Efficient Resource Allocation with Secretive Agents

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Abstract

We consider the allocation of homogeneous divisible goods to agents with linear additive valuations. Our focus is on the case where some agents are secretive and reveal no preference information, while the remaining agents reveal full preference information. We study distortion, which is the worst-case approximation ratio when maximizing social welfare given such partial information about agent preferences. As a function of the number of secretive agents k relative to the overall number of agents n, we identify the exact distortion for every p-mean welfare function, which includes the utilitarian welfare ($p = 1$), the Nash welfare ($p \to 0$), and the egalitarian welfare ($p \to -\infty$).

1 Introduction

We study a resource allocation problem in which divisible goods must be allocated to agents with linear additive valuations.[1] Treating goods as divisible captures cases where they are inherently divisible (such as land or food), and where they are indivisible (such as jewelry or artwork) but can be allocated randomly or timeshared. Formally, an allocation is a matrix $\mathbf{x}$, where $x_{i,j} \in [0, 1]$ is the fraction of resource j given to agent i and $\sum_i x_{i,j} = 1$ for all j. The preferences of agent i are given by a valuation function v_i such that her utility from allocation $\mathbf{x}$ is $v_i(\mathbf{x}_i) = \sum_j v_i(j) \cdot x_{i,j}$.

A classic solution is to allocate the resources in a way that maximizes some *social welfare function*, which maps the utilities of the agents to a single aggregate measure of allocation quality. Common examples include the utilitarian welfare ($\frac{1}{n} \sum_i v_i(\mathbf{x}_i)$), the Nash welfare ($(\prod_i v_i(\mathbf{x}_i))^{1/n}$), and the egalitarian welfare ($\min_i v_i(\mathbf{x}_i)$), where n is the number of agents. In fact, these are members of the broader class of *p-mean* welfare functions, given by $(\frac{1}{n} \sum_i v_i(\mathbf{x}_i)^p)^{1/p}$, with $p = 1$, $p \to 0$, and $p \to -\infty$ respectively.

When we have complete information about the valuation function of each agent, finding an allocation that maximizes social welfare is conceptually trivial (algorithmically, however, some welfare functions may be challenging to maximize [26, 23, 14, 7]). But when we have only partial information, it is less clear what outcomes are prescribed by the social welfare maximization paradigm. One approach in the literature is to consider the *distortion*, which is the worst-case approximation ratio of the maximum social welfare that could be achieved with full information to the social welfare achieved by the allocation rule given partial information. Distortion can be viewed as the "price" of missing information, and minimizing distortion provably reduces the (worst-case) impact that the missing information has on the solution quality. Distortion was originally defined by Procaccia and Rosenschein [30] in the context of voting, where it has led to an extensive literature of follow-up work; we point the reader to the recent survey by Anshelevich et al. [4] for a summary. The approach has since been applied to other settings including matching [1, 27, 2] and resource allocation [25].

Traditionally, the distortion framework has been applied when every agent reports ordinal preferences [16, 3, 25]. In this paper, we introduce and study a different model, in which some agents provide complete cardinal valuation functions while others provide no information. We term the latter agents *secretive agents*. In practice, agents may be secretive because they do not want to disclose their valuations for privacy reasons, or because they are simply unresponsive to requests for information. For example, on a popular resource allocation website Spliddit.org, more than 10% of the goods division instances did not succeed because at least one user did not submit their valuation function.[2] Prior work in resource allocation has considered secretive agents [6, 22, 20, 5], but these

[1] We discuss allocation of indivisible goods in Appendix C.

[2] For chore division instances, it was even higher at over 32%.

focus on guaranteeing certain fairness properties in the presence of secretive agents, not on welfare maximization or distortion. Further, unlike in our work, none of them allow more than a single agent to be secretive because guaranteeing the fairness properties they seek becomes trivially impossible in this case.

In the presence of one or more secretive agents, it is not a priori clear what a "good" allocation looks like. On the one hand, if we assign any good to a secretive agent, she might turn out to have very low value for that good, resulting in the good effectively being wasted. On the other hand, if we allocate nothing to the secretive agents, we run the risk of facing high distortion due to instances where the secretive agents are the key to achieving high welfare. How do we balance these considerations? Should we allocate any resources to the secretive agents? If so, how do we determine how much of a resource should be allocated to the secretive agents? We answer these questions by identifying worst-case optimal allocation rules, which turn out to be surprisingly simple.

1.1 Our Results

Let n be the number of agents, k of whom are secretive. We present our results for divisible goods in the main body and defer the treatment of indivisible goods to Appendix C. For divisible goods, we provide a complete picture of the exact distortion for all p-mean social welfare functions. We introduce a family of allocation rules parameterized by $\alpha \in [0, 1]$ and show that all our upper bounds can be achieved by setting the right value of α as a function of p, n, and k. Given α, the corresponding rule allocates α fractions of all the goods to the non-secretive agents in such a way to maximize their social welfare and splits the remaining $1 - \alpha$ fraction of each good equally among the secretive agents. In each case, we can obtain an exactly matching lower bound. A summary of our results is presented in Table 1 and Figure 1 shows how the distortion varies with p, n, and k.

The distortion naturally increases as the number of secretive agents k increases; for every p, the distortion starts at 1 when $k = 0$ (full information) and increases to n at $k = n$ (no information). Interestingly, for $p = 1$ (the utilitarian welfare), $p \to 0$ (the Nash welfare), and $p \to -\infty$ (the egalitarian welfare), the distortion already becomes n at $k = n - 1$, meaning that knowing the valuation function of a single agent is not helpful for these welfare functions, but this is not the case for intermediate values of p. When $k = \Theta(n)$, the distortion is $\Theta(n)$ for $p \leqslant 1$ and $\Theta(n)^{1/p}$ for $p > 1$. When $k \ll n$, it is worth noting that the distortion for the Nash welfare is $\approx 1 + k \ln n/n$, which grows linearly in k like for the utilitarian and egalitarian welfare, but at a lower rate. More generally, the Nash welfare leads to a surprisingly low distortion; see Figure 1.

Finally, we conduct simulations on synthetic data and real data from Spliddit.org to evaluate the empirical performance of our algorithms with respect to the utilitarian social welfare. While every $\alpha \in [1/(k+1), 1]$ is optimal in the worst case, we find that higher values of α perform better empirically.

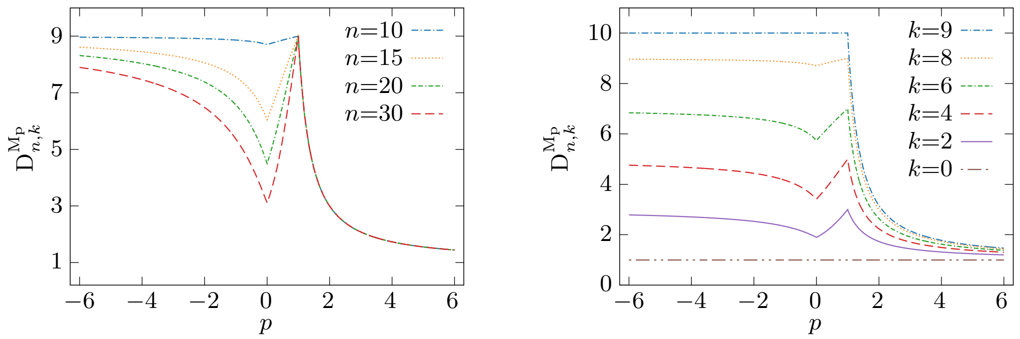
1.2 Related Work

In the voting literature, the idea of distortion has been analyzed under two primary frameworks, distinguished by what they assume the underlying expressive preference format to be: the utilitarian framework assumes that voters have utilities for candidates [16, 18, 12], while the metric framework assumes that voters have costs for candidates satisfying the triangle inequality [3, 24]. Following Halpern and Shah [25], our work follows the utilitarian framework as it is more applicable to allocating goods.

