Election Control for Euclidean Preferences

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Abstract

We report on some control problems for Euclidean elections. We prove that Plurality-CCAC is NP-hard for 2D-Euclidean elections. Furthermore, we demonstrate Approval-CCDV is NP-hard for 2D-Euclidean elections, and even in the case when all the voters share the same approval radius. At last, we provide polynomial-time algorithms for the Approval-CCDV and Approval-CCAV problems for the 1D-Euclidean elections. We show different proofs of these facts, using different reduction techniques.

This is a report concerning certain control problems with respect to both approval-based and ranking-based elections with respect to one-dimensional and two-dimensional Euclidean domains (defined and explained below). For a review of known results and state of the art in the field of election control and bribery, we refer the reader to the work by P. Faliszewski and J. Röthe in [2].

1 2D-Euclidean Elections

Elections. An *election* is a tuple $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ and $V = \{v_1, \dots, v_n\}$ are, respectively, the set of candidates and the set of voters. We assume that each voter $v \in V$ has a linear preference order over the set of candidates $\succ_v$ and we identify each voter with her preference order. Given a voter v and a candidate c, by $\mathrm{pos}_v(c)$ we denote the position of c in v's preference order, with the meaning that the candidate in the top position in v's order has position 1. Given an election $E = (C, V)$, under the Plurality Rule, each candidate $c \in C$ receives a score:

$$\mathrm{score}(c) = \sum_{v \in V} \alpha_1(\mathrm{pos}_v(c)),$$

where $\alpha_1 : \mathbb{N}_+ \to \mathbb{R}$ is a function defined as follows: $\alpha_1(t) = 1$ if $t = 1$, and $\alpha_1(t) = 0$, if $t > 1$, and the winner of the election $E = (C, V)$ is the candidate that gets the highest score.

The CONSTRUCTIVE CONTROL by ADDING CANDIDATES (CCAC) for Plurality Rule (hence denoted as Plurality-CCAC) is the following computational problem:

Given an election $(C \cup D, V)$, and a candidate $p \in C$, where C is the set of *registered* candidates and D is the set of *unregistered* candidates, and a number k, which is a maximum number of newly added candidates

Decide if there exists a subset $D' \subseteq D$ of size at most k such that in the election $(C \cup D', V)$ the candidate p is the winner under the Plurality Rule.

Given an election $E = (C, V)$, we say that the voters have tD-Euclidean preferences ($t \in \mathbb{N}$) if for each agent $a \in C \cup V$ (i.e., for each candidate and each voter) there exists a point $x_a = (x_{a,1}, \dots, x_{a,t})$ in $\mathbb{R}^t$ such that for every voter $v \in V$ and for all candidates $c_i, c_j \in C$ it holds that:

$$\text{if } \sqrt{\sum_{k=1}^{t}(x_{c_i,k} - x_{v,k})^2} \leq \sqrt{\sum_{k=1}^{t}(x_{c_j,k} - x_{v,k})^2},$$

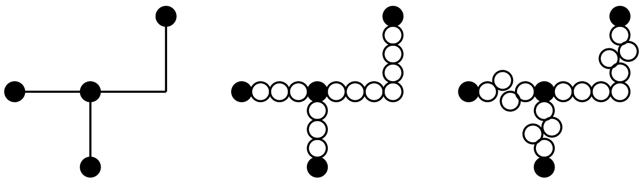

Figure 1: Construction of a penny graph.

$$\text{then } c_i \succeq_v c_j,$$

i.e., the closer the (ideal point of the) candidate c_i is to the (ideal point of the) voter v, the higher c_i is in v's preference order. Intuitively, the point x_a describes a's ideal position in a t-dimensional space of opinions.

In our NP-hardness proofs for the 2D Euclidean elections we use *penny graphs.* A penny graph is defined by a set of unit disks, i.e., balls of diameter one in $\mathbb{R}^2$, such that no two disks overlap (but they can touch). Each disk corresponds to a vertex and two vertices are connected by an edge if their disks touch (i.e., if their centers are exactly at distance 1). A graph is a penny graph if it has such a representation by unit disks (the name comes from the analogy between the disks and pennies laying on a flat surface). All penny graphs are planar. We will need the following algorithm of Valiant.

Lemma 1. *[7]. There is a polynomial-time algorithm that given a planar graph with maximum degree at most 4 computes its embedding on the plane so that its vertices are at integer coordinates and its edges are drawn with vertical and horizontal line segments.*

The following construction has been used in [4] devoted to the analysis of Winner Determination in Approval Multiwinner Voting Rules for Euclidean Elections.

Recall that in the Vertex Cover problem (VC) we are given a graph $G = (X, E)$ and a positive integer r. We ask if there exists a vertex cover of G—i.e., a subset of vertices $U \subseteq X$ of size at most r such that each edge $\{x, y\} \in E$ has an end in U, i.e., at least one of the vertices x, y is in U. It is known that the problem is NP-hard for cubic planar graphs [3]. Given an instance (G, r) of VC, where G is a cubic planar graph, we can construct an instance of VC for penny graphs as follows (we use the construction of Cerioli et al. [2011, Theorem 1.2]; we repeat it here as we need its specific properties).

