axe $\lhd$ tel que chaque bulletin d'approbation est un intervalle de $\lhd$, alors la règle retourne l'axe $\lhd$. Toutes les règles satisfont cet axiome.

	VD	MF	BC	MS	FT
Cohérence avec la linéarité	✓[1]	✓	✓	✓	✓
Indépendance aux clones	✓[1]	✗	✗	✗	✗
Stabilité	✓	✗	✗	✗	✗
Monotonie	✓[1]	✗	✓	✗	✗
Externalisation	✗	✗	✓	✓	✓
Cohérence des partitions	✗	✗	✓	✓	✓
Centralité des gagnants de Veto	✗	✗	✗	✓	✓
Proximité des clones	✗	✗	✗	✗	✓

Table B.4: Propriétés des règles d'axes. Les exposants [1] indiquent la caractérisation de VD.

Stabilité et monotonie. L'axiome de *stabilité* impose qu'une règle retourne au moins un axe en commun entre le profil P et le profil $P+\{A\}$ dans lequel un votant a été ajouté. Parmi les règles que nous avons défini, seul VD satisfait cet axiome (Proposition 6.4). Une propriété plus intéressante est celle de la *monotonie*, qui impose que si un axe $\lhd$ est retourné par la règle pour un profil P, alors si on prend un des bulletins $A \in P$ du profil qui n'est pas un intervalle de $\lhd$ et qu'on le complète pour qu'il devienne un intervalle de $\lhd$ (en le remplaçant par $A' = \{x \in C : \exists y, z \in A \text{ s.t. } y \trianglelefteq x \trianglelefteq z\}$), alors la règle retourne toujours l'axe $\lhd$. Parmi les règles que nous avons définies, seules VD et BC satisfont cet axiome (Proposition 6.5).

Centristes et extremistes. L'axiome d'*externalisation* (*clearance* dans sa version anglaise) impose qu'un candidat $x \in C$ qui n'est approuvé par aucun votant ne doit interferer avec aucun bulletin d'approbation $A \in P$. L'idée est donc que les candidats moins populaires, plus extremistes, soient poussés vers les extrémités de l'axe, là où ils ne « cassent » pas les intervalles. Les règles BC, MS et FT satisfont cet axiome, mais pas les autres (Proposition 6.6). L'axiome de *centralité des gagnants de Veto* impose quant à lui que dans un certain type de profil où chaque votant approuve exactement $m-1$ candidats, le candidat au centre de l'axe retourné par la règle soit celui qui est approuvé par le plus de votants. Cet axiome est satisfait par MS et FT, mais pas par les autres règles (Proposition 6.7).

Clones. Deux candidats sont des *clones* s'ils sont approuvés par exactement les mêmes votants. L'axiome d'*indépendance aux clones* impose que si l'on ajoute un clone d'un candidat dans un profil, l'ordre relatif des autres candidats soit le même sur l'axe retourné par la règle avec le clone et sur l'axe retourné par la règle sans le clone. Cet axiome est satisfait par VD, mais pas par les autres règles (Proposition 6.9). De plus, il est possible de caractériser VD (parmi les règles de score neutre) avec l'indépendance aux clones, la cohérence avec la linéarité et la monotonie (Theorem 6.10). L'axiome de *proximité des clones* impose quant à lui que si deux candidats sont des clones, alors ils doivent être adjacents sur l'axe retourné par la règle (ou bien séparés par d'autres clones). Cet axiome est satisfait par FT, mais pas par les autres règles (Proposition 6.8). Nous avons également montré que ces deux axiomes liés aux clones sont incompatibles, et qu'aucune règle de score neutre ne peut satisfaire la cohérence avec la linéarité, l'indépendance aux clones et la proximité des clones (Theorem 6.11).

Hérédité et cohérence des partitions. Enfin, l'axiome d'*hérédité* impose que lorsque l'on supprime un candidat d'un profil, l'ordre relatif des autres candidats reste le même sur les axes retournés par la règle. Cet axiome, proche de celui d'*indépendance aux alternatives non pertinentes*, est impossible à satisfaire par une règle cohérente avec la linéarité (Proposition 6.12). On ne peut donc pas espérer construire les axes de manière séquentielle, en ajoutant les candidats les uns après les autres. Cependant, si les candidats forment une partition $C_1, \ldots, C_k$ de sorte qu'il n'existe aucun « chemin de co-approbation » entre deux candidats de deux classes différentes, alors on peut diviser le problème en k sous-problèmes, un pour chaque classe. C'est l'idée de l'axiome de *cohérence des partitions*, qui est satisfait par BC, MS et FT, mais pas par VD et MF (Proposition 6.13).

