# Cutsets and EF1 Fair Division of Graphs

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**Abstract**

In fair division of a connected graph $G = (V, E)$, each of n agents receives a share of $G$'s vertex set $V$. These shares partition $V$, with each share required to induce a connected subgraph. Each agent uses her assigned valuation function to determine the non-negative numerical value of her share, and these values determine whether the allocation is fair in some specified sense. We show that *graph cutsets*, introduced here, constitute obstacles to divisions that are fair in the EF1 (envy-free up to one item) sense. If $G$ guarantees connected EF1 allocations for $n$ agents with valuations that are *CA* (common and additive), then $G$ contains no *cutset of gap* $\geq 2$ *and valence* $n-1$. If $G$ guarantees connected EF1 allocations for $n$ agents with valuations in the broader *CM* (common and monotone) class, then $G$ contains no *generalized* cutset of gap $\geq 2$ *and valence* $n-1$. These results rule out the existence of connected EF1 allocations in a variety of situations. They generalize one direction of the characterization, in Biló et al. [3], of graphs that guarantee connected EF1 allocations for $n = 2$ agents as those containing no *trident* (regardless of whether valuations are *CA* or *CM*), and suggest that the CA vs. CM distinction may be consequential for EF1 graph division when there are more than 2 agents. Additionally, we provide an example of a (non-traceable) graph on eight vertices that has no cutsets of gap $\geq 2$ at all, yet fails to guarantee connected EF1 allocations for three agents with *CA* preferences. We end by conjecturing a common pattern for all graphs, governing the values of $n$ for which connected EF1 allocations for $n$ agents are guaranteed to exist.

## 1 Introduction

In the original, continuous setting for fair division, a single divisible good or "cake," often modeled by the closed interval $[0, 1]$, is divided into $n$ pieces, with each agent allocated a different piece of the resulting partition. One thread of this literature studies allocations that are both envy-free (each agent values her assigned piece at least as highly as she values any of the other pieces) and connected (each piece forms a single subinterval of $[0, 1]$).

For the alternative setting of indivisible items, a finite set $O$ of indivisible goods is partitioned into disjoint subsets, with each agent allocated a different subset from the partition. This context precludes envy-freeness as a reasonable goal; for example, if $O$ contains but a single item, only one agent can get it. Budish [5] proposed a relaxation, *envy-freeness up to one good*, aka *EF1*, that circumvents this obstacle. It requires that whenever one agent $i$ envies the share of another agent $j$, there exists some item in $j$'s share whose removal would eliminate that envy.

Fair division of graphs, our context here, provides a natural way to import the connectivity requirement from the continuous world into the world of indivisible goods. The vertices of a finite connected graph $G = (V, E)$ are viewed as indivisible items, and we insist that the share of vertices allocated to each agent must form a connected subgraph. Natural applications represented by this model (and mentioned in [11]) include, for example, the problem of dividing cities connected by a road network among several parties, as when an island is partitioned and each party wishes to drive among its allocated cities without leaving its own territory. Alternately, consider offices allocated to several departments of an organization, where an edge represents a section of corridor joining a pair of offices in the organization's building, and each department must receive contiguous offices.

The graph cutsets introduced here provide a general tool for obtaining negative results when $n$ agents share the vertices of some finite, connected graph $G$ and each agent is required to receive a connected set of vertices. The cutset itself is a set of subgraphs; when the vertices in those subgraphs are excised, $G$ falls into a number of disconnected sections. Some agent $j$ will wind up with a share $A_j$ that fails to include enough critical vertices from the deleted subgraphs, so that $A_j$ lacks vertices to form a path connecting any pair of disconnected sections. This confines $A_j$, so that if one has chosen the valuations appropriately, then $A_j$'s value is too small to be fair, in the EF1 sense.

Let $N = \{1, 2, \ldots, n\}$ be a finite set of *agents* and $G = (V, E)$ be an undirected finite graph. A subset $I$ of $V$ is *connected* if it induces a connected subgraph of $G$. We write $\mathcal{C}(V)$ for the set of connected subsets of $V$, and we call a set $I \in \mathcal{C}(V)$ a (connected) *piece*. Each agent $i \in N$ has a *valuation* – a function $v_i \colon \mathcal{C}(V) \to \mathbb{R}^+$ assigning non-negative real values to connected pieces, with $v_i(\emptyset) = 0$. A valuation $v_i$ is *monotone* if for all $X, Y \in \mathcal{C}(V)$ it holds that $X \subseteq Y$ implies $v_i(X) \leq v_i(Y)$. Monotone valuations treat vertices as *goods*; we do not consider bads (or chores) here. The valuation functions of the agents are called *common* if $v_i = v_j$ holds for all $i, j \in N$, and are *arbitrary* if not required to be common. Valuations are *additive* if $v_i(I) = \sum_{x \in I} v_i(\{x\})$ for each agent $i$ and each piece $I \in \mathcal{C}(V)$. We will use abbreviations *CM* for "common and monotone", and *CA* for "common and additive." Additive valuations form a proper sub-class of monotone valuations (because of the non-negativity constraint on valuations), and CA forms a proper sub-class of CM. A (connected) *allocation* $A = \{A_i\}_{i \in N}$ of $G$ assigns each agent $i \in N$ a connected piece $A(i) \in \mathcal{C}(V)$, with these pieces *partitioning* $V$, so that $\bigcup_{i \in N} A_i = V$ and $A_i \cap A_j = \emptyset$ when $i \neq j$.

In fair division of a graph $G$, we ask whether there exists such an allocation that is *fair*, in some well-defined sense. *Maximin share fairness* was the principal fairness criterion studied in Bouveret et al. [4], which first introduced the topic of graph fair division. Three later works – Biló et al. [3], Igarashi [10], and Igarashi and Zwicker [11] – instead focus (as does this paper) on the two variants of *envy-freeness* defined below. The original definition of *envy-freeness* requires, of an allocation $A$ that $v_i(A_i) \geq v_i(A_j)$ hold for every pair $i, j \in N$ of agents – that each agent thinks her piece is, in her view, a best piece in the allocation. With indivisible objects, envy-free allocations may not exist, and so we instead use the following notions.

**Definition 1** *An allocation $A$ of graph vertices is* envy-free up to one good*, aka* EF1*, if for any pair $i, j$ of agents, either $v_i(A_i) \geq v_i(A_j)$, or there is an element $x$ of $A_j$ such that $v_i(A_i) \geq v_i(A_j \setminus \{x\})$; $A$ is* envy-free up to one outer good*, aka* $\text{EF1}_{\text{outer}}$ *if for any pair $i, j$ of agents, either $v_i(A_i) \geq v_i(A_j)$, or there is an element $x$ of $A_j$ such that $A_j \setminus \{x\}$ is connected and $v_i(A_i) \geq v_i(A_j \setminus \{x\})$.*

Here, EF1 is the original relaxation introduced by Budish [5]. Graph fair division, however, only allows connected shares, so the version of the property introduced in [3], and used here requires that $A_j$ remains connected after removing the vertex in question.[1] The additional demands made by $\text{EF1}_{\text{outer}}$ seem to be appropriate, at least so far, with positive results typically establishing the stronger $\text{EF1}_{\text{outer}}$ requirement and negative results defeating the weaker EF1.