First, we use Lemma 1 to obtain a planar representation of G, where the vertices are at integer coordinates and the edges consist of vertical and horizontal line segments (see the left-hand side of Figure 1; note that in this figure the vertices have degrees at most three, and not exactly three). Second, we multiply vertex coordinates by four, ensuring that the lengths of the line segments forming the edges also are multiples of four. Third, for each vertex v we put a unit disk centered at the position of v, and we replace all the line segments forming the edges by sequences of consecutive unit disks (located on the integral points within these lines; see the center of Figure 1). This way, each edge becomes a sequence of $4t - 1$ disks, where t is an integer (possibly different for each edge). Finally, for each edge we introduce a single *local displacement*, which consists of replacing the second disc that lies on the edge with two tangent disks (it does not matter from which end we start counting the disks); these two disks are also tangent to the disks on the two sides of the disk that we replaced (see the right-hand side of Figure 1). Local displacements ensure that disks on the edges come in multiples of four. All in all, we obtain a penny graph.

Let G' be the penny graph that we constructed. Each vertex of G' has either two or three adjacent vertices. The vertices with two neighbors correspond to disks put on the edges and we refer to them as *intermediate*. We call a vertex locally displaced if it corresponds to a disk that was introduced as a result of a local displacement. Let L be the total number of intermediate vertices. One can easily verify that G has a vertex cover of size r if and only if G' has a vertex cover of size $r' = r + L/2$. We refer to the penny graphs obtained by this construction as *almost integral* and we use the fact that VC is NP-hard for them.

Theorem 1. *Deciding Plurality-CCAC is NP-hard, even if the voters have 2D-Euclidean preferences.*

Proof. The proof goes by reduction from the Vertex Cover problem for almost integral penny graphs (where the graph is given by its geometric representation). Let (G, r) be an instance of VC, where G is an almost integral penny graph and r is an integer.

Let n be the number of vertices in G and let m denote the number of edges in G. Let ϵ be any positive real number from the interval $(0, 1/4)$. We distinguish the following points in $\mathbb{R}^2$ (we will use them as the ideal points of the agents):

Vertex Points: For each vertex x_i, we have a vertex point located in the center of x_i's disk,

Edge Points: For each edge $e = \{x_i, x_j\}$ in G, we have a point in the middle of e, to which we refer as e_{ij} (we view edges in G as straight, unit-length line segments).

Intermediate Points: For each edge $\{x_i, x_j\}$ in G, we take the unit line segment $\overline{x_i x_j}$ and let q_{ij} and q_{ji} be points on the line $\overline{x_i x_j}$ such that the distance of q_{ij} from the vertex x_i is ϵ and the distance of q_{ji} from the vertex x_j is ϵ as well.

Given (G, r), we construct an instance of Plurality-CCAC for 2D-Euclidean Elections consisting of an election $E = (C \cup D, V)$ and a distinguished candidate $p \in C$, with the following candidates and voters:

1. The set of registered candidates C consists of *edge candidates* and the *distinguished candidate*: In each edge point e_{ij} we put one edge candidate, called c_{ij}, and we put the distinguished candidate p in any point of the plane that makes her distant from the graph G, i.e., we put p in any point p such that for any point x in the graph (where we abuse notation and by 'in the graph' we mean that x is either a vertex or a point lying on one of the edges of G)

$$d(p, x) > \max(\{d(y, z) : y, z \in G\}).$$

   Together with the distinguished candidate p, there are thus $m+1$ registered candidates.

2. The set of unregistered candidates D consists of n candidates d_i (with their ideal points) located in the vertex points x_i.

3. We have the following three groups of voters

   (a) The distinguished voters: exactly 13 voters located in the point p.

   (b) The intermediate voters: exactly $8m$ voters located in the intermediate points q_{ij} and q_{ji} - we put 4 voters in each point q_{ij} ($u^1_{ij}, u^2_{ij}, u^3_{ij}, u^4_{ij},$), so there are 8 intermediate voters on each edge of the graph)

   (c) The edge voters: In each edge point e_{ij} we put six voters, $v^1_{ij}, v^2_{ij}, \dots, v^6_{ij}$.

Therefore, there are $m' = 1 + m + n$ candidates and $n' = 13 + 14m$ voters. We let k to be equal to r.

Now observe that in the election $E' = (C, V)$, the distinguished candidate gets 13 votes, whereas each registered edge candidate c_{ij} obtains exactly 14 votes - 8 votes from the intermediate voters, each of which is in the distance $1/2 - \epsilon$ from c_{ij}, and 6 votes from the edge voters located at the same point as c_{ij}.

It remains to show that the reduction is correct. Let us assume that G is a "yes" instance of VC. Take any vertex cover U of size r and define the corresponding subset D' of the set D of unregistered candidates to be $D_U = \{d_i \mid x_i \in U\}$, i.e., let D_U consist of the vertex candidates corresponding to the members of U.