Halpern and Shah [25], like us, assume that agents have additive cardinal valuations, but they study the case where every agent reports a ranking of her t most favorite goods. They analyze the best possible distortion with respect to the utilitarian social welfare as a function of t in relation to the number of goods m. In particular, when every agent ranks all the goods (i.e., $t = m$), they show

Welfare	Distortion with $0 \leqslant k < n$	Optimal α
Egal. W.	$k+1$	$\alpha = \frac{1}{k+1}$
$(-\infty, 0)$	$n^{\frac{1}{p}} \left((n-k)^{\frac{1}{1-p}} + k\right)^{\frac{p-1}{p}}$	$\alpha = \frac{(n-k)^{\frac{1}{1-p}}}{k+(n-k)^{\frac{1}{1-p}}}$
Nash W.	$n\,(n-k)^{-\frac{n-k}{n}}$	$\alpha = \frac{n-k}{n}$
$(0,1)$	$n^{1-\frac{1}{p}} \left((n-k)^{1-p} + k\right)^{\frac{1}{p}}$	$\alpha = \frac{n-k}{n}$
Util. W.	$k+1$	$\alpha \in [\frac{1}{k+1}, 1]$
$(1,\infty)$	$(k+1)^{\frac{1}{p}}$	$\alpha = 1$

Table 1: Summary of results for divisible goods. For $k=0$, all distortion values in the table evaluate to 1. However, for $k=n$ the correct distortion value is n for $p \leqslant 1$ and $n^{1/p}$ for $p > 1$.



(a) Vary n with $k=8$.

(b) Vary k with $n=10$.

Figure 1: Distortion value with divisible items as a function of p.

that the best possible distortion (with a randomized rule) is n, which is what one can achieve with no preference information whatsoever. That is, they argue that having access to ordinal preference information is not helpful for welfare maximization. In contrast, our distortion bound is better when $k \leqslant n-2$, i.e., even when we have access to the valuation functions of just *two* agents. In a sense, this shows the usefulness of eliciting cardinal preferences as opposed to ordinal preferences in resource allocation settings.

Finally, we note that the idea of secretive agents is also explored in the voting literature, albeit with very different motivations. Borodin et al. [15, Lemma 4] show that constant metric distortion can be achieved in elections where any subset of voters that is at least a constant fraction of the electorate participate and submit ordinal preferences; such a strong guarantee is known to be impossible to achieve in the utilitarian framework, but may be possible if the participating subset of voters is assumed to be drawn at random. Micha and Shah [28] study voting rules which have access to the votes of only a subset of voters, but instead of analyzing the distortion, their aim is to predict what popular voting rules would have returned given all the votes. One of their primary motivations is to design voting rules to apply to polls in order to predict the outcome of an upcoming election.

2 Preliminaries

A resource allocation instance $(\mathcal{N}, \mathcal{M}, \mathbf{v})$ consists of a set of n agents $\mathcal{N}$, a set of m goods $\mathcal{M}$, and a utility profile $\mathbf{v} = (v_1, \dots, v_n)$, where $v_i \colon \mathcal{M} \to \mathbb{R}_{\geqslant 0}$ is the valuation function of agent i. We normalize the valuations of each agent i to sum up to one, i.e., $\sum_{j\in\mathcal{M}} v_i(j) = 1$. This unit-sum normalization is widely used in welfare maximization [8].[3]

Allocations. An allocation is a division of the goods among the agents, denoted by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, where $x_{i,j}$ is the fraction of good j allocated to agent i and each good j is fully allocated (i.e., $\sum_{i\in\mathcal{N}} x_{i,j} = 1$ for each j). We consider the class of linear additive utilities, where the utility of agent i for her share $\mathbf{x}_i$ is, with slight abuse of notation, defined as $v_i(\mathbf{x}_i) = \sum_{j\in\mathcal{M}} v_i(j) \cdot x_{i,j}$.

Welfare functions. A welfare function W aggregates the utilities to the agents under an allocation $\mathbf{x}$ into a single non-negative real number measuring the efficiency of the allocation. Following Barman et al. [9], we consider the following class of welfare functions.

Definition 1 (p-Mean Welfare)**.** For $p \in \mathbb{R}$, the *p-mean welfare* of allocation $\mathbf{x}$ is defined as

$$\mathrm{M_p}(\mathbf{x}) = \left(\frac{1}{n}\sum_{i\in\mathcal{N}} v_i(\mathbf{x}_i)^p\right)^{1/p}.$$

This class contains three popular welfare functions:

- Choosing $p = 1$ induces the *utilitarian welfare*, given by $\mathrm{UW}(\mathbf{x}) = (1/n) \cdot \sum_{i\in\mathcal{N}} v_i(\mathbf{x}_i)$,
- The limit $p \to 0$ induces the *Nash welfare*, given by $\mathrm{NW}(\mathbf{x}) = \left(\prod_{i\in\mathcal{N}} v_i(\mathbf{x}_i)\right)^{1/n}$,
- The limit $p \to -\infty$ induces the *egalitarian welfare*, given by $\mathrm{EW}(\mathbf{x}) = \min_{i\in\mathcal{N}} v_i(\mathbf{x}_i)$.

It is known that p-mean welfare functions are characterized by five natural axioms [29, pp. 66-69], and further imposing the Pigou-Dalton principle induces $p \leqslant 1$. It is interesting that our result for the warm-up case of $k = n$ also differs depending on whether $p \leqslant 1$ or $p > 1$ (see Section 3.1).

2.1 Secretive Agents & Distortion

In our setting, we assume that we have no information about the valuation functions of k agents, whom we term *secretive agents*, while we have complete information on the valuation functions of the remaining agents, whom we term *non-secretive agents*. Our goal is to find an allocation that minimizes the worst-case multiplicative loss of efficiency measured by a p-mean welfare function.

More formally, let $\mathcal{N}_{\mathtt{sec}}$ and $\mathcal{N}_{\mathtt{nonsec}}$ denote the sets of secretive and non-secretive agents, respectively. An instance of resource allocation with secretive agents $(\mathcal{N}, \mathcal{M}, \mathbf{v}_{\mathtt{nonsec}})$ consists of a set $\mathcal{N}$ of agent, a set $\mathcal{M}$ of items, and a valuation function v_i for each non-secretive agent (the valuation functions implicitly define the sets $\mathcal{N}_{\mathtt{sec}}$ and $\mathcal{N}_{\mathtt{nonsec}}$). We aim to find an optimal strategy for the following game:

1. The adversary chooses the valuation functions of the non-secretive agents, denoted by $\mathbf{v}_{\mathtt{nonsec}} = (v_i)_{i\in\mathcal{N}_{\mathtt{nonsec}}}$.
2. The player chooses an allocation $\mathbf{x}$ of the goods to all agents (secretive and non-secretive).
3. The adversary chooses the valuation functions of the secretive agents, denoted by $\mathbf{v}_{\mathtt{sec}} = (v_i)_{i\in\mathcal{N}_{\mathtt{sec}}}$, as well as an allocation $\mathbf{x}^*$.
4. The player incurs the (multiplicative) loss $\mathrm{W}(\mathbf{x}^*)/\mathrm{W}(\mathbf{x})$.