Negative results in fair division seek counterexamples with agent valuations from as narrow a class as possible (often the class CA, sometimes with additional restrictions on the number of distinct vertex values). Positive results aim to apply to valuations from the broadest class possible (with monotonicity often being the only requirement). In general, positive results for a graph $G$ seem to be linked to whether $G$ is *traceable* (which means that $G$ has a *Hamiltonian path* – a path visiting each vertex exactly once) or satisfies some weak version of this property. In particular, the following theorem from [10] is of this kind. It improves an earlier version in [3].[2]

**Theorem 1** *([3], [10] ) Let G be any traceable graph and $n \geq 1$ be any integer. Given any assignment of monotone valuation functions to $n$ agents, there exists a connected* $\text{EF1}_{\text{outer}}$ *allocation.*

We restate this result informally, as follows: *Under arbitrary monotone preferences, traceable graphs universally guarantee connected $EF1_{outer}$ allocations.* Here *universally* conveys that the result holds for arbitrarily many agents; *guarantee* conveys that it holds for an arbitrary assignment of a (monotone) valuation function to each agent.

**Question 1** *Does there exist a non-traceable graph that offers the same universal guarantee?*

[1]Note that we do not even require that an agent's valuation function be defined for non-connected sets of vertices.

[2]For $n \geq 5$, the result in [3] only promises *$EF2_{outer}$* (which allows two items to be deleted, in removing any envy).

**Question 2** *Does the answer to Question 1 change if we restrict valuation functions to be CA?*

These questions remain open. One result from [11] leans negative, however: if $G$, as well as every subdivision of $G$, universally guarantees connected $EF1_{outer}$ allocations under CA valuations, then $G$ is traceable, as are all its subdivisions. Here, a *subdivision* of $G$ is obtained by placing new vertices of degree 2 along $G$'s edges.[3]

Our focus in this paper is on *particular* results rather than universal ones; given a particular integer $n$ and a graph $G$, does $G$ guarantee connected $EF1_{outer}$ allocations for $n$ agents with valuations from some specified class? Two previous results of this kind stand out. The first is a complete characterization for the two-agent case; the second applies to three agents, but is narrower in scope. The first result requires two additional notions.

**Definition 2** *A* bipolar ordering *of a graph $G = (V, E)$ is an enumeration (non-repeating and exhaustive list) $x_1, x_2, \ldots, x_k$ of $G$'s vertices such that every initial segment $x_1, x_2, \ldots, x_m$ ($1 \leq m \leq k$) of the list is connected, as is every final segment $x_m, x_{m+1}, \ldots, x_k$. Equivalently, each vertex $x_i$ is adjacent to some vertex $x_j$ appearing earlier on the list (unless $i = 1$) and is adjacent to some vertex $x_j$ appearing later on the list (unless $i = k$).*

Every Hamiltonian path *is* a bipolar ordering, but the converse fails.[4] The other notion needed is that of a *trident* – loosely, a subgraph whose removal cuts the graph into 3 or more disconnected pieces (expressed as "three or more connected components," below – see examples in Section 2).

**Definition 3** *For $G = (V, E)$ a finite connected graph, let $C \subseteq V$ be a set of vertices of $G$, and let $G \setminus C$ denote the subgraph of $G$ induced by the vertex set $V \setminus C$.*

*(i) If $C = \{c\}$ contains a single vertex, and $G \setminus C$ has three or more connected components, then $C$ is a* type 1 trident.

*(ii) If $C$ contains more than one vertex; $G \setminus C$ has exactly three connected components $H_1, H_2$, and $H_3$; for each $H_j$ exactly one vertex $s_j \in C$ (referred to as $H_j$'s* contact vertex*), is adjacent to any vertices of $H_j$; and the vertices $s_1, s_2, s_3$ are distinct, then $C$ is a* type 2 trident.

**Theorem 2** *([3]) The following are equivalent for any finite connected graph $G$:*

*(i) $G$ guarantees connected* $\text{EF1}_{\text{outer}}$ *allocations for $n = 2$ agents with monotone valuations that are arbitrary (they need not be common).*

*(ii) $G$ guarantees connected* $\text{EF1}_{\text{outer}}$ *allocations for $n = 2$ agents with CA valuations.*

*(iii) $G$ contains no tridents.*

*(iv) $G$ has a bipolar ordering.*

**Theorem 3** *([11]) The lips graph,[5] as well as all of its subdivisions, guarantees connected* $\text{EF1}_{\text{outer}}$ *allocations for $n = 3$ agents with monotone valuations.*

The proof of Theorem 3 uses a discretization of a modified version of Stromquist's famous moving knife argument for continuous fair division of the $[0, 1]$ interval. The technique works for a few other graphs, but has not yielded any general result for 3 agents, and we know of no characterization for 3 agents analogous to Theorem 2. For *positive* results, the situation for 4 or more agents is worse yet – there are none, except those for traceable graphs already implied by Theorem 1, and for the very specialized result we present in the Appendix.

The graph cutsets defined here provide new negative results for a variety of specific graphs and values of $n \geq 3$. Speaking loosely, a cutset $\mathcal{C}$ is a set containing several tridents; if any of these are

[3] That is, an edge from $x$ to $y$ is replaced by two edges – from $x$ to $z$ and from $z$ to $y$, where $z$ is a newly inserted *subdivision vertex* of degree 2 – and such insertions may be made repeatedly.

[4] Graph IV of Figure 2 is not traceable, but has a bipolar ordering – for example, order the vertices from left to right, with the middle pair of vertices ordered either way.

[5] The lips graph, depicted in [11], has vertices $a$, $b$, and $c$; two edges join $a$ to $b$, two join $b$ to $c$, and one joins $a$ to $c$.

type 2, then $\mathcal{C}$ is a *generalized cutset*, and if exactly one is type 2, then $\mathcal{C}$ is a *tame generalized cutset*. Cutsets generalize the role of tridents in Theorem 2, which applied only to $n = 2$, and are related to cutsets for *tangles* ("continuous graphs" – see [11]), but with differences.[6]

**Paper outline** The rest of the paper is organized as follows. Section 2 presents some examples of tridents and graph cutsets, motivating the definitions of Section 3, which also contains our two main results: if a graph $G$ contains either an "ordinary" cutset, or a *generalized* cutset that is *tame*, then there exist $n$ agents with CA valuations such that no connected *EF*1 allocation exists. From a non-tame generalized cutset, we obtain a weaker result: there exist $n$ agents with CM valuations (that are superadditive but not subadditive, hence not additive) such that no connected *EF*1 allocations exist. In Section 4 we present a counterexample to any converse, in the form of a graph $G$ containing no cutsets of any kind, yet there exist CA valuations for 3 agents that rule out the existence of a connected *EF*1 allocation. Thus while cutsets explain, for many graphs, why *EF*1 allocations can fail to exist, they are not the only explanation, and cannot, in their current formulation, provide anything like a characterization. Finally, in Section 6 we provide an analysis that settles completely, for Graph IV of Figure 2, and all values of $n$, whether connected $EF1_{outer}$ allocations for $n$ agents are guaranteed[7] (the only "no" being for $n = 3$), and we conjecture what such a complete analysis might look like for finite graphs in general.