In the election $(C \cup D_U, V)$, the distinguished candidate p gets 13 votes again, each of the newly added candidates d_i gets either:

- 8 votes from the intermediate voters $u^1_{ij}, \ldots, u^4_{ij}, u^1_{it}, \ldots, u^4_{it}$, in the case that the vertex x_i has 2 neighbors (x_j and x_t), in the graph G,
- 12 votes from the intermediate voters $u^1_{ij}, \ldots, u^4_{ij}, u^1_{ti}, \ldots, u^4_{it}, u^1_{il}, \ldots u^4_{il}$,in the case that the vertex x_i has 3 neighbors (x_j, x_t, and x_ℓ) in the graph G.

Since U is a vertex cover of G, each registered edge candidate c_{ij} gets either:

- 6 votes from the edge voters located in the same edge point e_{ij}, if both of the vertices x_i, x_j are members of U — in this case, both unregistered candidates d_i, d_j get registered or,
- 10 votes: 6 votes from the edge voters located in the same edge point e_{ij}, and 4 votes from the intermediate candidates u_{it} or u_{jl}, if only one of the vertices x_i, x_j is a member of U — in this case, only one of the unregistered candidates d_i, d_j, gets registered and obtains the votes of 4 intermediate voters located within ϵ from her,

and since U is a vertex cover of G, these are the only possibilities - at least 4 votes have to be *taken away* from each registered candidate c_{ij} by either d_i or d_j, since at least one of the points x_i or x_j has to enter the set U.

All in all, if there is a vertex cover U of size r in G, then adding r candidates from D to the election (C, V) results in p getting the highest Plurality score and making her the unique winner of the election $C \cup D_U, V)$.

For the other direction, assume that there is no size-r vertex cover of G. Then, for any size-r set $D' \subset D$ of unregistered candidates there is at least one edge e_{ij} such that neither x_i nor x_j are in the size-r set of vertices, therefore there must also be at least one registered edge candidate c_{ij} such that she still obtains 14 votes, since neither d_i nor d_j get registered. Then, p cannot be even a non-unique winner of the election $(C \cup D', V)$, which ends the proof. □

We also consider control problems for other election rules. An *approval-based election* is a pair $E = (C, V)$, where $C = \{c_1, \ldots, c_m\}$ and $V = \{v_1, \ldots, v_n\}$ are, respectively, the set of candidates and the set of voters. For each voter $v \in V$, by $A(v)$ we denote the *approval set* of v, i.e., the set of those candidates that voter v finds acceptable. Conversely, by $V(c)$ we denote the set of voters who approve candidate c, i.e., $V(c) = \{v \in V \mid c \in A(v)\}$.

Given an approval-based election E, under the Approval Rule, each candidate $c \in C$ receives a score:

$$\text{score}_E(c) = |\{v \in V : c \in A(v)\}|.$$

However, we note that whenever election E is clear from the context, we use a more concise notation, that is, $score(c)$. The winner of the election is the candidate that gets the highest score.

The CONSTRUCTIVE CONTROL by DELETING VOTERS (CCDV) for Approval Rule (hence denoted as Approval-CCDV) is the following computational problem:

Given an approval-based election (C, V), and a candidate $p \in C$, *Decide* if there exists a subset $V' \subseteq V$ of size at most k such that in the election $(C, V \setminus V')$ the candidate p is the winner under the Approval Rule.

Given an approval-based election $E = (C, V)$, we say that the voters have tD-Euclidean preferences ($t \in \mathbb{N}$) if for each agent $a \in C \cup V$ (i.e., for each candidate and each voter) there exists a point $x_a = (x_{a,1}, \ldots, x_{a,t})$ in $\mathbb{R}^t$ and a nonnegative real value $r_a \in \mathbb{R}$ such that:

$$c \in A(v) \iff \sqrt{\sum_{j=1}^{t}(x_{c,j} - x_{v,j})^2} \le r_c + r_v.$$

For a candidate $c \in C$, r_c can be seen as c's charisma: It specifies which positions surrounding his or her ideal one the candidate can accommodate credibly. For a voter $v \in V$, r_v specifies v's willingness to compromise, i.e., the positions around his or her ideal one that the voter still accepts. Two special cases of Euclidean Preferences are:

1. The voter range model (VR), where we require that all the candidates have radii equal to zero.
2. The candidate range model (CR), where all the voters have radii equal to zero.

We refer to the full model as the voter/candidate range model (VCR).

We first prove the weak form of our theorem, giving a hardness result for the widest class of election scenarios. We include it, since we believe the technique we employ to obtain it is of its independent interest and might be useful to other researchers working on similar problems.

Theorem 2. *Deciding Approval-CCDV is NP-hard, even if the voters have 2D-Euclidean preferences.*

Proof. The proof goes by reduction from the Planar Exact 3-Hitting Set Problem.

Recall the Set Cover problem. Given a ground set $U = \{1, \ldots, n\}$, a family $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq \mathcal{P}(U)$, and a positive integer k, decide if there exists a set $J \subseteq \{1, \ldots m\}$ s.t. $|J| \le k$, and $\bigcup_{j \in J} F_j = U$.