[3] Our results for the Nash welfare hold independently of any normalization. However, for every other p-mean welfare function, the optimal distortion is $k + 1$ in the absence of any normalization.

This game is formalized via the notion of *distortion*.

Definition 2 (Distortion with Secretive Agents)**.** Given the number of agents n, the number of secretive agents k, and a welfare function W, the *distortion* is defined as

$$\mathrm{D}^{\mathrm{W}}_{n,k} = \sup_{\mathbf{v}_{\mathtt{nonsec}}} \inf_{\mathbf{x}} \sup_{\mathbf{v}_{\mathtt{sec}}, \mathbf{x}^*} \frac{\mathrm{W}(\mathbf{x}^*)}{\mathrm{W}(\mathbf{x})}.$$

Note that the distortion is always at least 1 as the adversary can always return the same allocation as the player returns, i.e., $\mathbf{x}^* = \mathbf{x}$.

A strategy for the player corresponds to an *allocation rule* that maps instances to allocations. Because we express distortion values that depend on n and m (that is, we are typically interested in varying $\mathbf{v}_{\mathtt{nonsec}}$), we suppress the dependence on $\mathcal{N}$ and $\mathcal{M}$ and simply write $\mathtt{A}(\mathbf{v}_{\mathtt{nonsec}})$ to denote the output of allocation rule $\mathtt{A}$ on instance $(\mathcal{N}, \mathcal{M}, \mathbf{v}_{\mathtt{nonsec}})$.

Definition 3 (Distortion of an Allocation Rule)**.** Given n agents, k secretive agents, and a welfare function W, the *distortion of an allocation rule* $\mathtt{A}$ is defined as

$$\mathrm{D}^{\mathrm{W}}_{n,k}(\mathtt{A}) = \sup_{\mathbf{v}_{\mathtt{nonsec}}, \mathbf{v}_{\mathtt{sec}}, \mathbf{x}^*} \frac{\mathrm{W}(\mathbf{x}^*)}{\mathrm{W}(\mathtt{A}(\mathbf{v}_{\mathtt{nonsec}}))}.$$

If $\mathrm{D}^{\mathrm{W}}_{n,k}(\mathtt{A}) = \mathrm{D}^{\mathrm{W}}_{n,k}$ then we refer to $\mathtt{A}$ as an optimal strategy for the player.

3 Distortion Values

In this section, we present allocation rules that provide provable guarantees on the distortion with respect to p-mean welfare functions.

3.1 Warm-up: $\mathbf{k = 0}$ and $\mathbf{k = n}$

First, let us consider two extreme cases where $k = 0$ and $k = n$ which provide us some intuition for the general case.

Case $k = 0$. If there are no secretive agents, then we have full information of the utilities and we can return the allocation that maximizes the welfare for all agents. The adversary cannot obtain a welfare higher than us, therefore, the distortion value is 1. As $\mathcal{N} = \mathcal{N}_{\mathtt{nonsec}}$ in this case, we may say our strategy was maximizing the welfare for the *non-secretive* agents. Denote this strategy by

$$\mathtt{OPT}_{\mathtt{nonsec}}(\mathbf{v}_{\mathtt{nonsec}}) = \arg\max_{\mathbf{x}} \left(\frac{1}{n} \sum\nolimits_{i \in \mathcal{N}_{\mathtt{nonsec}}} v_i(\mathbf{x}_i)^p \right)^{1/p}.$$

Case $k = n$. Suppose all agents are secretive and we do not have any information from their utilities. To assist with intuition, assume $p \leqslant 1$. Our best response would be to return a uniform allocation, i.e. allocate $1/n$ of each item to each (secretive) agent. Intuitively speaking, this follows from the concavity of $\mathrm{M_p}$ for $p \leqslant 1$. If we act differently, the adversary can use the asymmetry in our allocation to incur a higher distortion (details provided in the full version). Denote this strategy by

$$\mathtt{Uniform}_{\mathtt{sec}}(\mathbf{v}_{\mathtt{nonsec}}) = \{x_{i,j} = \frac{1}{|\mathcal{N}_{\mathtt{sec}}|} \mid \forall j \in \mathcal{M}, i \in \mathcal{N}_{\mathtt{sec}}\}.$$

Regardless of the utilities, for all agents we have $v_i(\mathbf{x}) = 1/n$. Hence, the welfare obtained is $1/n$. The adversary cannot achieve mean welfare of more than 1. Thus, we get an upper bound of $\mathrm{D}^{\mathrm{W}}_{n,k=n} \leqslant n$. In Appendix A, we also show a matching lower bound.

Lemma 1. *For all p-mean welfare functions with $p \in (-\infty, 1]$ (including* NW *and* UW*) and* EW*, the distortion with n secretive agents is $\mathrm{D}^{\mathrm{W}}_{n,n} = n$.*

The analysis presented does not hold for $p > 1$. By the convexity of $\mathrm{M_p}$ when $p > 1$, our best response is to allocate all items to one agent. Then, only one agent will have a utility of 1 while others get 0 utility. Therefore, $(\frac{1}{n}\sum_{i \in \mathcal{N}} v_i(\mathbf{x})^p)^{1/p} = (1/n)^{1/p}$ leading to an upper bound of $\mathrm{D}^{\mathrm{M_p}}_{n,n} \leqslant \frac{1}{n^{-1/p}} = n^{1/p}$.

Lemma 2. *For all p-mean welfare functions with $p \in (1, \infty)$, the distortion is $\mathrm{D}^{\mathrm{W}}_{n,n} = n^{1/p}$.*

3.2 Results for $1 \leqslant k \leqslant n - 1$

In general, our strategy for the general case is to mix the two strategies described for the extreme cases of $k \in \{0, n\}$. That is, our allocation rule is one from the following class of allocation rules,

$$\mathtt{A}_\alpha = \alpha \, \mathtt{OPT}_{\mathtt{nonsec}} + (1 - \alpha) \, \mathtt{Uniform}_{\mathtt{sec}}, \tag{1}$$

where we allocate $\alpha \in [0, 1]$ portion of each item according to the $\mathtt{OPT}_{\mathtt{nonsec}}$ rule, and the rest uniformly among the secretive agents. The proper choice of α however, depends on the chosen welfare function.

We begin with a lemma that provides an upper bound on the adversary's welfare. Throughout this paper, we often use $\beta = \sum_{i \in \mathcal{N}_{\text{nonsec}}} v_i(\mathtt{OPT}_{\mathtt{nonsec}}(\mathbf{v}_{\mathtt{nonsec}}))^p$ to refer to the unnormalized welfare (i.e., the p-th power of the p-mean welfare) that the non-secretive agents receive under $\mathtt{OPT}_{\mathtt{nonsec}}$.