Analyse expérimentale

Malgré le fait que les règles d'axes soient NP-difficiles à calculer, nous les avons implémentées, et grâce à un algorithme optimisé, il est possible de calculer les résultats de ces règles pour des profils ayant jusqu'à une douzaine de candidats en un temps raisonnable. Nous avons testé ces règles sur de très nombreux jeux de données d'approbation, à la fois synthétiques et réels. Nous en avons également profité pour comparer nos règles basées sur les votes d'approbation avec les règles basées sur des classements de votants (notamment celles présentées par Tydrichová (2023)). Ci-dessous, nous donnons un aperçu des résultats que nous avons obtenus.

Modèles probabilistes. Nous avons tout d'abord étudié les règles sur des jeux de données synthétiques, generés selon des modèles probabilistes pour lesquels il existe un axe sous-jacent. Ainsi, les bulletins d'approbation peuvent être vus comme une approximation bruitée d'un intervalle de cet axe sous-jacent. Nous avons ensuite calculé la distance de Kendall-tau moyenne entre les axes retournés par les règles et l'axe sous-jacent. On observe que différentes règles sont plus efficaces pour différents modèles probabilistes (Figure 6.12). Cependant, la règle VD retourne de bien moins bon axes que les autres règles pour les modèles probabilistes qui ne sont pas « taillés » pour cette règle, tandis que les autres règles semblent plus robustes.

Modèle euclidien. De manière similaire, nous avons étudié un modèle euclidien à une dimension, pour lequel il existe un axe sous-jacent des candidats (selon leurs positions dans l'espace unidimensionnel), et les votants font des observations bruitées de ces positions, et basent leurs préférences sur ces observations bruitées (en approuvant tous les candidats proches de leur position, à une distance inférieur à un paramètre r). On observe que pour ce modèle, FT est la règle qui retourne les axes les plus proches de l'axe sous-jacent (suivie dans l'ordre de MS, BC, MF et VD), et que les règles basées sur les votes d'approbation sont souvent meilleures que celles basées sur les classements (Figure 6.13).

Voter Autrement. Nous avons ensuite calculé les axes pour les différents jeux de données recoltés lors des expérimentations du projet *Voter Autrement*, qui ont eu lieu lors des présidentielles françaises depuis 2002. On obtient alors des axes très proches des axes gauche-droite communément utilisés par les médias et par les institut de sondages (Table 6.2 et 6.3). On observe également que la majorité du coût des axes retournés, et des désaccords entre les différentes règles sont dus à une poignée de « petits » candidats dont le positionnement politique est plus flou (Figures 6.15 and 6.16).

Cour Suprême des États-Unis. Nous avons calculé les axes des juges de la Cour Suprême des États-Unis, en utilisant les décisions des juges sur les affaires de la Cour Suprême. Ici, c'est donc plutôt un axe des votants que des candidats, mais cela ne change rien au modèle, qui est totalement symétrique puisque toutes les données sont binaires. On compare les axes que l'on obtient à la *baseline* actuelle, basée sur la méthode de *Martin-Quinn* (Martin and Quinn, 2002). Parmi nos règles, FT semble être la meilleure, et celle qui se rapproche le plus des axes de Martin-Quinn (Table 6.4). De plus, elle semble être plus stable que les autres au fil des années (Figure 6.19).

Tierlists. Enfin, nous avons testé les règles sur des jeux de données de *tierlists*, qui sont des classements de candidats (ou d'objets) en catégories. Nous avons collecté de nombreuses tierlists sur des sujets variés (Table 6.5), qui admettent généralement un axe « naturel », par exemple des séries de films (avec un axe chronologique). Si l'axe « attendu » n'est pas toujours parfaitement préservé, les règles parviennent néanmoins à grouper les éléments similaires ensemble (Table 6.6 et Figure 6.23), par exemple les films d'une même trilogie. Cependant, on observe que les règles ont tendance à « pousser » les éléments les plus populaires vers le centre de l'axe, et les éléments les moins populaires vers les extrémités, ce qui n'est pas toujours souhaitable (Figure 6.21).