An instance of Exact Cover by 3-Sets problem, **X3C** is a special case of an instance of Set-Cover. Given a ground set U with the property that $|U|$ is divisible by 3, a family $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq \mathcal{P}(U)$ s.t. $\forall i \le m \; |F_i| = 3$ and $k = |U|/3$, the problem is to decide if there exists a set $J \subseteq \{1, \ldots m\}$ s.t. $|J| = k$, and $\bigcup_{j \in J} F_j = U$.

An instance of the Planar Exact Cover by 3-Sets (Planar-X3C) problem consists of a bipartite planar graph $G = (X \cup \mathcal{S}, E)$, where $X = \{x_1, \ldots, x_{3k}\}$, $\mathcal{S} = \{S_1, \ldots, S_m\}$, and every element in $\mathcal{S}$ is connected to exactly 3 elements in X. The problem is to decide if there is a subset $\mathcal{S}' \subset \mathcal{S}$ of size k s.t. every element of X is connected to exactly one element of $\mathcal{S}'$.

The following modification of Planar-X3C was used in [5]. We follow the modification to apply it to our investigation. The modified version of Planar-X3C, the so-called Planar-X3C*, there are additional geometric constraints imposed on graph G from the Planar-X3C. Namely, when embedded on the plane, the graph has the following properties:

1. For every $x \in X$, $S_i, S_j \in \mathcal{S}$ s.t. $x \in S_i \setminus S_j$ it holds that $d(x, S_i) < d(x, S_j)$.
2. For every $x, x' \in X$, $S_i \in \mathcal{S}$ s.t. $x \in S_i$ but $x' \notin S_i$, it holds that $d(x, S_i) < d(x', S_i)$.
3. For every $x \in X$, $S_i, S_j \in \mathcal{S}$ s.t. $x \in S_i \cap S_j$ it holds that $d(x, S_i) < 2d(x, S_j)$.

4. For every $x, x' \in X$, $S_i \in \mathcal{S}$ s.t. $x, x' \in S_i$, it holds that $d(x, S_i) < 2d(x', S_i)$.
5. For every two elements x, x' that belong to the same set $S_i \in \mathcal{S}$, the angle (x, S_i, x) is between $\frac{\pi}{3}$ and $\frac{5\pi}{6}$.
6. Every 3 elements that share a set induce a triangle, and the triangles do not overlap.

Last, but not least, recall the Exact 3-Hitting Set (X3HS) problem. Given a ground set U with the property that $|U|$ is divisible by 3, a family $\mathcal{F} = \{F_1, \ldots, F_m\} \subseteq \mathcal{P}(U)$ s.t. $\forall i \in [m]$ $|F_i| = 3$, and $k = {}^{|U|}/_{3}$, the problem is to decide if there exists a subset U' of U of size k such that for each $i \in [m]$ $|F_i \cap U'| = 1$.

The problem of X3HS is polynomially equivalent to X3C, and Planar-X3HS and Planar-X3HS*, which are analogously defined for planar graphs and planar graphs with geometric constraints, are polynomially equivalent to Planar-X3C, and Planar-X3C*, correspondingly.

Since Planar-X3C* was shown to be NP-hard, the corresponding Planar-X3HS* problem is NP-hard as well.

We thus reduce from Planar-X3HS*. Let $I = (G = (X \cup \mathcal{S}, E), k)$ be an instance of Planar-X3HS*. Let $n = 3k$ be the number of vertices in G, and let m denote the number of edges in G. Without loss of generality, we may assume that $m > 2k$.

We distinguish the following points in $\mathbb{R}^2$ that, in the embedding of G on the plane, correspond to elements if I (we will use them again as the ideal points of the agents):

Vertex Points: For each vertex x_i, we have a point in the location of the vertex.

Set Points: For each set S_j in $\mathcal{S}$, we have a point in the location of the set.

Given (G, k), we construct an instance of Approval-CCDV for 2D-Euclidean Elections consisting of an election $E = (C, V)$ and a distinguished candidate $p \in C$, with the following candidates and voters:

1. The set of candidates C consists of *set candidates* and the *distinguished candidate*: In each set point S_j we put one set candidate, called c_j, and we put the distinguished candidate p in any point of the plane that makes her distant from the graph G, i.e., we put p in any point p such that for any point x in the graph:

$$d(p, x) > \max(\{d(y, z) : y, z \in G\}).$$

   Together with the distinguished candidate p, there are thus $m + 1$ candidates.

2. We have the following two groups of voters
   (a) The distinguished voters: exactly 5 voters located in the point p.
   (b) The vertex voters: exactly $6k$ voters located in the vertex points x_i - we put 2 voters in each point x_i (v_i^1, v_i^2). For each i, consider the set $B_i = \{S_j : x_i \in S_j\}$, and define v_i's approval radius as $r_i = \max_{S_j \in B_i} d(x_i, S_j)$.