Lemma 3. *For any valuations of the non-secretive agents $\mathbf{v}_{\mathtt{nonsec}}$ and $p \in \mathbb{R}$, it holds that*

$$\mathrm{M_p}(\mathbf{x}^*) \leqslant \left(\frac{1}{n}(\beta + k)\right)^{\frac{1}{p}} = (k+1) \cdot \mathrm{M_p}\left(\mathtt{A}_{1/(k+1)}(\mathbf{v}_{\mathtt{nonsec}})\right), \tag{2}$$

where $\beta = \sum_{i \in \mathcal{N}_{\text{nonsec}}} v_i(\mathtt{OPT}_{\mathtt{nonsec}}(\mathbf{v}_{\mathtt{nonsec}}))^p$. That is, the welfare achieved by the adversary is at most $k + 1$ times higher than the welfare achieved by the allocation rule $\mathtt{A}_{1/(k+1)}$.

Proof. Let u_i denote the utility achieved by non-secretive agent i under $\mathtt{OPT}_{\mathtt{nonsec}}$. Note that each secretive agent receives utility at most 1 due to the unit-sum normalization. Hence,

$$\begin{aligned}
\mathrm{M_p}(\mathbf{x}^*) &= \left(\frac{1}{n} \cdot \sum_{i \in \mathcal{N}} v_i(\mathbf{x}^*)^p\right)^{\frac{1}{p}} \\
&\leqslant \left(\frac{1}{n}\left(\sum_{i \in \mathcal{N}_{\text{nonsec}}} u_i^p + \sum_{i \in \mathcal{N}_{\text{sec}}} 1\right)\right)^{\frac{1}{p}} = \left(\frac{\beta + k}{n}\right)^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, note that the utility vector under $\mathtt{A}_{1/(k+1)}(\mathbf{v}_{\mathtt{nonsec}})$ is $(\frac{u_1}{k+1}, \ldots, \frac{u_{n-k}}{k+1}, \frac{1}{k+1}, \ldots, \frac{1}{k+1})$ because $\frac{1}{k+1}$ fraction of each item is allocated according to $\mathtt{OPT}_{\mathtt{nonsec}}$ and $\frac{1}{k+1}$ fraction of each item is allocated to each of k secretive agents. Hence, $\mathrm{M_p}\left(\mathtt{A}_{1/(k+1)}(\mathbf{v}_{\mathtt{nonsec}})\right) = \frac{1}{k+1} \cdot \left(\frac{\beta+k}{n}\right)^{\frac{1}{p}}$, which yields the desired relation. □

Lemma 3 immediately implies the following Corollary.

Corollary 1. *For all p-mean welfare functions* W*, the allocation rule $\mathtt{A}_{1/(k+1)}$ has distortion $\mathrm{D}^{\mathrm{W}}_{n,k}(\mathtt{A}_{1/(k+1)}) \leqslant k + 1$ for all $n \geqslant k \geqslant 0$.*

It turns out that the upper bound of $k+1$ is only tight for the egalitarian and utilitarian welfare functions. For other values of p, we can achieve lower distortion by tailoring our strategy to the particular welfare function. The next two lemmas contain common parts to the analysis that we will use to prove our guarantees.

Lemma 4. *Consider a resource allocation instance with secretive agents. For all $\alpha \in [0,1]$ and any p-mean welfare $\mathrm{M_p}$ we have*

$$\frac{\mathrm{M_p}(\mathbf{x}^*)}{\mathrm{M_p}(\mathtt{A}_\alpha(\mathbf{v}_{\mathtt{nonsec}}))} \leqslant \left(\frac{\beta + k}{\alpha^p \beta + \left(\frac{1-\alpha}{k}\right)^p k}\right)^{\frac{1}{p}} \triangleq f_p(\beta, \alpha), \tag{3}$$

where $\beta = \sum_{i \in \mathcal{N}_{\mathrm{nonsec}}} v_i(\mathtt{OPT}_{\mathtt{nonsec}}(\mathbf{v}_{\mathtt{nonsec}}))^p$.

Proof. By Lemma 3, the welfare achieved by the adversary is upper bounded by $\mathrm{M_p}(\mathbf{x}^*) \leqslant (\frac{1}{n}(\beta + k))^{1/p}$. Next, using the allocation rule $\mathtt{A}_\alpha$, the player achieves a welfare of $\mathrm{M_p}(\mathtt{A}_\alpha(\mathbf{v}_{\mathtt{nonsec}})) = \left(\frac{1}{n}\left(\alpha^p \beta + \left(\frac{1-\alpha}{k}\right)^p k\right)\right)^{\frac{1}{p}}$. □

Lemma 4 immediately implies that

$$D^{\mathrm{M_p}}_{n,k} \leqslant \mathrm{D}^{\mathrm{M_p}}_{n,k}(\mathtt{A}_\alpha) \leqslant \max_{\mathbf{v}_{\mathrm{nonsec}}} \left(\frac{\beta + k}{\alpha^p \beta + \left(\frac{1-\alpha}{k}\right)^p k}\right)^{\frac{1}{p}}.$$

Lemma 5. *Let $f_p(\beta, \alpha)$ be as defined in* (3)*. Then, for a fixed $\alpha \geqslant \frac{1}{k+1}$, f_p is non-increasing over $\beta \geqslant 1$.*

Proof. As $f_p(\beta, \alpha) \geqslant 1$ and since log preserves monotonicity, it is sufficient to show $\frac{d}{d\beta} \log f_p(\beta, \alpha) \leqslant 0$ for all $\beta \geqslant 1$.

$$\frac{d}{d\beta} \log f_p(\beta, \alpha) = \frac{1}{p}\left(\frac{1}{\beta + k} - \frac{1}{\beta + \left(\frac{1-\alpha}{\alpha k}\right)^p k}\right).$$

By $\alpha \geqslant \frac{1}{k+1}$, we have $\frac{1-\alpha}{\alpha k} \leqslant 1$. Then, we can check this expression is non-positive both for $p > 0$ and $p < 0$. □

As f_p is non-increasing, to obtain an upper bound on the distortion, we need a lower bound on β. This value, as well as the proper choice of α, depends on p. In the rest of this section, we will find the proper choices for α and β based on the welfare function. We begin with $p \in (-\infty, 0)$.

Theorem 1. *For $p \in (-\infty, 0)$, the allocation rule $\mathtt{A}_{z/(k+z)}$ with $z = (n-k)^{\frac{1}{1-p}}$ achieves $\mathrm{D}^{M_p}_{n,k}(\mathtt{A}_\alpha) \leqslant n^{\frac{1}{p}} (z+k)^{\frac{p-1}{p}}$.*

Proof. Lemma 5 requires $\frac{z}{k+z} \geqslant \frac{1}{k+1}$. The function $f(x) = \frac{x}{k+x}$ is increasing over x, and $z \geqslant 1$ due to $n - k \geqslant 1$ and $p < 0$. Hence, $\frac{z}{k+z} \geqslant \frac{1}{k+1}$.

By Lemma 4, we can bound the distortion by (3). Furthermore, by Lemma 5, this bound is maximized when β is minimized. We know for all $i \in \mathcal{N}_{\mathtt{nonsec}}$, $v_i \leqslant 1$. Therefore, $v_i^p = (1/v_i)^{-p} = (1/v_i)^{|p|} \geqslant 1$. Consequently, $\beta \geqslant n - k$.