Conclusion et perspectives

Dans ce chapitre, nous avons étudié comment construire des axes de candidats à partir de votes d'approbation. Nous avons introduit un modèle formel pour ces règles, et nous avons étudié leurs propriétés axiomatiques. Nous avons également implémenté les règles et les avons comparées sur de nombreux jeux de données réels et synthétiques. Nos résultats semblent montrer que la règle des *triplets interdits* (FT) est la plus efficace pour construire des axes, et qu'elle possède de meilleures propriétés. Cependant, elle est moins explicable et plus difficile à calculer que d'autres règles. La règle de *completion de bulletins* (BC) semble être un compromis intéressant entre la simplicité de calcul et de l'explicabilité, et la qualité des axes retournés.

Ce modèle admet de nombreuses extensions qu'il serait intéressant d'étudier. On pourrait par exemple s'intéresser aux axes *« circulaires »*, dans le sens ou les deux extrémités sont reliées, ou bien vouloir augmenter le nombre de dimensions pour représenter les candidats. Enfin, une autre piste intéressante est celle de définir des règles calculables en temps polynomial pour les cas d'applications avec plus de candidats (par exemple en adaptant les règles que nous avons introduites).

B.7 Identifier les paires de candidats sources de conflit

Dans la section précédente, nous avons vu que l'on pouvait appliquer les méthodes et concepts de la théorie du choix social pour résoudre des problèmes plus larges que la simple sélection de gagnants. En particulier, nous avons étudié une manière d'*apprendre* la structure des candidats à partir des préférences des votants. Dans cette section, nous allons étudier un autre problème qui est également motivé par l'idée que l'on peut apprendre des informations sur l'électorat et les candidats à partir des préférences des votants. En effet, nous nous intéressons à la question de savoir comment identifier des *paires* de candidats qui sont le plus sources de conflit entre les votants, à partir de leurs classements.

L'étude de ce problème est motivée par l'augmentation de la polarisation et de la conflictualité dans nos sociétés (Boxell et al., 2024; Draca and Schwarz, 2024), ce qui peut mettre à mal nos systèmes démocratiques. *Identifier* les sources de polarisation au sein d'une population semble être une étape nécessaire pour *résoudre* ces conflits. On pourrait alors se concentrer sur la résolution de ces conflits en priorité. L'identification des plus grosses sources de conflit dans une population peut également avoir une tout autre motivation, comme par exemple l'organisation d'un affrontement (par exemple un débat, ou un match sportif) qui soit le plus intéressant possible pour les spectateurs.

Il est important de souligner que nous souhaitons ici identifier des *paires* de candidats qui sont sources de conflit, dans le sens où la population sera divisée en deux groupes, l'un préférant le premier candidat de la paire, et l'autre préférant le second candidat de la paire. Ce problème se distingue donc d'autres problèmes déjà abordés dans la littérature, comme celui d'identifier les candidats individuellement les plus polarisants (Colley et al., 2023b), ou celui de mesurer le degré de polarisation d'un profil de préférences, tous candidats confondus (Can et al., 2015; Faliszewski et al., 2023a). Notre objectif semble donc plus proche de l'élection de comités. Cependant, aucun des objectifs classiques des règles d'élection de comités (excellence, proportionnalité et diversité) ne correspond réellement à notre objectif de conflictualité. Il va donc falloir définir de nouvelles règles et de nouveaux axiomes pour ce modèle.

Mesures de conflit

Pour ce problème, nous supposons que nous avons un profil de préférences $P = (\succ_1, \dots, \succ_n)$, où $\succ_i$ est le classement du votant i. Pour des candidats $a, b \in C$, on note $V^{a \succ b} = \{i \in V : a \succ_i b\}$ le sous-ensemble des votants qui préfèrent a à b. Pour un votant $i \in V$, on note également $d_i(a, b) = \text{rank}_i(a) - \text{rank}_i(b)$ la différence de rang entre les candidats a et b dans le classement du votant i. Cette différence de rang est positive si le votant préfère a à b et négative si le votant préfère b à a.

Avant de définir des règles de sélection de paires de candidats conflictuels, il est nécessaire de définir ce que l'on entend par *conflit* entre deux candidats. Nous allons voir qu'il existe différentes notions de conflit, qui ne sont pas toujours compatibles entre elles.