There are thus exactly $m + 1$ candidates and $6k + 5$ voters. Each vertex voter v_i^α, for $\alpha \in \{1, 2\}$ approves of exatly $deg(x_i)$ candidates. Observe that, by construction, each set candidate S_j obtains exactly six votes: one from each point v_i^1 and v_i^2, for all three points x_i with $x_i \in S_j$. By the geometric constraints imposed on the instance of Planar-X3HS*, the preferences of the voters satisfy the Euclidean condition.

Now, let $k' = 2k$. We claim that there exists a hitting set of size at most k in G if and only if it is possible to delete at most $k' = 2k$ voters so that p becomes a unique winner in the constructed election.

To see this, first assume (G, k) is a "yes" instance of Planar-X3HS*. Thus, it is possible to find a set $H = \{x_{i_1}, \dots, x_{i_k}\}$ of k vertices in the graph G, such that for each $j \in [m]$, the set S_j contains exactly one of the members of H, i.e., $\forall i \in [m]\ |S_j \cap H| = 1$. Let us delete the $2k$ voters corresponding to the vertices from H, i.e., delete all the voters from the set: $V' = \{v_i^\alpha : x_i \in H; \alpha \in \{1, 2\}\}$. Since H is a hitting set for G, each set candidate loses exactly two votes thanks to deletion of the voters from V'. In effect, the score of each set candidate is now equal to 4, which makes p still having score equal to 5, the unique winner of the election.

For the other direction, assume (G, k) is a "no" instance of Planar-X3HS*. This means that for any k-size set H of vertices of G, there exists a set $S_j \in \mathcal{S}$ such that it has an empty intersection with H. This implies that for any $2k$-size set of voters $V' \subseteq V$, there has to be a set candidate c_j such that in the election $(C, V \setminus V')$ he obtains at least five votes.

□

We prove two variants of the stronger hardness result concerning the CCDV problem for Approval-based elections, for two versions of the control problem. In the first one, let's call it the unique-winner model, we are asked if it is possible to delete voters in such a way that the designated candidate p becomes the unique winner of the election, i.e., having strictly more votes than any other candidate. For the other one, which we refer to as the non-unique-winner model, we ask if it is possible to make the candidate p be in the set of (possibly tied) winners of the election. i.e., delete voters in such a way that no other candidate has strictly more votes than p. The proofs of NP-hardness for both of versions differ slightly, so we include the arguments for both of the theorems.

Theorem 3. *Deciding Approval-CCDV is NP-hard for the non-unique winner model, even if the voters have 2D-Euclidean preferences in the VR model and all the approval radii of the voters are identical.*

Proof. The proof goes again by reduction from the Vertex Cover problem for almost integral penny graphs (where the graph is given by its geometric representation). Let (G, r) be an instance of VC, where G is an almost integral penny graph and r is an integer. Let ϵ be any positive real number strictly smaller than $1/4$.

Let n be the number of vertices in G and let m denote the number of edges in G. Without loss of generality, we assume that $m > 2r$. We distinguish the following points in $\mathbb{R}^2$ (we will use them again as the ideal points of the agents):

Vertex Points: For each vertex x_i, we have a vertex point located in the center of x_i's disk,

Edge Points: For each edge $e = \{x_i, x_j\}$ in G, we have a point in the middle of e, to which we refer as e_{ij} (we view edges in G as straight, unit-length line segments).

Given (G, r), we construct an instance of Approval-CCDV for 2D-Euclidean Elections consisting of an approval-based election $E = (C, V)$ and a distinguished candidate $p \in C$, with the following candidates and voters:

1. The set of candidates C consists of *edge candidates* and the *distinguished candidate*: In each edge point e_{ij} we put one edge candidate, called c_{ij}, and we put the distinguished candidate p in any point of the plane that makes her distant from the graph G, i.e., we put p in any point p such that for any point x in the graph:

$$d(p, x) > \max(\{d(y, z) : y, z \in G\}).$$

Together with the distinguished candidate p, there are thus $m+1$ registered candidates.

2. We have the following two groups of voters
   (a) The distinguished voter: exactly 1 voter located in the point p.
   (b) The vertex voters: exactly 1 voter located in each point x_i (v_i) with the approval radii equal to $1/2$.

   Let the approval radii of all voters be equal to $1/2$.

Each voter approves of either one candidate (the voters located in the point p), two candidates (the vertex voters corresponding to the vertices of G of degree equal to two), or three candidates (the vertex voters corresponding to the vertices of G of degree equal to three).

Therefore, there are $m' = m + 1$ candidates and $n' = 1 + n$ voters. We let k to be equal to r.

Now observe that in the election $E' = (C, V)$, the distinguished candidate gets 1 vote, whereas each candidate c_{ij} obtains exactly 2 votes. Namely, each c_{ij} gets one vote from each of the vertex voters v_i, and v_j.

It remains to show that the reduction is correct. Let us assume that G is a "yes" instance of VC. Take any vertex cover U of size r and define the corresponding set of voters V' to be $V' = \{v_i : x_i \in U\}$, i.e., let V' consist of the vertex voters corresponding to the members of U. Delete the voters V' from V and consider the election $(C, V \setminus V')$.