# 2 Examples of tridents and cutsets

## 2.1 Trident Examples

Figure 1 shows three connected graphs, each of which contains a *trident*; see Definition 3. For each graph, we will consider vertex allocations to two agents. In Graph I, deleting the one point $a$ would disconnect the graph into three components (each of which consists of a single vertex, for this simplest possible example). Only one agent can receive a share including $a$; we will say that this agent *dominates* $a$. The share of the second "deprived" agent must be contained within a single one of the three components, as she cannot use $a$ to forge a connected share from vertices belonging to different components. Next, assume that each of the four vertices has value 1 to both agents, and the value of a set of vertices is obtained by summing the values of the individual vertices. Then the deprived agent receives a single vertex of value 1, while the other agent receives three vertices, each worth 1 to the deprived agent, thus leaving her envious of the other agent by more than one item. In some other example, deleting vertex $a$ might leave more than three connected components, each with more than one vertex . We can similarly force envy by assigning value 1 to $a$ and to one vertex from each component, with value 0 assigned to all other vertices. In these situations, vertex $a$ is an example of a *type 1 trident* – an obstacle to connected *EF*1allocations for two agents, when paired with a suitable choice of CA valuations.

Deleting any single vertex from Graph II yields at most two disconnected components, so this graph has no type 1 trident. It does have what we will call a *type 2 trident*, however, in the form of the central subgraph $\mathcal{C}_{II}$ induced by vertex set $\{b, c, d\}$, which acts collectively in a manner similar to a type 1 trident. Deleting the vertices in $\mathcal{C}_{II}$ would yield three disconnected components, and only one agent can *dominate* $\mathcal{C}_{II}$ by being allocated at least two of the three vertices $b, c, d$. The share of a second, deprived agent contains at most one of these three vertices – not enough for her to form a connected share containing vertices from more than one of the components. Note that this last argument requires that each component have its own distinct *contact point* $s \in \mathcal{C}$, with $s$ adjacent to a

[6] We say that an edge from $x$ to $y$ in $G$ is *saturated* if $x$ or $y$ has degree 1 or degree 2; $G$ is saturated if all its edges are. Any graph can be made saturated by inserting a new subdivision vertex along each edge. Envy-free fair division of tangles tells us a lot about EF1 fair division of saturated graphs, but less about non-saturated ones. Differences between the graph and tangle definitions of cutset seem driven, in part, by additional subtlety in defining graph cutsets for non-saturated graphs.

[7] Equivalently, whether *EF*1 allocations are guaranteed for Graph IV – see comment in the Appendix.

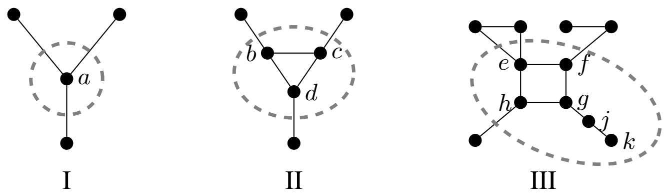


Figure 1: Graph I has a type 1 trident; II and III have type 2 tridents.

vertex in that component. For a type 2 trident, the counterexample CA valuations that defy connected *EF*1 allocations for 2 agents are a bit different. We assign value $\frac{1}{3}$to each of the three contact points $b, c$ and $d$ of $\mathcal{C}_{II}$, value $1$ to one vertex from each of the three components, and value $0$ to all other vertices (from the components, or from $\mathcal{C}_{II}$). The deprived agent now receives a share with at most two valuable vertices, whose values are $1$ and $\frac{1}{3}$, while the agent who dominates $\mathcal{C}_{II}$ is left with at least four valuable vertices with values $1$, $1$, $\frac{1}{3}$, and $\frac{1}{3}$, so he is envied by more than one item.

What about variants of Graph II for which deleting a subgraph $\mathcal{C}$ leaves more than three components? This happens in Graph III if we declare the trident to be the central square induced by vertices $e, f, g,$ and $h$. We would then get 4 components, each with its own distinct contact point in the square. However, we disallow tridents with more than three contact points (and they are not needed for the 2-agent characterization). In the case of Graph III we can instead enlarge the square by having our trident $\mathcal{C}_{III}$ absorb the fourth component (with vertices $j$ and $k$ in Figure 1) completely, as suggested by the dashed gray ellipse in the figure. For the 2 agent case, the same can be done any time the proposed type 2 trident has more than 3 contact points, or has 3 contact points with multiple components sharing a common contact point.

## 2.2 Graph Cutset Examples

Cutsets provide obstacles to connected *EF*1 allocations for more than just 2 agents, generalizing tridents. The consecutive numbers $1, 2$ and $3$ played a critical role in a type 1 trident; we removed $1$ point from a connected graph, we had $2$ agents, and the point's removal yielded $3$ subgraphs that were disconnected from one another. Suppose instead we remove $2$ points from the graph, we have $3$ agents, and when we remove both of the points we get $4$ disconnected subgraphs? This is exactly the situation for Graph IV in Figure 2, when removing the circled points $a$ and $b$. We will declare the *gap* $\geq 2$ *cutset* here to be $\mathcal{C}_{IV} = \{a, b\}$. The word "gap" here refers to the difference between the number of points removed and the number of disconnected subgraphs that result.

How can connected *EF*1 allocations for three agents be blocked by a cutset similar to the one for Graph IV? Given any partition of the vertices into three connected shares, at most one agent dominates vertex $a$ (meaning $a$ is in his share) and at most one other dominates $b$. With three agents, this leaves some third, *deprived* agent dominating neither member of the cutset, and as before his share will only contain vertices from one of the four disconnected subgraphs. Consider the CA valuation that assigns value $1$ to $a$, $1$ to $b$, $1$ to exactly one vertex selected from each of the four disconnected subgraphs, and value $0$ to all remaining vertices (if there are any – of course, there are none for Graph IV). The deprived agent gets at most one valuable vertex, worth $1$, with $5$ valuable vertices, worth $1$ each, to be split between the remaining two agents. One gets at least three of these, and the deprived agent envies her by more than one item.

Graph V is different; removing two vertices from the graph never yields more than *three* disconnected subgraphs.[8] But the subgraph induced by $c, d,$ and $e$ acts like a type 2 trident; at most one agent $x$ can dominate the subgraph (meaning $x$'s share contains at least two of the three contact points $c, d,$ and $e$). We set the gap $\geq 2$ cutset to be $\mathcal{C}_V = \{\{c, d, e\}, \{f\}\}$, our first example of a

[8]For a traceable graph, deleting $k$ vertices never yields more than $k + 1$ disconnected subgraphs.

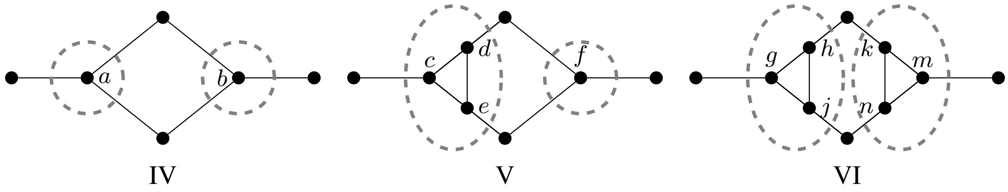


Figure 2: Three examples of gap $\geq 2$ cutsets with valence 2.