La partition du conflit. La première notion de conflit que nous introduisons est celle de *partition du conflit*. L'idée est que le conflit induit par deux candidats est maximum si exactement la moitié des votants préfère le premier candidat de la paire, et l'autre moitié préfère le second candidat de la paire. Cela motive la définition du *ratio de partition* α :

$$\alpha(a, b) = \frac{2}{n} \min(|V^{a \succ b}|, |V^{b \succ a}|) \in [0, 1].$$

L'intensité du conflit. Une seconde notion de conflit est celle de l'*intensité du conflit*. L'idée est que le conflit est plus important si les votants ont des opinions très tranchées sur les deux candidats. En particulier, l'intensité du conflit entre deux candidats est maximale pour un votant si celui-ci classe un des candidats en première position, et l'autre en dernière position. On peut donc définir l'*intensité du conflit* β entre deux candidats a et b comme suit :

$$\beta(a, b) = \frac{1}{n \cdot (m-1)} \sum_{i \in V} |d_i(a, b)| \in [0, 1].$$

L'équilibre du conflit. Enfin, à ces deux notions s'ajoute une troisième, qui est l'*équilibre du conflit*. L'idée est que l'intensité du conflit doit être équilibrée entre les deux groupes de votants, c'est-à-dire que l'on ne doit pas avoir un groupe aux préférences très tranchées, et l'autre avec des préférences plus nuancées. On peut alors définir deux notions d'équilibre du conflit que nous notons γ et ϕ. La première est définie comme suit :

$$\gamma(a,b) = \min(\mu_{a\succ b}/\mu_{b\succ a}, \mu_{b\succ a}/\mu_{a\succ b}) \in [0,1],$$

où $\mu_{a\succ b}$ est l'intensité moyenne de conflit dans un groupe de votants :

$$\mu_{a\succ b} = \frac{\sum_{i\in V^{a\succ b}} |d_i(a,b)|}{|V^{a\succ b}|}.$$

La seconde est définie comme suit :

$$\phi(a,b) = \frac{|\sum_{i\in V} d_i(a,b)|}{\sum_{i\in V} |d_i(a,b)|} \in [0,1].$$

Une paire parfaitement conflictuelle est donc une paire de candidats (a,b) telle que $\alpha(a,b) = 1$, $\beta(a,b) = 1$, $\gamma(a,b) = 1$ et $\phi(a,b) = 0$ (Table 7.1). Cependant, il y a généralement un compromis à faire entre les différentes notions de conflit, qui peuvent être maximales pour des paires de candidats différentes dans un même profil. C'est pour cela que l'on définit des règles de sélection de paires de candidats conflictuels, dont on va voir qu'elles ont chacune tendance à favoriser certaines notions de conflit plutôt que d'autres.

Règles de conflit

On peut désormais définir différentes règles (irrésolues) de sélection de paires de candidats conflictuels. Ces règles prennent en entrée un profil de préférences (sous forme de classement) et retournent un ensemble de paires de candidats.

MaxPolarisation. Pour une valeur de $p \in \mathbb{R}_{>0}$, la règle de *p-MaxPolarisation* retourne les paires de candidats (a,b) qui maximisent la valeur de $\alpha(a,b) \times \beta(a,b)^p$.

Pairwise conflict. Pour définir les deux règles suivantes, on utilise la notion de *pairwise conflict* (*conflit par paires* en français), qui mesure le conflit induit par une paire de candidats (a,b) pour une paire de votants (i,j), et est défini comme suit :

$$\text{conf}^{\circ}_{i,j}(a,b) = \begin{cases} 0 & \text{si } d_i(a,b)\cdot d_j(a,b) > 0 \\ |d_i(a,b)| \circ |d_j(a,b)| & \text{autrement} \end{cases}$$

avec $\circ \in \{+,\times\}$. On peut alors définir des règles qui sélectionnent les paires de candidats qui maximisent la somme de ces conflits par paires $\sum_{i,j\in V} \text{conf}^{\circ}_{i,j}(a,b)$. Avec l'opérateur $\circ = +$, on obtient la règle *MaxSumConflict*, et avec l'opérateur $\circ = \times$, on obtient la règle *MaxNashConflict*.