Since U is a vertex cover of G, each edge is incident to at least one vertex x_i from U. Therefore, each edge candidate receives at most 1 votes. This is so, since at least two voters corresponding to an element of U x_i is incident to got deleted.

All in all, if there is a vertex cover U of size at most r in G, deleting the corresponding $2r$ voters results in p getting the highest Approval score equal to 1 and becoming a non-unique winner.

For the other direction, suppose there is no vertex cover of G of size at most r. This means that for any size-r set $U \subseteq X$ of vertices in G there is at least one edge e_{ij} such that neither x_i nor x_j are in U. Therefore, for any size-r set $V' \subseteq V$ of voters that can be deleted there must be at least one candidate C_{ij} such that she still obtains at least 2 votes, so p cannot be even a non-unique winner of the election $(C, V \setminus V')$. This completes the proof. □

As announced, a modification of the proof gives also:

Theorem 4. *Deciding Approval-CCDV is NP-hard for the unique winner model, even if the voters have 2D-Euclidean preferences in the VR model and all the approval radii of the voters are identical.*

Proof. We follow exactly the construction above, and modify one detail. Instead of one distinguished voter, we put exactly 2 voters located in the point of the distinguished candidate p, and we keep all the other voters.

The rest of the proof remains the same. If (G, r) is a "yes" instance of the Vertex Cover, then all the edge candidates obtain at most 1 vote after deleting the voters corresponding to the vetices of the vertex cover, and p becomes the unique winner.

If there is no vertex cover of size at most r of G, then for any choice of $k = r$ voters to be deleted, there will always be an edge candidate such that he or she still obtains 2 votes from the vertex candidates. Therefore, in this case, the candidate p will not become a unique winner of the election, wchich completes the proof. □

2 1D-Euclidean Elections

Definition 1. *In the 1D-VCR model, every agent a is represented by their point x_a and radius r_a. We define a's range beginning and range end as:*

$$b(a) = x_a - r_a,$$

$$e(a) = x_a + r_a.$$

Definition 2. *Having an approval election E and two candidates a and b. Under the approval-voting scoring rule, we say that:*

$$scoreDelta(a, b) = score(a) - score(b).$$

2.1 Approval-CCDV

In this section we describe our algorithm and consider all the caveats. Let us consider the problem of Approval-CCDV. The input consists of a VCR election $E = (C, V)$, a distinguished candidate $p \in C$, and a maximum number of voter deletions k. First, we discuss some concepts and observations, that will help us explain the algorithm. We use the representation of an agent in a VCR election by his or hers range beginning and end (see Definition 1). Next, for p to become a unique winner, we need to defeat every candidate that has at least as many votes as p. We call this subset of candidates a set of opponents, formally:

$$\widetilde{C} = \{c \in C \setminus \{p\} |\ scoreDelta(c, p) \geq 0\}.$$

Another observation is that we never delete a voter who approves of p. Clearly, it would not bring us any closer to a solution. Thus, when deleting voters, we consider only the ones from the set:

$$V^{''} = \{v \in V |\ p \notin A(v)\}.$$

Our algorithm is a greedy one. It repeats the following steps until p becomes a unique winner, or it fails by exceeding the maximum number of voter deletions k. First, we find

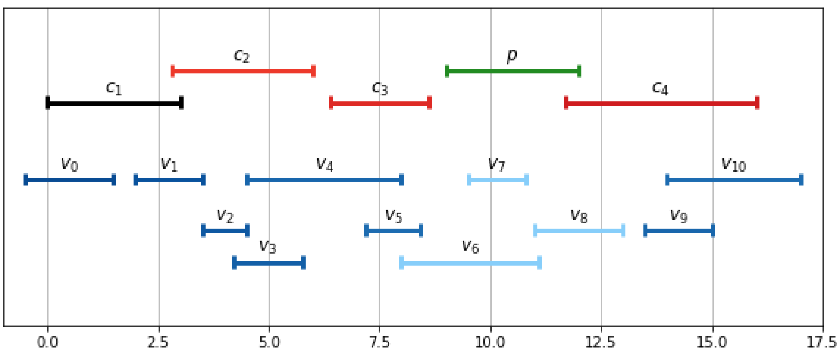

Figure 2: A 1D-VCR election for the Approval-CCDV problem. The candidates are labeled with c_i (expect for the preferred candidate p), and the voters with v_i. We use colors to distinguish some key groups of agents. That is, p is in green, and voters that approve of him or her (we never delete them) are in light-blue color. Additionally, all the initial opponents are in red.

a candidate that we will try to defeat. Then, to achieve this goal, we decide which voters to delete. After each iteration, we recompute approvals and the set of opponents $\widetilde{C}$. We describe these steps in detail below:

1. **Choosing opponents.** The only candidates that we consider are from the set of opponents $\widetilde{C}$. We pick the leftmost candidate, that is, the one with the minimal beginning of the range, $b(c)$ (we break ties arbitrarily). We refer to the selected candidate as c_L. This strategy together with recalculating $\widetilde{C}$ after each iteration, ensures that we perform only the necessary steps to find a solution.