*generalized cutset*. The additional set braces clarify that $\mathcal{C}_V$ has two members – a *type I member* resembling a type 1 trident, and a *type II member* resembling a type 2 trident. Deleting the vertices from both members yields 4 disconnected subgraphs. We can think of each member as a kind of ticket, on which is printed a face value of 1:

**1 pass-through allowed**.

Any agent who dominates a member gets to use the ticket, granting her the possibility to obtain a share containing vertices from two of the four subgraphs.

The *valence* of a cutset is the sum of the face values on the tickets, so $\mathcal{C}_V$ has valence 2, as does cutset $\mathcal{C}_{IV}$ for Graph IV. (Shortly we will consider tickets with face values $k > 1$, that can be used by $k$ agents, and that correspond to cutset members with enough contact points to be passed through by $k$ agents.) When the number of agents is one more than the valence of the cutset (for graphs IV or V, when the number of agents is three), we know that some deprived agent has no ticket to use, so his share has vertices from at most one of the disconnected subgraphs.

We use $\mathcal{C}_V$ to block connected *EF*1*outer* allocations for three agents with CA valuations as follows: assign weight 1 to each of the four unlabeled vertices in Figure 2 (that's one vertex from each disconnected subgraph), weight 1 to vertex $f$ and weight $\frac{1}{3}$ to each of the vertices $c, d, e$; if there were any additional vertices, we would have assigned them weight 0. Now the deprived agent receives at most *two* valuable vertices, with values of 1 and $\frac{1}{3}$. The remaining agents receive at least three unlabeled vertices, so some agent $y$ gets at least two of these. These two come from different disconnected subgraphs, so to connect them agent $y$ must also get a "ticket," meaning $y$'s share either includes $f$ or includes at least two of the three vertices. So agent $y$ either gets three vertices valued at 1, 1, and 1, or gets four vertices valued at 1, 1, $\frac{1}{3}$ and $\frac{1}{3}$. Either way, the deprived agent envies $y$ by more than one item.

This more complicated weighting works in general for generalized cutsets that are *tame*, meaning they are limited to a single type II member, but falls apart with two or more such members, as we see next. For Graph VI we will consider the cutset $\mathcal{C}_{VI} = \{\{g, h, j\}, \{k, m, n\}\}$, with 2 type II members, and valence of 2. With three agents, there will again be a deprived agent who dominates neither member of $\mathcal{C}_{VI}$, meaning his share contains fewer than 2 vertices from $\{g, h, j\}$ and fewer than 2 from $\{k, m, n\}$. However there is a connected allocation in which the deprived agent's share is $\{h, k, T\}$ (where $T$ denotes the top middle vertex of Graph VI), and the other two shares are $\{L, g, j\}$ and $\{B, n, m, R\}$ (where $L, B$ and $R$ denotes the leftmost, bottom middle, and rightmost vertices, respectively). Suppose we use a weighting similar to that from Graph V, by assigning weight $\epsilon$ to each of the six vertices $g, h, j, k, m$, and $n$, and weight 1 to the other 4 vertices. Then the specified allocation turns out to be *EF*1*outer* for any value of $\epsilon \geq 0$.[9]

We can, however, use $\mathcal{C}_{VI}$ to construct common *monotone* preferences that are not additive and that block connected *EF*1 allocations of Graph VI for 3 agents. We set the common value of a share $A$ to be the number of unlabeled vertices in $A$ plus the number of cutset elements of $\mathcal{C}_{VI}$ dominated by $A$. For example, $h$ alone adds nothing to a share, nor does $h$ along with $k$, but $h$ along with $j$ (or $h$ along with $j$ and $g$) adds 1 to the value. As the deprived agent dominates neither member of $\mathcal{C}_{VI}$, her share is worth 1. Once again some agent $y$ gets at least two unlabeled vertices from different pieces,

[9]Other weightings are possible, of course, but so far we found no CA valuations that defeat *EF*1*outer* for Graph VI.

each worth 1, and dominates some member of $\mathcal{C}_{VI}$. So $y$ additionally owns at least two vertices from a single cutset element. Taken together, they add another 1 to the value of her share, for a total value of 3. Again, the deprived agent envies $y$ by more than one item.

Graph VII has a cutset that blocks connected $EF1_{outer}$ allocations for 4 agents with certain CA valuations. Our cutset for this graph will be $\mathcal{C}_{VII} = \{\{a\}, \{b, c, d, e, f\}\}$, with two members; $D = \{a\}$ is type I and $E = \{b, c, d, e, f\}$ is type II, but has 5 contact points rather than only 3. Consider any connected share that contains 2 of the 5 unlabeled vertices of Graph VII, but omits $a$. To connect those two, it must include at least two of $E$'s vertices – that is, it must dominate $E$. But $E$ has only 5 vertices, so it can be dominated by at most two agents. Its ticket reads

**2 pass-throughs allowed**

and enables as many as two agents to pass through $E$, while a third ticket enables one more agent to pass through $D$. The valence of $\mathcal{C}_{VII}$ is the sum $1 + 2 = 3$ of the pass-through numbers on these two tickets. The rest of the argument is similar to the one for Graph V. [10]

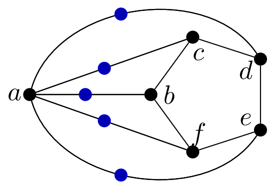


Figure 3: Graph VII has a cutset of valence 3.

The seven examples presented here portend all of the main ideas used in the general proofs of Section 4, as well as most – but not all – of the features found in the precise definitions of cutset and of generalized cutset. Other examples, not presented here, have stretched these definitions in two directions, relaxing some requirements so as to be satisfied by these other examples, while forcing the imposition of additional requirements not needed for graphs I – VII. We seek a definition as broad as possible, to encompass any situation for which a version of the deprived agent argument applies. For example, we want to allow cutsets for which the "gap" (between valence of some cutset $\mathcal{C}$ and the number of connected components that remain after excising $\mathcal{C}$) is more than two, but the requirement that contact points be unique and distinct must also be imposed on these excess components, and these requirements then sometimes rule out cutset examples that we might think to include. In some cases we can bypass that problem by merging several components into one. The merged version is a subgraph that might no longer be internally connected, so we will not call it a "component" any more; note that the deprived agent argument only needs these subgraphs to be disconnected from each other, and never needs them to be internally connected as subgraphs.[11] Other requirements are necessitated for generalized cutsets that are not tame (meaning they have more than one type II member). For example, we need to be sure that the deprived agent cannot put together a path from one section to a different one by crossing from one type II member to a different type II member while using only one contact point from each.

# 3 Cutsets for Graphs: Definitions and Main Theorem

Here we give precise definitions for *graph cutset* (aka cutset *simpliciter*) and for *generalized graph cutset* (tame or not), and prove that for $n$ agents with specified valuations, cutsets with valence $n - 1$ block the existence of connected *EF*1 allocations for $n$ agents. These proofs are straightforward and very similar to those for the examples in the previous section.

The definition of cutset *simpliciter* is also quite simple – much more so than that for generalized cutset to follow. A cutset *simpliciter* is just a set of $t$ vertices, that – when excised – break the graph into at least $t + 2$ pieces that are disconnected from one another (expressed as "at least $t + 2$ connected

[10]With four agents, some deprived agent dominates neither member of $\mathcal{C}_{VII}$ and their share contains at most one unlabeled vertex. If we assign weight 1 to $a$ and to each unlabeled vertex, and assign weight $\frac{1}{3}$ to each of the five contact vertices of $H_2$, then the deprived agent will envy one of the others by more than one item.