Maximum d'échanges. Enfin, la dernière règle que nous introduisons, *MaxSwap* (*Maximum d'échanges* en français), sélectionne les paires de candidats qui maximisent le nombre minimum d'échanges (de *swap*) de candidats adjacents dans les classements des votants nécessaire pour que tous les votants préfèrent le même candidat de la paire. C'est donc une mesure de distance entre le profil de préférences et un profil où il y aurait un consensus sur cette paire de candidats. On peut définir formellement le *swap score* d'une paire de candidats $\{a,b\}$ comme suit :

	MaxSum	MaxNash	MaxSwap	MaxPolar
Stabilité par l'inverse	✓	✓	✓	✓
Cohérence au conflit	✓	✓	✓	✓
Monotonie par rapport au conflit	✗	✗	✗	✗
Cohérence à l'antagonisation	✓	✓	✓	✓
Efficacité d'intensité	✓	✓	✗	✓
Priorité à l'équilibre	✗	✓	✓	✗

Table B.5: Résumé des résultats axiomatiques.

$$\min\left(\sum_{i\in V^{a\succ b}} d_i(a,b), \sum_{i\in V^{b\succ a}} d_i(b,a)\right).$$

MaxNash et *MaxSwap* peuvent être exprimées à partir uniquement des mesures de conflit β et ϕ (Proposition 7.1 et 7.2). Enfin, à l'aide d'un exemple simple (Figure 7.1), il est possible de montrer que certaines de ces règles favorisent les paires avec une haute valeur d'intensité du conflit (c'est le cas de MaxNash) tandis que d'autres favorisent les paires avec une valeur élevée de partition du conflit (c'est le cas de MaxSwap).

Analyse axiomatique

Encore une fois, nous avons d'abord étudié les propriétés axiomatiques des règles que nous venons d'introduire. Les résultats sont résumés dans la Table B.5, et détaillés ci-dessous.

Axiomes fondamentaux. Nous introduisons deux axiomes fondamentaux pour notre modèle. La *stabilité par l'inverse* impose que si une paire de candidats $\{a, b\}$ est sélectionnée par la règle pour un profil de préférences P, alors la même paire $\{a, b\}$ doit être sélectionnée dans le profil P' dans lequel tous les classements ont été inversés. Cet axiome part de l'idée que l'*adoration* et la *haine* envers les candidats sont symétriques. Le deuxième axiome fondamental est celui de *cohérence au conflit*, qui impose que pour tout profil de préférence P non unanime (c'est-à-dire qu'il existe au moins deux votants aux classements différents), la règle retourne uniquement des paires de candidats qui sont conflictuelles, c'est-à-dire pour lesquelles il n'y a pas de consensus des votants sur l'un des candidats de la paire (autrement dit, $\alpha(a, b) > 0$). L'idée est simple : s'il existe une paire qui induit du conflit, il ne faut pas choisir une paire qui n'en induit pas. Toutes les règles que nous avons définies satisfont ces deux axiomes (Proposition 7.3).

Efficacité d'intensité. Cet axiome repose sur la notion de *domination d'intensité* entre deux paires de candidats $\{a, b\}$ et $\{c, d\}$, qui s'inspire de la domination de Pareto (voir Example 7.3). Nous avons prouvé que MaxSum, MaxNash et p-MaxPolarisation satisfont cet axiome, tandis que MaxSwap ne le satisfait pas (Proposition 7.4).

Monotonie et antagonisation. L'axiome de *monotonie par rapport au conflit* impose que si une paire de candidats $\{a, b\}$ est sélectionnée par la règle pour un profil de préférences P, alors si l'on augmente l'écart $|d_i(a, b)|$ entre a et b dans le classement d'un votant sans toucher à l'ordre relatif des autres candidats, la paire $\{a, b\}$ doit toujours être sélectionnée. Cet axiome est très fort, et aucune des règles que nous avons définies ne le satisfait. En fait, cet axiome est même incompatible avec la cohérence au conflit et la domination d'intensité (Theorem 7.5). Cependant, on peut définir un axiome plus faible, l'axiome de *cohérence à l'antagonisation*, qui impose que si une paire de candidats $\{a, b\}$ est sélectionnée par la règle pour un profil de préférences P, alors si l'on change chaque classement de sorte que ces deux candidats soient désormais en première et dernière position du classement (et dans le même ordre qu'auparavant), sans changer l'ordre relatif des autres candidats, alors la paire $\{a, b\}$ doit toujours être sélectionnée. Cet axiome, bien plus faible, est satisfait par toutes les règles que nous avons définies (Proposition 7.7).