2. **Deleting voters.** Having currently processed opponent c_L. We consider only the voters who approve of him or her, i.e., $\widehat{V} = \{v \in V'' |\, c_L \in A(v)\}$. To defeat c_L, we need to delete $\mathit{scoreDelta}(c_L, p) + 1$ voters from $\widehat{V}$. If $|\widehat{V}| < \mathit{scoreDelta}(c_L, p) + 1$, then we fail (it is impossible to delete enough voters to defeat c_L). We proceed with deleting the rightmost voters, that is, the ones with the maximum end of the range $e(v)$ (we break ties arbitrarily). We believe that this is an optimal strategy, because choosing the rightmost voter maximizes the number of other opponents who might be contained inside each voter's range. Furthermore, it is perfectly safe to do so for two reasons: The first one is our leftmost candidate picking strategy. At this point any opponent to the left of c_L would have already been defeated. The second reason is related to the special case of contained candidates that we discuss in detail in Section 2.2.

We illustrate the algorithm in Example 1, which describes an election from Figure 2.

Example 1. *Let us discuss an instance of the Approval-CCDV problem. We consider the VCR election from Figure 2, the designated candidate p with 3 votes (i.e., $\{v_6, v_7, v_8\}$) and the set of opponents $\widetilde{C} = \{c_2, c_3, c_4\}$. We proceed to find the smallest number of voters that we need to delete for p to win the election. First, we choose the leftmost opponent c_2. We need to delete two voters ($\mathit{scoreDelta}(c_2, p) + 1$). We pick the rightmost voters v_4 and v_3. Now, an important observation is that by removing v_4 we defeated the next opponent c_3. We finish this iteration, and recompute approvals and the set $\widetilde{C}$. The only opponent left is c_4, and we just need to delete one voter. We pick v_{10} (in this case it would not matter if we chose v_9). We succeed, p is the winner, and in total we removed three voters, i.e, v_4, v_3 and v_{10}.*

2.2 Special Case of Contained Candidates

In the VCR model it is possible that some candidates have their interval fully contained inside another candidate's interval (see Figure 3). If so, every voter who would approve the smaller candidate (i.e., the internal one) would also approve the larger one (i.e., the external one), but not necessarily the other way round. For this case, our algorithm's strategy of choosing rightmost voters to delete, might seem suboptimal at first glance. We define additional terminology and symbols to help us explain why we do not have to worry about this issue.

Definition 3. *Consider an instance of the CCDV problem, with an approval election E, a set of candidates C, and the designated candidate p. We extend the representation of the score difference between two candidates (see Definition 2). For each candidate $c \in C \setminus \{p\}$ we represent a number of voter deletions required for p to defeat c with:*

$$\Delta(c) = \mathit{score}(c) - \mathit{score}(p) + 1.$$

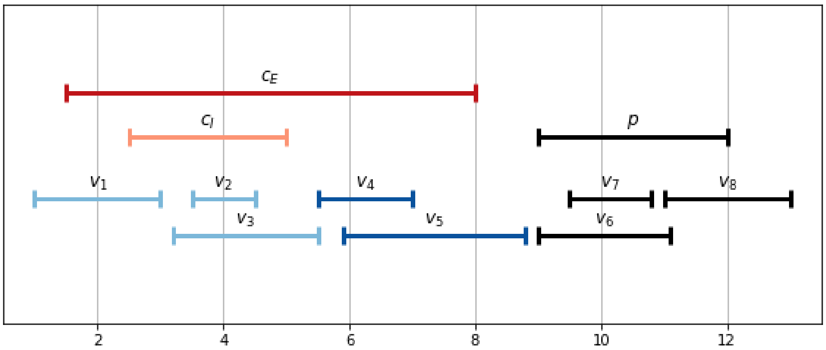

Figure 3: A 1D-VCR election with two candidates contained inside each other's ranges. The larger one c_E (external), and the smaller one c_I (internal). Furthermore, voters from the set $\widehat{V}$, are divided into two groups, i.e., $\widehat{V}_E = \{v_4, v_5\}$ a subset of voters who approve of c_E but not c_I, and a subset of remaining voters $\widehat{V}_I = \{v_1, v_2, v_3\}$. Because we discuss the CCDV problem, there is also the preferred candidate p and his voters.