[11]Curiously, the argument also does not require that a type II member of a cutset be connected as a subgraph. But it is certainly easier to find cutsets hiding within a graph when their type II element are connected, and we do not know of any examples where imposing connectivity would limit the consequences for connected *EF*1 allocations.

components” in the precise version, below). The *valence*, in this case, is just the number $t$ of vertices that are cut (but valence will become more complicated for generalized cutsets). This simpler type of cutset was exemplified in graphs I (with valence 1) and IV (with valence 2) of the previous section, and can be thought of as a generalization of the type 1 tridents from [3].

**Definition 4** *(graph cutset) For $G = (V, E)$ a finite connected graph, let $\mathcal{C} = \{c_1, c_2, ..., c_t\}$ be a set of $t$ vertices in $V$. If the subgraph of $G$ induced by the vertex set $V \setminus \mathcal{C}$ contains at least $t + 2$ nonempty connected components $H_1, H_2, \ldots, H_t, H_{t+1}, \ldots, H_{t+r}$, then $\mathcal{C}$ is a* graph cutset of gap $\geq 2$ and valence $t$.

Before we define generalized cutset, we need some additional terminology:

**Definition 5** *Let $K = (V, E)$ be a finite graph, not necessarily connected. Two sets $L, M \subseteq V$ are* independent in $K$ *if $L \cap M = \emptyset$ and no vertex in $L$ is adjacent to any vertex in $M$; they are* isolated in $K$ *if there is no path in $K$ from a vertex in $L$ to a vertex in $M$ . Given a set $H = \{H_1, H_2, \ldots H_s\}$ with $H_i \subseteq V$ for each $i$ with $1 \leq i \leq s$, $H$ is* independent in $K$ *if every two members in $H$ are independent in $K$; $H$ is* isolated in $K$ *if every two members in $H$ are isolated in $K$.*

**Observation 1** *Suppose $K = (V, E)$ is a finite graph, and $H = \{H_1, H_2, \ldots H_s\}$ is a set of subsets of $V$. Then if $H$ is isolated in $K$, it is independent in $K$. Suppose we additionally assume that $H$ partitions $K$'s vertex set $V$. Then $H$ is isolated in $K$ if and only if it is independent in $K$.*

The definition of generalized cutset is complex, so we will start with a slightly simpler, preliminary version of the definition that excludes some cases covered by the final version. Specifically, the cutset for Graph VII (in the previous section) will not satisfy our preliminary definition. Recall that this cutset introduced a new feature; one of its cutset elements had 5 contact vertices, rather than 3, allowing as many as two agents to own shares containing 2 of them. Those two agents could both use their vertices to connect a pair of the unlabeled vertices in the diagram. This is why the corresponding (imaginary) “ticket” had a face value of 2 and why we needed 4 agents rather than 3 (and 5 unlabeled vertices rather than 4) to make the *deprived agent* argument go through. Our preliminary version will only apply to cutsets for which the corresponding tickets each have face value 1. For these cutsets “valence” is just equal to the number of members of a cutset, allowing us to write a definition in which “valence” is synonymous with cardinality:

**Preliminary Definition 1** *(preliminary version, generalized graph cutset) For $G = (V, E)$ a finite connected graph, let*

- *$\mathcal{C} = \{C_1, C_2, ..., C_t\}$ be a set of $t$ pairwise disjoint, nonempty subsets of $V$,*
- *$G \setminus \mathcal{C}$ be the subgraph of $G$ induced by the vertex set $V \setminus \bigcup \mathcal{C}$ (where $\bigcup \mathcal{C} = C_1 \cup C_2 \cup \cdots \cup C_t$),*
- *and $H = \{H_1, H_2, \ldots, H_{t+r}\}$ partition $V \setminus \bigcup \mathcal{C}$, with $H$ isolated in the graph $G \setminus \mathcal{C}$.*

*Assume, in addition, that*

- *for each $C_i$ and $H_j$ there is at most one vertex $s_{i,j}$ in $C_i$ adjacent to any vertices in $H_j$, with $s_{i,j}$ referred to as the* contact vertex *for $C_i$ and $H_j$,*
- *each $C_j \in \mathcal{C}$ is either a “type I member” containing a single vertex, or a “type II member” containing more than one,*
- *for each type II member $C_i$, exactly three of the $H_j$ have a contact vertex in $C_i$ and these three vertices are distinct,*
- *the set containing all type II members of $\mathcal{C}$ is independent in $G$, and*
- *$r \geq 2$.*

*Then $\mathcal{C}$ is a generalized graph cutset of gap $\geq 2$ and valence $t$, with witness $H$.*

Note that each $H_j$ is a union of the vertex sets from one or more of the connected components of $G \setminus \mathcal{C}$, so that $H_j$ itself is disconnected if more than one component is used. Our final version of the definition now allows for cutset elements with pass-through numbers greater than 1.

**Definition 6** *(final version, generalized cutset) For $G = (V, E)$ a finite connected graph, let*

- *$\mathcal{C} = \{C_1, C_2, ..., C_t\}$ be a sequence of $t$ pairwise disjoint, nonempty subsets of $V$,*
- *$\tau = \{\tau_1, \tau_2, ..., \tau_t\}$ be a sequence of natural numbers, with $\tau_j$ called $C_j$'s* pass-through number *and the sum $\Sigma\tau$ of all $\tau_j$ called the* valence*,*
- *$G \setminus \mathcal{C}$ be the subgraph of $G$ induced by the vertex set $V \setminus \bigcup \mathcal{C}$ (where $\bigcup \mathcal{C} = C_1 \cup C_2 \cup \cdots \cup C_t$),*
- *and $H = \{H_1, H_2, \ldots, H_{\Sigma\tau+r}\}$ partition $V \setminus \bigcup \mathcal{C}$, with $H$ isolated in the graph $G \setminus \mathcal{C}$.*

*Assume, in addition, that*

- *for each $C_i$ and $H_j$ there is at most one vertex $s_{i,j}$ in $C_i$ adjacent to any vertices in $H_j$, with $s_{i,j}$ referred to as the* contact vertex *for $C_i$ and $H_j$,*
- *each $C_j \in \mathcal{C}$ is either a "type I member" containing a single vertex, or a "type II member" containing more than one,*
- *for each type II member $C_i$, the number of $H_j$ for which there exists some contact vertex $s_{i,j} \in C_i$ is either $2\tau_j + 1$ or $2\tau_j$ and these contact vertices are distinct,*
- *the set containing all type II members of $\mathcal{C}$ is independent in $G$, and*
- *$r \geq 2$.*

*Then $\mathcal{C}$ is a generalized graph cutset of gap $\geq 2$ and valence $\Sigma\tau$, with witness $H$. Such a cutset is* tame *if it contains at most one type II member.*

**Observation 2** *Every cutset $\mathcal{C}$ (of gap $\geq 2$ and valence $t$) satisfies the preliminary definition for generalized cutset (of gap $\geq 2$ and the same valence $t$), once each vertex $c_i \in \mathcal{C}$ is converted into the corresponding singleton set $\{c_i\}$ and $H$ is set equal to the set of connected components in $G \setminus \mathcal{C}$. Every pair $\mathcal{C}$ and $H$ satisfying the preliminary definition for generalized cutset (with witness $H$, of gap $\geq 2$ and valence $t$) also satisfies the final definition (with the same witness $H$, of gap $\geq 2$ and valence $\Sigma\tau$), once $\tau$ is set equal to $(1, 1, \ldots, 1)$, with $\Sigma\tau = t$.*

Observation 2 explains why we present only one version of the main theorem that follows.