Priorité à l'équilibre. Enfin, nous introduisons l'axiome de *priorité à l'équilibre*, qui fonctionne également sur une relation de domination entre paires de candidats (la *domination d'équilibre*, voir Example 7.4). L'idée est de favoriser les paires de candidats dont l'intensité du conflit est plus équilibrée entre les deux groupes de votants. Cet axiome est satisfait par MaxNash et MaxSwap, mais pas par MaxSum et p-MaxPolarisation (Proposition 7.8).

Analyse expérimentale

Nous nous penchons maintenant sur l'analyse expérimentale des règles de conflit que nous avons définies. Nous avons implémenté les différentes règles que nous venons d'introduire, et nous les avons testées sur de nombreux jeux de données réels et synthétiques. Nous avons également comparé les résultats qu'elles retournent à ceux de règles d'élection de comités plus classiques, comme la règle de *k-Borda* ou celle de *Chamberlin-Courant*.

Positions dans l'espace euclidien. Nous avons tout d'abord testé les règles sur un modèle euclidien à deux dimensions, dans lequel votants et candidats sont associés à des positions dans l'espace, et les votants classent les candidats en fonction de leur distance à leur position (le plus proche en premier, le plus loin en dernier). Nous avons d'abord observé *où* se situaient les candidats sélectionnés par différentes règles pour différentes distributions de positions dans l'espace (Figure 7.2). On observe notamment que les règles de conflit, comme *MaxNash*, retournent des paires de candidats éloignés du centre de l'espace, mais aussi éloignés l'un de l'autre, tandis que les règles classiques d'élection de comités, qu'elles favorisent l'excellence comme *k-Borda*, ou la diversité comme *Chamberlin-Courant*, retournent des candidats plus proches du centre de l'espace.

Mesures de conflit. Nous avons ensuite calculé les valeurs des différentes mesures de conflit pour les paires de candidats sélectionnées par les différentes règles de conflit. Ces analyses ont été faites sur des jeux de données de préférences euclidiennes (Figure 7.4), sur des préférences générées à partir de modèles de Mallows (Figure 7.7), et sur des préférences issues de données réelles (Figure 7.10). On observe bien que certaines règles sélectionnent des paires de candidats dont la *partition* du conflit α est plus forte, comme MaxSwap, tandis que d'autres règles favorisent des paires de candidats avec une *intensité* du conflit β plus forte, comme MaxNash. On observe également que les règles de p-MaxPolarisation ont tendance à ignorer l'*équilibre* du conflit (γ et ϕ).

Voter Autrement. Enfin, nous nous sommes intéressé aux résultats des règles de conflit sur les données des expériences du projet *Voter Autrement* de 2017 et 2022, pour lesquels nous possédons de nombreux classements de votants. On observe encore une fois de grandes différences entre les résultats des règles de conflit et ceux des règles d'élection de comités classiques. Si l'on se réfère aux axes gauche-droite calculés dans la section précédente, on observe que les règles de conflit sélectionnent des candidats aux extrémités de l'axe (Table 7.3). De plus, MaxNash (qui favorise l'intensité du conflit) sélectionne parfois des candidats plus « populaires » (c'est-à-dire ayant reçu un meilleur score à l'élection) mais moins « extrêmes » que les candidats sélectionnés par MaxSwap (qui favorise la partition du conflit).

Conclusion et perspectives

Dans ce chapitre, nous avons étudié comment identifier des paires de candidats qui sont le plus sources de conflit entre les votants, à partir de leurs classements. Nous avons introduit différentes notions de conflit, et nous avons défini des règles de sélection de paires de candidats conflictuels. Après avoir étudié les propriétés axiomatiques de ces règles, nous les avons testées sur de nombreux jeux de données réels et synthétiques. Nos résultats semblent montrer que MaxNash est plus efficace pour sélectionner des paires de candidats avec une forte intensité du conflit, tandis que MaxSwap est plus efficace pour sélectionner des paires avec une forte partition du conflit.

Une possibilité d'extension de ce travail serait de définir des règles de sélection d'*ensemble* de candidats conflictuels de taille $k > 2$. Pour cela, on pourrait définir par exemple des règles séquentielles, qui sélectionnent les candidats un par un, en maximisant à chaque étape le conflit avec les candidats déjà sélectionnés, ou bien on pourrait généraliser le principe des règles proposées (ce qui se fait plutôt facilement pour MaxSwap, par exemple).