Putting all together, by substituting β and α we get

$$\begin{aligned}
\mathrm{D}^{\mathrm{M_p}}_{n,k}(\mathtt{A}_\alpha) &\leqslant \left(\frac{n-k+k}{\alpha^p(n-k)+\left(\frac{1-\alpha}{k}\right)^p k}\right)^{\frac{1}{p}} && \text{(sub. } \beta\text{)} \\
&= n^{\frac{1}{p}}\left(\frac{z^p(n-k)}{(z+k)^p}+\frac{k}{(z+k)^p}\right)^{-\frac{1}{p}} && \text{(sub. } \alpha\text{)} \\
&= n^{\frac{1}{p}}\left(\frac{(n-k)^{\frac{1}{1-p}}}{(z+k)^p}+\frac{k}{(z+k)^p}\right)^{-\frac{1}{p}} \\
&= n^{\frac{1}{p}}\left(\frac{z+k}{(z+k)^p}\right)^{-\frac{1}{p}} = n^{\frac{1}{p}}\,(z+k)^{\frac{p-1}{p}}\,. && \square
\end{aligned}$$

Taking the limit as $p \to 0$ and $p \to -\infty$ in Theorem 1 suggests upper bounds of $n(\frac{1}{n-k})^{\frac{n-k}{n}}$ and $k+1$ for the Nash and egalitarian welfare respectively. For the egalitarian welfare, we have already shown an upper bound of $k+1$ in Corollary 1, and the following lemma proves that this upper bound is achievable for the Nash welfare.

Theorem 2. *For the Nash welfare, the allocation rule* $\mathtt{A}_{(n-k)/n}$ *achieves* $\mathrm{D}^{\mathrm{NW}}_{n,k}(\mathtt{A}_\alpha) \leqslant n\left(\frac{1}{n-k}\right)^{\frac{n-k}{n}}$.

Proof. Note that $\beta = \max_{\mathbf{x}} \prod_{i\in\mathcal{N}_{\text{nonsec}}} v_i(\mathbf{x})$ is the maximum product possible for the nonsecretive agents. Following the definition of $\mathtt{A}_{\alpha=(n-k)/n}$, each secretive agent will have a utility of $(1-\alpha)/k$, and

$$\mathrm{NW}(\mathbf{x}) = \left(\alpha^{n-k}\beta\cdot\left(\frac{1-\alpha}{k}\right)^k\right)^{\frac{1}{n}} = \beta^{\frac{1}{n}}\left(\left(\frac{n-k}{n}\right)^{n-k}\cdot\left(\frac{1}{n}\right)^k\right)^{\frac{1}{n}}$$

Furthermore, by Lemma 3, we have $\mathrm{NW}(\mathbf{x}^*) \leqslant (\beta\cdot 1^k)^{1/n} = \beta^{\frac{1}{n}}$. Hence,

$$\mathrm{D}^{\mathrm{NW}}_{n,k} \leqslant \frac{\mathrm{NW}(\mathbf{x}^*)}{\mathrm{NW}(\mathbf{x})} = n\left(\frac{1}{n-k}\right)^{\frac{n-k}{n}}. \qquad \square$$

Now, we will focus on the range $p \in (0,1]$.

Theorem 3. *For* $p \in (0,1]$, *the allocation rule* $\mathtt{A}_{(n-k)/n}$ *achieves* $\mathrm{D}^{\mathrm{M_p}}_{n,k}(\mathtt{A}_\alpha) \leqslant n\left(\frac{(n-k)^{1-p}+k}{n}\right)^{\frac{1}{p}}$.

Proof. The requirement of Lemma 5 is met, as for $k<n$, $(n-k)(k+1) \geqslant n \Rightarrow \frac{n-k}{n} \geqslant \frac{1}{k+1}$.

By Lemma 4, distortion is bounded by (3), and by Lemma 5, this bound is maximized when β is minimized. For any given $\mathbf{v}_{\texttt{nonsec}}$, one suboptimal allocation is $\texttt{Uniform}_{\texttt{nonsec}}$. Each agent gets $v_i = \frac{1}{n-k}$ utility from this rule. Hence, $\beta \geqslant (n-k)(\frac{1}{n-k})^p = (n-k)^{1-p}$.

By substituting β and α in (3), we have

$$\begin{aligned}
\mathrm{D}^{\mathrm{M_p}}_{n,k}(\mathtt{A}_\alpha) &\leqslant \left(\frac{(n-k)^{1-p}+k}{\left(\frac{n-k}{n}\right)^p(n-k)^{1-p}+\frac{k}{n^p}}\right)^{\frac{1}{p}} \\
&= n\left(\frac{(n-k)^{1-p}+k}{n-k+k}\right)^{\frac{1}{p}}. \qquad \square
\end{aligned}$$

Note that for $p=1$, Theorem 3 implies an upper bound of $k+1$ for the utilitarian welfare, matching the upper bound from Corollary 1. Moreover, by taking the limit $p \to 0$ in Theorem 3, we get the same upper bound proven in Theorem 2.

In fact, for the case of utilitarian welfare, a range of strategies all yield a distortion of $k+1$.

Proposition 1. For the utilitarian welfare and for all $\alpha \in [\frac{1}{k+1}, 1]$, the allocation rule $\mathtt{A}_\alpha$ achieves $\mathrm{D}^{\mathrm{UW}}_{n,k}(\mathtt{A}_\alpha) \leqslant k+1$.

Proof. By Lemma 4 we have $\mathrm{D}^{\mathrm{UW}}_{n,k} \leqslant \max_\beta \frac{\beta+k}{\alpha\beta+(1-\alpha)}$. As the utilitarian welfare in any instance is at least 1, e.g. by giving all items to one agent, by Lemma 5 and setting $\beta = 1$ we have $\mathrm{D}^{\mathrm{UW}}_{n,k} \leqslant \frac{1+k}{\alpha+(1-\alpha)} = k+1$. ☐

Lastly, the following theorem treats the case of $p > 1$.

Theorem 4. *For $p \in (1, \infty)$, the allocation rule $\mathtt{A}_1$ achieves $\mathrm{D}^{\mathrm{M_p}}_{n,k}(\mathtt{A}_1) \leqslant (k+1)^{\frac{1}{p}}$.*

Proof. By Lemma 4 and our choice of $\alpha = 1$, the distortion value is bounded by $\max_\beta \left(\frac{\beta+k}{\beta}\right)^{\frac{1}{p}}$. This term is maximized when β is minimized. Moreover, $\beta \geqslant 1$ because one suboptimal allocation is to give all items to one agent and obtain $\sum_{i \in \mathcal{N}_{\text{nonsec}}} v_i^p = 1$. By substituting $\beta = 1$, we get the desired bound. ☐

In Appendix B, we present matching lower bounds for all of the upper bounds proven in this section.

4 Experiments

In this section, we measure the average-case approximation ratio of *utilitarian* welfare achieved by different rules based on synthetic and real-world data. In principle, one could conduct a similar analysis with other welfare measures, but we focus on utilitarianism for simplicity and conciseness.

Rules. We compare the following allocation rules motivated by Section 1.1: `Uniform` (allocate items uniformly to all agents), $\mathtt{A}_\alpha$ with $\alpha = \frac{1}{k+1}$, $\alpha = \frac{n-k}{n}$, and $\alpha = 1$. Recall that $\mathtt{A}_1$ returns a utilitarian welfare maximizing allocation for the nonsecretive agents, and all three of the $\mathtt{A}_\alpha$ rules tested achieve the optimal distortion for utilitarian welfare.