**Theorem 4** *(Main Theorem) Let $G = (V, E)$ be a finite, connected graph. Suppose $\mathcal{C} = \{C_1, C_2, ..., C_t\}$ is a generalized cutset for $G$, of gap $\geq 2$ and valence $\Sigma\tau$, with witness $H = \{H_1, H_2, \ldots, H_{\Sigma\tau+r}\}$. Let $n = 1 + \Sigma\tau$ be the number of agents under consideration. Then*

- *if $\mathcal{C}$ is tame, there exist common additive valuations for the $n$ agents for which no connected* EF1 *allocations exist (whence no connected* $\text{EF1}_{\text{outer}}$ *allocations exist).*
- *whether or not $\mathcal{C}$ is tame, there exist common monotone valuations for the $n$ agents for which no connected* EF1 *allocations exist (whence no connected* $\text{EF1}_{\text{outer}}$ *allocations exist).*

Proof: We will say that a set $S \subseteq V$ *dominates* a type I member $C_i = \{c_i\} \in \mathcal{C}$ if $c_i \in S$; $S$ *dominates* a type II member $C_i \in \mathcal{C}$ if $S$ includes two or more of $C_i$'s contact vertices $s_{i,j}$. Given an allocation $A = \{A_1, A_2, ..., A_n\}$ of $G$'s vertices to the $n$ agents, we will say that agent $i$ dominates a member $C_i \in \mathcal{C}$ if their assigned share $A_i$ dominates $C_i$. An agent who dominates no member of $\mathcal{C}$ will be called *deprived*.

<u>Claim 1</u> For every allocation $A$, a deprived agent exists. Claim 1 follows because at most $\tau_i$ agents can dominate any single member $C_i \in \mathcal{C}$, so the number of agents who dominate members of $\mathcal{C}$ is at most $\Sigma\tau$. But there are $1 + \Sigma\tau$ agents.

<u>Claim 2</u> For every connected allocation $A$, the share of a deprived agent contains no two vertices coming from different members $H_i \in H$ (and also contains no vertex $c_i$ with $\{c_i\} \in \mathcal{C}$, and no two contact points $s_{i,j}$ from the same type II member of $C$). Note that the parenthetical part merely repeats the definition of "deprived." For the first part, assume $A_i$ is a connected share that contains vertices from two different members of $H$. We will show that $A_i$ is not deprived. Let $\rho = x_1, x_2, \ldots, x_{k-1}, x_k$ be a shortest path possible consisting entirely of vertices from $A_i$ and joining members $x_1, x_k$ from different sets in $H$. As $H$ is an independent collection , $x_1$ and $x_k$ are not adjacent, so there must be at least one vertex in the "middle" part $x_2, \ldots x_{k-1}$ of $\rho$. None of

those middle vertices are in $\bigcup H$, else we would get a shorter path of the desired kind, so they all come from $\bigcup \mathcal{C}$. If any middle vertex is some $c_j \in \{c_j\} = C_j \in \mathcal{C}$ then $A_i$ dominates that type I member $C_j$, so $A_i$ is not deprived, as desired. If not, then all of the middle vertices come from type II members of $\mathcal{C}$. But the type II members form an independent collection, so all of the middle vertices come from the same type II member $C_j$. Thus $x_2$ and $x_{k-1}$ are each contact vertices from the same $C_j$, but for distinct sets from $H$. We cannot have $x_2 = x_{k-1}$, because contact vertices for different sets in $H$ and the same $C_j$ are required to be distinct. So $x_2$ and $x_{k-1}$ are two contact points from the same type II member $C_j$, establishing that $A_i$ dominates $C_j$, as desired.

The common valuation $v$ of a set of vertices will be defined as the sum $v = v_H + v_{\mathcal{C}}$ of an $H$-*part* and a $\mathcal{C}$-*part*. For the $H$-part, we choose one vertex $x_j$ from each $H_j \in H$, set $v_H(x_j) = 1$ for each such $x_j$, and set $v_H(y) = 0$ for every other vertex $y \in \bigcup H$. Then $v_H(A_i)$ is defined to be sum of these values for all vertices in the set $A_i \cap (\bigcup H)$. Observe that at least $2 + \Sigma\tau$ of the vertices in $\bigcup H$ have been given value 1. The definition of $v_{\mathcal{C}}$ now depends on whether or not $\mathcal{C}$ is tame:

<u>Case 1</u> Assume $\mathcal{C}$ is tame. Then for each $x \in \bigcup \mathcal{C}$ we set $v_{\mathcal{C}}(x) = 1$ if $\{x\}$ is a type I member of $\mathcal{C}$, $v_C(x) = \frac{1}{3}$ if $x$ is one of the contact vertices in the only type II member of $\mathcal{C}$, and $v_{\mathcal{C}}(y) = 0$ for every other vertex $y \in \bigcup \mathcal{C}$. Then $v_{\mathcal{C}}(A_i)$ is defined to be sum of these values for all vertices in the set $A_i \cap (\bigcup \mathcal{C})$. In this case, $v = v_H + v_{\mathcal{C}}$ is additive as well as common and the total value of the deprived agent's share $A_j$ is at most $1 + \frac{1}{3}$, thanks to Claim 2.

<u>Case 2</u> Assume $\mathcal{C}$ is not tame. Then $v_{\mathcal{C}}(A_i)$ is defined to be the number of members of $\mathcal{C}$ dominated by $A_i$. In this case, $v = v_H + v_{\mathcal{C}}$ is still common and monotone, but is not additive (because any single contact point from a type II member adds no value to a share, whereas two or more contact points from the same type II member adds 1 to the value). The total value of the deprived agent's share $A_j$ is at most 1, as $v_C$ contributes no value to a deprived agent.

In both cases the number of vertices in $H$ that have been given value 1 is $r + \Sigma\tau \geq 2 + \Sigma\tau$. These valuable vertices have been distributed to $1 + \Sigma\tau$ many agents, so some agent $k$ must have two of them in her connected share. The proof of Claim 2 tells us that $A_k$ is not deprived – $A_k$ must dominate some member of the cutset, meaning $A_k$ either includes some vertex $x$ for which $\{x\}$ is a type I member of $C$, or includes some two contact points from the same type II member $C_j$ of $\mathcal{C}$.

In Case 1 we conclude that $A_k$ has (additive) value of at least $1 + 1 + \frac{1}{3} + \frac{1}{3}$ or $1 + 1 + 1$, so removing any single vertex leaves $A_k$ with value at least $1\frac{2}{3}$. The deprived agent thus envies agent $k$ by more than 1 item, as her own share is worth at most $1\frac{1}{3}$ for Case 1.