B.8 Conclusion et perspectives de la thèse

Dans cette thèse, nous avons étudié cinq problèmes, tous motivés par des applications concrètes d'agrégation de préférences. Dans les trois premiers problèmes, nous avons proposé des manières d'améliorer les systèmes de vote utilisés en pratique, en utilisant des bulletins de vote plus *expressifs* que ceux utilisés actuellement. Dans les deux derniers problèmes, nous avons proposé des outils qui utilisent les préférences des votants pour *apprendre* des informations sur la structure de l'électorat et de l'ensemble des candidats, ce qui peut être particulièrement intéressant pour l'analyse politique. Pour chaque problème, nous avons introduit un modèle formel ainsi que différentes règles, avant d'étudier leurs propriétés axiomatiques. Nous avons également comparé ces règles expérimentalement sur de nombreux jeux de données réels et synthétiques.

Chacun des problèmes introduits dans cette thèse admet de nombreuses extensions intéressantes, que nous avons déjà évoquées dans les conclusions respectives de chaque section. Mais l'étape la plus importante de la suite de ce travail est probablement d'introduire ces méthodes dans le débat public, et dans le débat politique. Pour cela, il est possible de directement s'adresser aux acteurs de la vie politique (députés, conseillers locaux, etc.), comme j'ai commencé à le faire pendant ma thèse, mais aussi de sensibiliser le grand public à ces questions, et de les convaincre de l'intérêt des alternatives que nous proposons. C'est d'ailleurs un des objectifs du projet *Voter Autrement*, en plus de la collecte de données. J'essaie également de participer à cette sensibilisation en écrivant des articles de vulgarisation sur mon blog. Enfin, il est crucial de rendre les jeux de données de préférences expressives (classements, approbation, etc.) ouverts et accessibles librement, afin de permettre à d'autres chercheurs (ou simplement des curieux) de les utiliser pour étudier ces questions, et bien d'autres. Ainsi, tous les jeux de données collectés dans le cadre de cette thèse sont accessibles (Delemazure et al., 2024a; Delemazure and Bouveret, 2024).

Enfin, une autre piste de recherche que j'aimerais explorer pour poursuivre le travail que j'ai commencé, serait d'appliquer les méthodes utilisées dans les différents problèmes de ma thèse à d'autres cadres démocratiques. Il serait par exemple très intéressant d'appliquer la méthode axiomatique pour étudier les propriétés normatives des processus de délibération, qui gagnent en popularité ces dernières décennies, notamment à travers la multiplication des *conventions citoyennes* dans de nombreux pays. Ces analyses axiomatiques pourraient également être complétées par des analyses expérimentales, en utilisant les données collectées lors de ces processus de délibération.

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RÉSUMÉ

Les systèmes de vote utilisés en pratique, tels que le scrutin uninominal majoritaire à deux tours ou les systèmes proportionnels avec seuils électoraux, ne parviennent souvent pas à saisir la complexité des préférences des électeurs. Ce manque d'expressivité des bulletins de vote est à l'orgine de défauts étudiés et documentés, comme la division des voix entre candidats similaires, et la non prise en compte des votes pour les partis n'atteignant pas les seuils électoraux, incitant les électeurs au vote stratégique (dit "vote utile") et conduisant potentiellement à des résultats indésirables. Cette thèse soutient que des formats de préférences expressifs (tels que les bulletins d'approbation et les classements, avec ou sans égalités) peuvent atténuer significativement ces problèmes et potentiellement améliorer le processus démocratique, et qu'ils fournissent en outre des données pertinentes pour mener de nouvelles formes d'analyses descriptives du paysage politique.

MOTS CLÉS

Choix social, Théorie du vote, Intelligence artificielle, Analyse de données, Modes de scrutin, Préférences, Simulations, Axiomes, Politique, Représentation proportionnelle, Collecte de préférences

ABSTRACT

Voting systems that are used in practice, such as plurality with runoff or proportional systems with thresholds, often fail to capture the complexity of voter preferences. This lack of expressiveness in preference formats contributes to well-documented flaws, including the spoiler effect where similar candidates split the vote, and wasted votes for parties falling below electoral thresholds, pushing voters to vote strategically and potentially leading to the selection of undesirable outcomes. This thesis argues that expressive preference formats (such as approval ballots, rankings, and weak orders) can significantly mitigate these issues and potentially improve the democratic process, and that they additionally provide insightful data for new types of descriptive analyses of the political landscape.

KEYWORDS

Social choice, Voting theory, Artificial intelligence, Data analysis, Voting systems, Preferences, Simulations, Axioms, Politics, proportional Representation, Preference collection