Measurement. For a resource allocation instance with secretive agents, we measure the ratio between the maximum feasible welfare by full information and the welfare obtained by the rule, averaged over many instances. This provides us with an average-case analogue of the (worst-case) distortion.

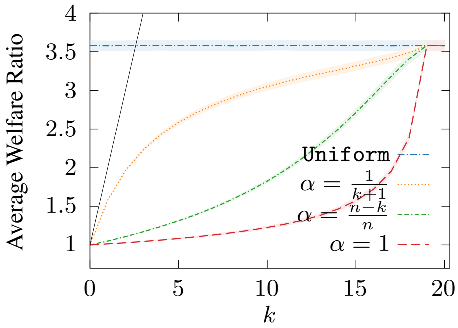
4.1 Synthetic Data

Data Generation. We generate utilities for each agent, either secretive or nonsecretive, sampled i.i.d. from a Dirichlet distribution with m concentration parameters all set at 1, i.e. $\mathrm{Dir}(1, \dots, 1)$. Each reported datum is the average of welfare ratios over 1000 randomly generated instances.

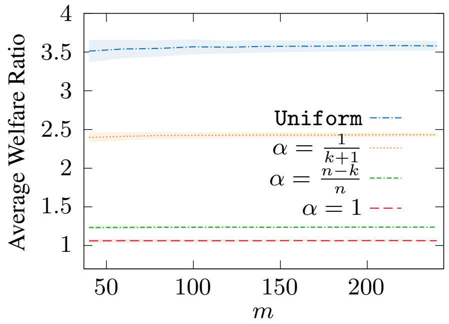
Experiments. We conduct three experiments each varying a parameter while fixing the others: vary k (Figure 2a), vary n with a fixed k (Figure 2c), vary n with a fixed ratio of k/n (Figure 2d), and vary m (Figure 2b).

Results. In all four figures we see a consistent relationship between the rules: rules with higher α outperform rules with lower α and all three of the $\mathtt{A}_\alpha$ rules outperform `Uniform`. This is perhaps not surprising, since higher values of α more heavily exploit the information available to the rule from the non-secretive agents, with the `Uniform` rule being one example of an extreme case that ignores all available information about the utility functions.

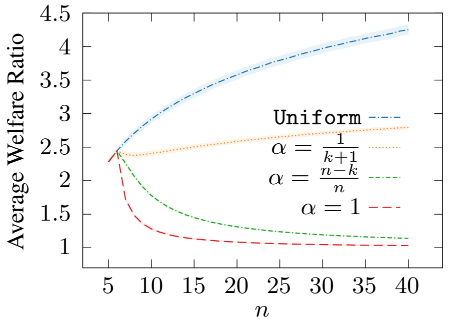
In Figure 2a we see all three $\mathtt{A}_\alpha$ rules achieve average welfare ratio 1 when $k = 0$, with the welfare ratio converging to that of `Uniform` when $k = n$, as expected. Of particular note is $\mathtt{A}_1$, which achieves an average welfare ratio close to 1 even for relatively large values of k (for example, the average welfare ratio is ~ 1.23 when $k = 10$) before rapidly increasing for large k. Of note is



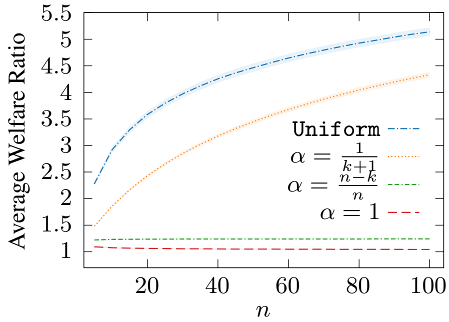
(a) Vary k with $n = 20$. The solid line is the line $y = k + 1$, i.e. the distortion value.



(b) Vary m with $n = 20$ and $k = 4$.



(c) Vary n with $k = 5$.



(d) Vary n with $k = \lceil 0.2n \rceil$.

Figure 2: Average welfare ratio achieved by different strategies. Error bands indicate the standard deviation. In all plots $m = 200$ (except in Figure 2b).

that all algorithms significantly outperform the worst-case bound of $k + 1$ displayed with a dotted line in the figure.

Figure 2c reveals an interesting separation when $\mathtt{A}_1$ and $\mathtt{A}_{(n-k)/n}$ are compared to $\mathtt{A}_{1/(k+1)}$ and `Uniform`. The average welfare ratio of the former rules decreases to 1 as n increases (with $k = 5$) while the average welfare ratio of the other rules actually increases with n. Figure 2d suggests that this increase persists even when the ratio k/n is held (approximately) constant.

4.2 Spliddit Data

Data Generation. We also used the real-world goods division instances from Spliddit.org. For each instance with n agents and a fixed k, we randomly sampled k (secretive) agents, hid their utilities from the allocation rules and measured the welfare ratio based on the actual utilities. Similar to the simulated experiments, we report the average of 1000 such simulations.

Data Statistics. The report is based on 4679 Spliddit instances. The distribution of the number of agents n is $\{2 : 27.5\%, 3 : 67.3\%, 4 : 2.4\%, 5 : 1.7\%, 6 : 0.4\%$, and $n \geqslant 7 : 0.7\%\}$. The number of goods m was in the range $[2, 96]$ with the mean and std. dev. of 31.1 ± 26.3.

Experiments. We divided instances based on n and varied k from 1 to $\min(5, n-1)$. The average welfare ratio is presented in Figure 3and Figure 5 (in Appendix D).

Results. In line with the results on synthetic data, we see higher α outperform lower α (and all outperform `Uniform`). The dependence on n and k also follows similar patterns as the synthetic case. Additionally, it is interesting to note the magnitude of the welfare ratio achieved by our rules.

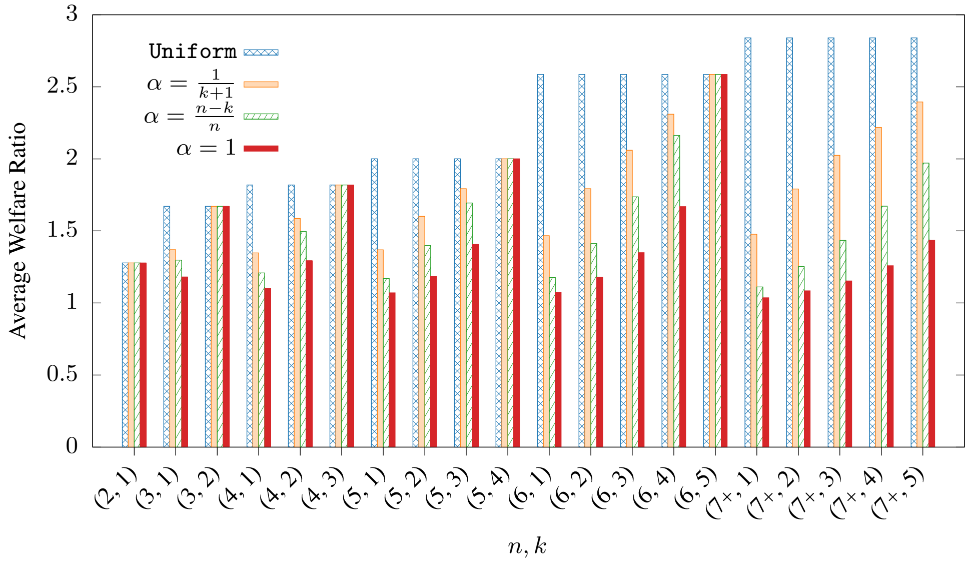


Figure 3: Average welfare ratio by different strategies on the Spliddit data. The x-axis is sorted by n and then k.