In Case 2 we conclude that $v_H$ awards a value of at least $1 + 1$ to $A_k$ with $v_{\mathcal{C}}$ providing an additional value of at least 1, for a total of at least 3. Removing any single vertex would not reduce that value below 2. The deprived agent again envies agent $k$ by more than 1 item, as her own share is worth at most 1 in Case 2. □

# 4 A Counterexample

Consider the following three conditions on a finite and connected graph $G$:
(C1) $G$ is traceable.
(C2) $G$ guarantees connected $EF1_{outer}$ allocations *universally*, for CM valuations.
(C3) $G$ contains no generalized cutsets of gap $\geq 2$.
We know these conditions satisfy "(C1) ⇒ (C2)" (from Theorem 1) and "(C2) ⇒ (C3)" (from the previous section of this paper). At one point, we did not know whether either arrow reversed. Recently we found a 10-vertex graph showing that "(C3) ⇒ (C1)" fails.[12]

[12] This was not a great surprise as "(C3) ⇒ (C1)" would imply that "(C3) ⇔ (C1)", and hence NP is contained in coNP; the latter, however, is widely considered as unlikely. The reason is that checking traceability is an NP-complete problem, while checking the non-existence of generalized cutsets is a coNP problem. We conjecture that it is indeed NP-complete to determine the existence of generalized cutsets (see Definition 6).

The example was a bit too large to check directly whether connected $EF1_{outer}$ allocations were guaranteed universally. Much more recently we were able to reduce the earlier example to the 8-vertex graph *JCS* presented here, which was small enough to yield to a trial-and-error search for a "bad" common additive valuation. It is worth noting that while the valuation used in the proof for *JCS* is not very complicated, it does seem quite different from the valuations used (in the previous section) to defeat $EF1_{outer}$ allocations in graphs that contain cutsets.

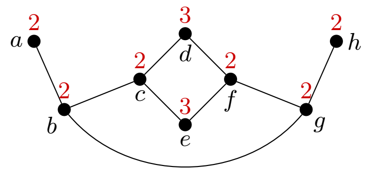


Figure 4: The JCS graph – a counterexample.

**Theorem 5** *The non-traceable* 8*-vertex* JCS *graph of Figure 4 has no graph cutsets of gap* $\geq 2$ *(of any kind), yet fails to guarantee connected* $\mathrm{EF1_{outer}}$ *allocations for three agents, even for agents with common additive valuations. Thus "(C3)* $\Rightarrow$ *(C2)" fails.*

Proof: It is straightforward to check by inspection that $JCS$ contains no cutsets of gap $\geq 2$, and it is also easy to see that no Hamiltonian path exists (which, alternately, follows from Theorem 1). To see that connected *EF*1 allocations may fail to exist, consider the vertex weights appearing directly above the vertices in Figure 4. Let the common value $v(S)$ assigned (by all agents) to a connected set $S$ of vertices be given by the sum of the weights of the vertices in $S$.

A partition $P$ of the JCS vertex set V into three connected pieces will be called a 3*−partition.* Let $X$ and $Y$ be pieces of a 3−partition $P$, and $X^\star$ denote the set of vertices that remain in $X$ after $X$'s most valuable vertex is removed. We will write $X >> Y$ if $v(X^\star) > v(Y)$. If a 3−partition $P$ contains two such pieces $X$ and $Y$, then any assignment of $P$'s pieces to the agents clearly fails to be *EF*1 (whence $EF1_{outer}$ also fails). In this case we will say simply that $P$ *fails.* Our goal, then, will be to show that every 3−partition fails in this way.

Claim Let $P$ be a 3-partition, $A$ denote the piece of $P$ containing vertex $a$ in Figure 4, and $H$ denote the piece containing vertex $h$. Then for $P$ to avoid failure we must have that $A \neq H$, with

(i) $A$ containing $b$ and $c$, and omitting $d$ and $e$, and
(ii) $H$ containing $g$ and $f$, and omitting $d$ and $e$.

The theorem follows immediately from this claim, because the third piece of P is now forced to be $\{d, e\}$, which is disconnected. To prove the claim, note that if $A = \{a\}$ or $A = \{a, b\}$ then $v(A)$ is only 2 or 4, and the value of the remaining vertices in $V \setminus A$ is at least 14. But there exists no partition of $V \setminus A$ into two connected pieces of value 7 each, so one of these pieces $X$ must have value at least 8, whence $X >> A$. So $A$ contains $a$, $b$, and at least one more vertex. Similarly, $H$ contains $h, g$ and at least one more vertex. If $A$ contains $g$ then $A = H$, with $v(A) \geq 8$. But then the value of the remaining vertices is only 10 – small enough to force $A >> Y$ for some $Y \in P$. So for $P$ to avoid failure we must have $a, b$, and $c$ in $A$; $f, g$, and $h$ in $H$; and $A \neq H$. If $A$ also contained $d$ or $e$, we would again get $A >> Y$ for some $Y \in P$, and the same reasoning applies to $H$. This establishes the claim, and the theorem. □

# 5 The EF-1 Spectrum of a Graph

Suppose that we fix a finite graph $G = (V, E)$ along with some class $\mathcal{V}$ of valuations (all monotone valuations, for example) and ask: for which natural numbers $n$ are $EF1_{outer}$ allocations of $G$ guaranteed for $n$ agents with valuations in $\mathcal{V}$ ? We will record the answer in the form of an infinite sequence of *yes-no* answers, with the $n^{th}$ member of the sequence being a "*yes*" if the $EF1_{outer}$ guarantee holds for $n$, and a "*no*" if the *EF*1 guarantee fails for $n$. We will refer to that sequence as the $EF1_{outer}$ *spectrum* of $G$ for the class $\mathcal{V}$ (or just as the *spectrum*, when the context is clear). For example, we know from Theorem 1 that the spectrum of a traceable graph is ⟨yes, yes, . . . , yes . . .⟩.

Are there any general patterns that hold for all such spectra? When $n$ is one less than the number $|V|$ of vertices, one can impose a picking order and have agents pick their most preferred vertex among those still available after agents earlier in the order have picked. The one remaining vertex is no more valuable to any agent than is the vertex picked earlier by that agent, so it can be given to any agent for which connectivity is maintained. The resulting allocation is connected and $EF1_{outer}$ for arbitrary monotone preferences. When $n$ is equal to or greater than the number $|V|$ of vertices, we can give each vertex to a different agent (with some agents possibly getting no vertices); the result is $EF1_{outer}$ regardless of $\mathcal{V}$. Thus, the spectrum of a graph is all *yes* from the $(|V|-1)^{th}$ place on. Each spectrum with any *no*s thus has a last *no* at some location $j$. We will record a *YES* in capital letters at location $j+1$ to indicate that the sequence is all *yes* from then on.

Also, as long as $G$ is connected, its spectrum clearly has an initial *yes* for $n = 1$. These two observations summarize everything we know for certain: every spectrum for a connected graph starts with a *yes*, and is all *yes* from some point on. However, we do have a few examples that suggest the following conjecture:

**Conjecture 1** *The spectrum of any connected graph G consists of an initial* yes *string, followed by a (possibly empty)* no *string, followed by an unending* yes *string.*

A few examples, discussed below, suggest some support for the conjecture:

(1) The 5-pointed star has spectrum ⟨yes, no, no, no, YES⟩.

(2) Graph IV in this paper, aka the *friendly diamond* graph, has spectrum ⟨yes, yes, no, YES⟩.

(3) The version $L_8$ of the Lips graph found in Figure 11 of [11] has 8 vertices, with spectrum ⟨yes, yes, yes, no, ?, ?, YES⟩.