Definition 4. *Consider an election E, where a candidate is contained inside another one. The two candidates are c_E and c_I, where the range of c_I is contained in that of c_E. The voters who approve of c_E or c_I are in the set $\widehat{V}$. We divide the voters into two groups. First, a subset of voters who approve of c_E but not c_I:*

$$\widehat{V}_E = \{v \in \widehat{V} \mid c_E \in A(v) \wedge c_I \notin A(V)\},$$

and the remaining ones:

$$\widehat{V}_I = \widehat{V} \setminus \widehat{V}_E.$$

We know the size of these subsets based on the approval scores of each candidate:

$$|\widehat{V}_E| = score(c_E) - score(c_I),$$

$$|\widehat{V}_I| = |\widehat{V}| - |\widehat{V}_E|.$$

Let us consider an instance of the Approval-CCDV problem, with an election that has this special case. We keep in mind that the internal candidate c_I might not even be a dangerous candidate at all (i.e., $\Delta(c_I) \leq 0$). For p to defeat c_E we need to delete $\Delta(c_E)$ voters, and to defeat c_I (independently of c_E), we need to delete $\Delta(c_I)$ voters (see Definition 3). Now we show that in the process of defeating the larger candidate we always automatically defeat the smaller one. Using simple transformations in Equation 1 we show that:

$$\begin{aligned} \Delta(c_E) = score(c_E) - score(p) + 1 = \\ |\widehat{V}_E| + score(c_I) - score(p) + 1 = |\widehat{V}_E| + \Delta(c_I). \end{aligned} \tag{1}$$

It means that to defeat c_E we have no other choice but to delete at least $\Delta(c_I)$ voters from $\widehat{V}_I$. The remaining $|\widehat{V}_E|$ voters can be deleted from any of the subsets (i.e., either $\widehat{V}_E$ or $\widehat{V}_I$). This will result in defeating both c_E and c_I (automatically).

Theorem 5. *The 1D-VCR Approval-CCDV algorithm finds an optimal solution in polynomial time.*

Proof. Let S be the solution found by the algorithm and let S^* be an optimal solution to the problem, that is, a minimal-size set of removed voters. We consider every dangerous candidate $c \in \widetilde{C}$. They are ordered by their range's beginnings, ascending (i.e., left to right). We choose the first candidate in this order, for whom there is a voter $v_r \in S - S^*$. When v_r was added to S, it was the rightmost voter (max $e(v_r)$). Since S^* is optimal, then there must be some voter $v_x \in S^* - S$, such that v_x approves the first candidate and $e(v_x) \leq e(v_r)$, because v_r is the rightmost voter. Let the set $S^{*\prime} = S^* - \{v_x\} \cup \{v_r\}$ be S^* but substituting v_r for v_x. We know that $S^{*\prime}$ is a correct solution, because the approval set of the voter v_r contains at least as many candidates as the approval set of the voter v_x. Since $|S^{*\prime}| = |S^*|$, and we can repeat the just-described process, we conclude that $S^{*\prime}$ is a minimal-size set of removed voters. □

2.3 Improving Approval-CCDV Algorithm

We can simplify our Approval-CCDV algorithm and make the special case of contained candidates more intuitive. To do so we modify the strategy of choosing oponents in the algorithm. We continue to pick the leftmost candidate, however, instead of choosing the one with the minimal beginning of the range, we choose the one with minimal end of the range, $e(c)$. This way we make sure, that if there is a contained (internal) candidate it is going to be processed before the external one.

2.4 Approval-CCAV

We extended the strategy from [6] for CR elections. Let us consider a problem of Approval-CCAV. The input consists of a VCR election $E = (C, V)$, a set of unregistered voters U, a distinguished candidate p, and a maximum number of voter additions k. We assume that the election $(C, V \cup U)$ has the VCR property, and that every voter in U approves of p (adding a voter who does not approve of p would be counterproductive). First, we form a new election $E' = (C, V \cup U)$, that is, we add all unregistered voters to the election. If p in not a winner then it is impossible for him or her to become a winner (after adding all U-voters, current score of p is the highest he or she can obtain). Now, we have to remove $k' = max(\|U\| - k, 0)$ of U-voters, so that, we do not exceed k voters addition limit when we form E'. After we remove k' voters, the score of p will decrease to $score(p) - k'$. To preserve p's victory we have to ensure that each candidate $c \in C \setminus \{p\}$ has a score lower than $score(p) - k'$. We can reformulate this problem to solving Approval-CCDV instance, and then we can utilize the previous algorithm (Section 2.1).

Let us consider an election formed from only newly added voters $E'' = (C, U)$, we have to delete exactly k' voters in such a way that every candidate $c \in C \setminus \{p\}$ has a score lower than $score_{E'}(p) - k'$. The Approval-CCDV algorithm considers a set of opponents $\widetilde{C}$ (i.e., candidates with score higher or equal to p's), and defeats them one by one. To do so the algorithm deletes $scoreDelta(c, p) + 1$ voters. Now, to solve Approval-CCAV, for each candidate $c \in C$ we define

$$\Delta(c) = score_{E'}(c) - (score_{E'}(p) - k') + 1,$$

intuitively $\Delta(c)$ is a number of U-voters who approve of c that we need to delete from E''. Next we simply redefine the set of opponents to

$$\widetilde{C} = \{c \in C \setminus \{p\} | \ \Delta(c) \geq 0\}.$$

The Approval-CCDV algorithm will ensure that every candidate has at most $(score_{E'}(p) - k') - 1$ approvals (by deleting $\Delta(c)$ voters for each opponent), and it will not exceed k' voter deletions (if it deletes fewer voters, then we can choose remaining ones arbitrarily).

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