For Spliddit instances with 5 or fewer agents and at least 2 non-secretive agents ($k \leqslant n-2$), the average welfare ratio is never higher than 1.5 for the rule $\mathtt{A}_1$. That is, on average, we could achieve two-thirds of the maximum possible utilitarian welfare even if one or two agents do not respond to requests for their utility information.

5 Discussion

We studied distortion in resource allocation when k of the agents are secretive. For the utilitarian welfare, we identified a family of rules parametrized by $\alpha \in [1/k+1, 1]$ as worst-case optimal. Among this family, we find the rule with $\alpha = 1$ to be particularly interesting because it allocates no resources to the secretive agents and thus, unlike $\alpha < 1$, provides no incentive to an agent to be secretive. This can lead to fewer agents being secretive, which can further reduce distortion.

It is known that maximizing the Nash welfare yields desirable fairness guarantees [19, 21]. Fortunately, we find that the Nash welfare is the most approximable among all p-mean welfare functions.

Our work opens the door for interesting directions for future work. It would be interesting to study instance-wise optimal allocations, that is, allocations that minimize the worst-case approximation ratio on a given instance. It is likely that such allocations would more carefully decide which (and how much of) resources to allocate to the secretive agents depending on how highly they are valued by the non-secretive agents.

One may wish to reconcile distortion (welfare maximization) with fairness in the presence of secretive agents. If the goal is to only ensure fairness among the non-secretive agents, one can easily modify the rules proposed in this work by replacing $\mathtt{OPT}_{\mathtt{nonsec}}$ (the welfare-optimal allocation to the non-secretive agents) by an allocation to the non-secretive agents that maximizes welfare *subject to* the fairness guarantee. The additional loss in welfare incurred is precisely the *price of fairness*, which is well understood [17, 13, 11, 10].[4] However, if the goal is to ensure fairness to *all* agents,

[4]For ensuring proportionality to the non-secretive agents, we would need $\alpha \geqslant (n-k)/n$, which can be set for $p \in [0, 1]$

it may be necessary that no more than a single agent is secretive, and even then, achieving fairness alone is already challenging [5].

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A Lower Bounds for Section 3.1

In Lemma 6, we show matching lower bound examples for the upper bounds shown in Section 3.1. Lemmas 1 and 2 also derive from the following.

Lemma 6. *Suppose $k = n$, i.e. all agents are secretive. Then,* $\mathrm{D}^{\mathrm{M_p}}_{n,n} \geqslant n$ *for* $p \leqslant 1$ *as well as* NW *and* EW*, and* $\mathrm{D}^{\mathrm{M_p}}_{n,n} \geqslant n^{1/p}$ *for* $p > 1$.

Proof. Consider an instance with n items and n agents. No information about the utilities are revealed due to all agents being secretive. Let $\mathbf{x}$ be the allocation returned by the player. Take any n disjoint matchings from agents to items. For example, suppose in the j-th matching for $j \in [n]$, agent $i \in [n]$ is matched to item $m_{j,i} = ((j + i) \bmod n) + 1$. Note that $\sum_{j\in[n],i\in[n]} \mathbf{x}_{j,m_{i,j}} = n$. Then, at least for one of the matchings $j^\star$ we have $\sum_{i\in[n]} \mathbf{x}_{i,m_{j^\star,i}} \leqslant 1$.

Setting Utilities. Suppose each agent i values her matched item $m_{j^\star,i}$ at 1 and the rest at 0. The adversary's allocation can be according to this matching, obtaining a utility of 1 for each agent and hence a welfare of 1. Therefore, $\mathrm{W}(\mathbf{x}^*) = 1$.

Lower Bounds. Denote $u_i = v_i(\mathbf{x})$ and observe that $u_i = \mathbf{x}_{i,m_{j^\star,i}}$. Let $\alpha = \sum_{i\in[n]} u_i$. Note that $\alpha = \sum_{i\in[n]} \mathbf{x}_{i,m_{j^\star,i}} \leqslant 1$.

Case $p \in (1, \infty)$. For $p > 1$, x^p is convex and $(\frac{1}{n} \sum_{i\in n} u_i^p)^{1/p}$ is maximized when $u_i = \alpha$ for an agent i and $u_j = 0$ for agents $j \neq i$. Hence, $\mathrm{M_p}(\mathbf{x}) \leqslant (\frac{1}{n})^{\frac{1}{p}} \Rightarrow \mathrm{D}^{\mathrm{M_p}}_{n,n} \geqslant n^{1/p}$.

Case $p \in (-\infty, 1)$. We will show $\mathrm{W}(\mathbf{x}) \leqslant 1/n$ based on W, and conclude that $\mathrm{D}^{\mathrm{W}}_{n,n} \geqslant n$.

- *Egalitarian Welfare.* $\mathrm{EW}(\mathbf{x}) = \min_{i\in[n]} u_i \leqslant \frac{1}{n} \sum_{i\in[n]} u_i \leqslant \frac{1}{n}$.
- $p \in (-\infty, 0)$. As $p < 0$, $(\frac{1}{n} \sum_{i\in[n]} u_i^p)^{1/p}$ is maximized when $\sum_{i\in[n]} u_i^p$ is minimized. Furthermore, by the convexity of x^p for $x > 0$, the sum is minimized when u_i's are equal to $\alpha/n \leqslant 1/n$. Hence, $\mathrm{M_p}(\mathbf{x}) \leqslant 1/n$.
- *Nash Welfare.* By the concavity of $\prod_{i\in[n]} u_i$, the Nash welfare is maximized when u_i's are equal, i.e. $u_i = \alpha/n \leqslant 1/n$. Therefore, $\mathrm{NW}(\mathbf{x}) \leqslant 1/n$.
- $p \in (0, 1]$. The result again follows by the concavity of x^p, i.e. $(\frac{1}{n} \sum_{i\in[n]} u_i^p)^{1/p}$ is maximized when u_i's are equal, i.e. $u_i = \alpha/n \leqslant 1/n \Rightarrow \mathrm{M_p}(\mathbf{x}) \leqslant 1/n$. □

B Lower Bounds for Section 3.2

In this section, we present matching lower bounds for the results in Section 3.2. We use a similar high-level approach for all welfare functions.

B.1 Approach

First, we present the common parts of the lower bound construction and its analysis.

Suppose we present an instance to the player based on the welfare function, and the player returns an allocation $\mathbf{x}$. For all item $j \in \mathcal{M}$, let p_j denote the total portion of j allocated to the nonsecretive agents combined. Sort the items in decreasing order of p_j, i.e. $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_m$. Take the top k items w.r.t. p_j, and define $\lambda = \frac{1}{k} \sum_{i=1}^{k} p_i$. As λ is the average of the top k values,

$$\frac{1}{m} \sum_{j\in[m]} p_j \leqslant \frac{1}{k} \sum_{i\in[k]} p_i = \lambda. \tag{4}$$

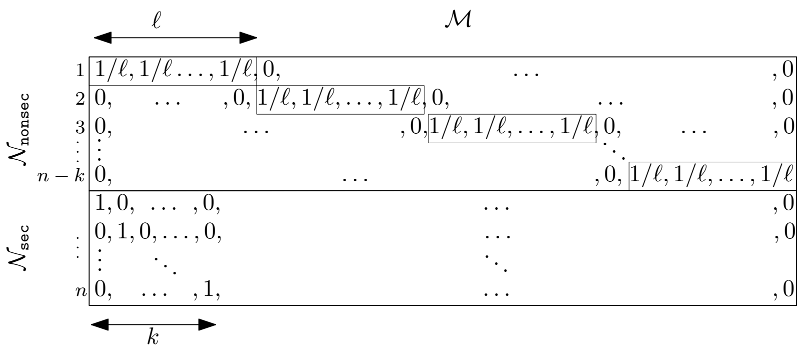

Figure 4: Lower bound example for the Egalitarian and $\mathrm{M}_{\mathrm{p}<0}$ welfare functions.