A 5-pointed star has a central vertex adjacent to 5 vertices of degree 1. The three *no*s in its spectrum follow from reasoning like that used for Graph I (in Section 2). The *YES* sits in the $5^{th}$ position because of the general rule (discussed above) for $n \geq |V| - 1$. Similar reasoning applies to a $k$-pointed star, yielding a string of $k-2$ *no*s. For these graphs, all the *yes* answers hold for arbitrary monotone valuations, and all *no* answers arise from CA counterexamples, so the class $\mathcal{V}$ of valuations for this spectrum can be taken to be any of the classes of valuations that we have discussed.

For the $L_8$ Lips graph, the second *yes* uses the existence of a bipolar ordering paired with Theorem 2. The third *yes*, proved in [11], uses the discretization (mentioned earlier) of a modified version of Stromquist's moving knife argument for three agents in [13]. The *no* in the $4^{th}$ position uses a graph cutset of valence 3, consisting of the three vertices of degree greater than 2 in $L_8$.[13] The unknown entries for 5 and 6 agents reflect the absence of any good techniques for proving positive $EF1_{outer}$ results for non-traceable graphs when the number of agents is greater than 3. The class $\mathcal{V}$ for this spectrum can again be any of those we have mentioned.

The surprise for Graph IV is that we are able to fill in *yes* answers for 4 agents, thus determining the entire spectrum for $\mathcal{V} = CM$. The argument, found in the appendix, rests on a detailed case analysis, and applies when $\mathcal{V}$ is limited to the class of *common* monotone valuations. Despite this limitation, the argument was probably only feasible because the graph has so few vertices. We conjecture that the *yes* answer for 4 agents holds even for arbitrary monotone valuations. The *yes* in location 2 follows from the bipolar ordering for Graph IV, and the *no* follows from the cutset of valence 2 (both discussed earlier).

[13]That argument, from "*Example 1 continued*" of [11], does not use the term "graph cutset," but the idea is the same.

## Acknowledgments

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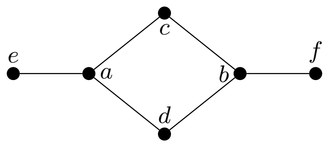
# Appendix

## A Friendly Diamond Graph Shared by 4 Agents




IV

Figure 5: A diamond graph with 6 vertices.

**Theorem 6** *The graph given in Figure 5 guarantees connected* $\text{EF1}_{\text{outer}}$ *allocations for* $n = 4$ *agents, under common and monotone valuations.*

Proof: We use the following notational conventions:

- Given a common monotone valuation function $v$ we write $x$, $xy$, and $xyz$ as shorthand for $v(\{x\}), v(\{x, y\})$, and $v(\{x, y, z\})$, respectively.
- We write "agent $i$ does not envy$_{>1}$ agent $j$" as shorthand for "either $i$ does not envy $j$, or some one vertex in $j$'s share would, if removed, eliminate any such envy while leaving $j$'s share connected."
- Greek letter subscripts on an inequality, such as $\min(e, a) <_\gamma \min(d, bf)$, allow us to refer the inequality when it is used to justify a later step.

Note that except for Subcase II, all of the specified allocations award shares consisting of one or two vertices each, and such allocations automatically satisfy the requirement that $j$'s share remain connected after removing any one vertex. In the exceptional case, $A_4 = \{d, b, f\}$, with $d$ being the vertex that gets removed in the argument. The remaining vertices $b$ and $f$ are indeed adjacent in Graph IV. This dispenses with the connectivity issue for Theorem 6.

Now, we proceed with showing how to find a connected $\mathit{EF1}_{\mathit{outer}}$ allocation for 4 agents. Without loss of generality, assume $\min(e, a) \geq_\alpha \min(b, f)$ and $c \geq_\beta d$. We consider four cases.

**Case I: Assume** $\min(e, a) <_\gamma \min(d, bf)$.
Allocate the vertices as follows: $A_1 = \{e, a\}$, $A_2 = \{c\}$, $A_3 = \{d\}$, $A_4 = \{b, f\}$.
As $d >_\gamma \min(e, a) \geq_\alpha \min(b, f)$ and $c \geq_\beta d$, players 2 and 3 do not envy$_{>1}$ players 1 or 4.
As $ea \geq \min(e, a) \geq_\alpha \min(e, f)$, agent 1 does not envy$_{>1}$ agent 4.
Moreover, $bf \geq_\gamma \min(a, e)$, so agent 4 does not envy$_{>1}$ agent 1.

**Case II: Assume** $\min(e, a) \geq_\delta \min(d, bf))$ **and** $\min(a, d) \geq_\varepsilon bf$**.**
Let $A_1 = \{e\}$, $A_2 = \{a\}$, $A_3 = \{c\}$, and $A_4 = \{d, b, f\}$.
As $\min(a, d) \geq_\varepsilon bf$ and $c \geq_\beta d$, agents 2 and 3 do not envy$_{>1}$ agent 4.
As $d \geq \min(a, d) \geq_\varepsilon bf$, we have that $\min(e, a) \geq bf$, so agent 1 does not envy$_{>1}$ agent 4.

**Case III: Assume** $\min(e, a) \geq_\delta \min(d, bf)$ **and** $\min(a, d) <_\zeta bf$ **and "**$d \geq_\eta bf$ **or** $c \geq_\theta \min(b, f)$**".**
Let $A_1 = \{e\}$, $A_2 = \{a, d\}$, $A_3 = \{c\}$, $A_4 = \{b, f\}$.

Agents 1 and 2 do not envy$_{>1}$ agent 4 because $\min(e, a) \geq_\alpha \min(b, f)$.
Agent 3 does not envy$_{>1}$ agent 2 because $c \geq_\beta d$, and does not envy$_{>1}$ agent 4 because "$d \geq_\eta bf$ or $c \geq_\theta \min(b, f)$", together with $c \geq_\beta d$, implies $c \geq \min(b, f)$.
Agent 4 does not envy$_{>1}$ agent 2 because $\min(a, d) <_\zeta bf$.
Now, if $d \geq_\kappa bf$, then $e \geq \min(e, a) \geq_{\kappa,\delta} bf >_\zeta \min(a, d)$, so agent 1 does not envy$_{>1}$ agent 2. Otherwise, $d < bf$, so $\min(e, a) \geq_\delta d$, and thus agent 1 does not envy$_{>1}$ agent 2.

**Case IV: Assume** $\min(e, a) \geq_\delta \min(d, bf)$ **and** $\min(a, d) <_\zeta bf$ **and** $d <_\lambda bf$ **and** $c <_\mu \min(b, f)$**.**

Let $A_1 = \{e\}$, $A_2 = \{a, d\}$, $A_3 = \{c, b\}$, and $A_4 = \{f\}$.

The assumptions imply that $\min(e, a) \geq_{\delta,\lambda} d$ and $\min(e, a) \geq_\alpha \min(b, f) >_\mu c$.

These implications tell us that agent 1 does not envy$_{>1}$ agents 2 or 3, and also that agent 2 does not envy$_{>1}$ agent 3. Agent 4 does not envy$_{>1}$ agents 2 or 3 because $\min(b, f) >_\mu c \geq_\beta d$. Finally, agent 3 does not envy$_{>1}$ agent 2 because $c \geq_\beta d$.

□