Deciding on $\mathbf{v}_{\texttt{sec}}$. Take any k disjoint matchings from the secretive agents to the selected items. For example, suppose in the t-th matching for $t \in [k]$, the secretive agent $i \in [k]$ is matched to item $m_i^t = ((t+i) \bmod k) + 1$. Note that $\sum_t \sum_i \mathbf{x}_{i,m_i^t} = \sum_{i=1}^k 1 - p_i = k(1-\lambda)$. Then, at least for one of the matchings $t^\star$ we have $\sum_i \mathbf{x}_{i,m_i^{t^\star}} \leqslant 1 - \lambda$.

Suppose the adversary then sets the utilities of the secretive agents according to this matching, i.e. secretive agent i values her matched item $m_i^{t^\star}$ at 1 and values the rest at 0. Then,

$$\sum_{i \in \mathcal{N}_{\texttt{sec}}} v_i(\mathbf{x}) = \sum_{i \in \mathcal{N}_{\texttt{sec}}} \mathbf{x}_{i,m_i^{t^\star}} \leqslant 1 - \lambda. \tag{5}$$

Deriving a Lower Bound. Next, we obtain an upper bound on $\mathrm{W}(\mathbf{x})$ based on λ. Optimize for the choice of λ to eliminate our dependence on it, and we finish the proof with a lower bound on $\mathrm{W}(\mathbf{x}^*)$.

B.2 Egalitarian and $\mathrm{M}_{\mathbf{p}\in(-\infty,\mathbf{0})}$ Welfares

We use the following example to show lower bounds for the egalitarian and $\mathrm{M}_{\mathrm{p}\in(-\infty,0)}$ welfares.

Example 1. Let $m = (n-k)\ell$ for a positive integer ℓ. Suppose the utility vector of the i-th nonsecretive agent is $1/\ell$ for items in range $[(i-1)\ell + 1, i\ell]$ and 0 for the rest of the items (see Figure 4).

In the following two lemmas, we take Example 1 and let $\ell \geqslant kd/\epsilon$, where d is the proposed distortion value for a fixed welfare function defined in the lemma statement.

Theorem 5. *For $k < n$ and $\epsilon > 0$, there exists an example such that any allocation incurs a distortion value with the egalitarian welfare of at least $d - \epsilon$ for $d = k + 1$.*

Proof. Deciding on $\mathbf{v}_{\texttt{sec}}$. Let $j_1 = \arg\max_{j \in \mathcal{M}} p_j$ and $\lambda' = p_{j_1}$. One of the secretive agents at most owns $(1-\lambda')/k$ fraction of j_1. Name this agent i_1 and set $v_{i_1,j_1} = 1$. Take any other $k-1$ items, and for each one let a unique secretive agent (other than i_1) value this item at 1.

Bounding $\mathrm{EW}(\mathbf{x})$. By the choice of j_1, we know at least $1-\lambda'$ fraction of any item is given to the secretive agents. Hence, the utility of any nonsecretive agent is at most λ'. Furthermore, $v_{i_1}(\mathbf{x}) = (1-\lambda')/k$. Hence, $\mathrm{EW}(\mathbf{x}) \leqslant \min\{(1-\lambda')/k, \lambda'\}$. The choice of $\lambda' = 1/(k+1)$ maximizes this amount. Therefore, $\mathrm{EW}(\mathbf{x}) \leqslant \frac{1}{k+1} = 1/d$.

Bounding $\mathrm{EW}(\mathbf{x}^*)$. As for the adversary's allocation, suppose we match each secretive agent with the item they value at 1 and assign the rest of the items to the nonsecretive agent that value it at $1/\ell$. This way, $\forall i \in \mathcal{N}_{\texttt{sec}}, v_i = 1$ and $\forall i \in \mathcal{N}_{\texttt{nonsec}}$, we have $v_i \geqslant 1 - k/\ell$. Therefore, $\mathrm{EW}(\mathbf{x}^*) \geqslant 1 - k/\ell \geqslant 1 - \epsilon/d$.

Therefore, $\mathrm{D}^{\mathrm{EW}}_{n,k} \geqslant \frac{\mathrm{EW}(\mathbf{x}^*)}{\mathrm{EW}(\mathbf{x})} \geqslant \frac{1-\epsilon/d}{1/d} = d - \epsilon$. □

Theorem 6. *For $k < n$, $p \in (-\infty, 0)$, and $\epsilon > 0$ there exists an example such that any allocation incurs $\mathrm{D}^{\mathrm{M_p}}_{n,k} \geqslant d - \epsilon$, where $d = n^{\frac{1}{p}}((n-k)^{\frac{1}{1-p}} + k)^{\frac{p-1}{p}}$.*

Proof. Suppose we set $\mathbf{v}_{\mathtt{sec}}$ according to Appendix B.1.

Bounding $\mathrm{M_p}(\mathbf{x})$. By (4), sum of the utilities for the nonsecretive agents is at most $\sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i \leqslant \frac{1}{\ell}\sum_{j\in\mathcal{M}} p_j \leqslant m\lambda/\ell$. By the concavity of $\mathrm{M_p}$, $\sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i^p$ is bounded by $\lambda^p(n-k)$ when v_i's are equal to $\frac{m\lambda}{\ell(n-k)} = \lambda$ for all $i \in \mathcal{N}_{\mathtt{nonsec}}$. Similarly, by (5) and the concavity of $\mathrm{M_p}$, $\sum_{i\in\mathcal{N}_{\text{sec}}} v_i^p$ is bounded by $\left(\frac{1-\lambda}{k}\right)^p k$ when v_i's are equal to $(1-\lambda)/k$. Hence,

$$\mathrm{M_p}(\mathbf{x}) \leqslant \left(\frac{1}{n}\left(\lambda^p(n-k) + \left(\frac{1-\lambda}{k}\right)^p k\right)\right)^{\frac{1}{p}}.$$

The above is maximized when $\lambda = \frac{z}{k+z}$ for $z = (n-k)^{\frac{1}{1-p}}$. Substituting λ, we have $\mathrm{M_p}(\mathbf{x}) \leqslant n^{-\frac{1}{p}}(z+k)^{-\frac{p-1}{p}} = 1/d$.

Bounding $\mathrm{M_p}(\mathbf{x}^*)$. For the adversary, assign each secretive agent i her matched item $m_{t^\star,i}$, hence $v_i = 1$. For the nonsecretive agents, we can assign each item to the single agent that values it at $1/\ell$ except the selected k items. For each nonsecretive agent i, $v_i \geqslant 1 - k/\ell$. By the fact that $\mathrm{M_p}(\mathbf{x}) \geqslant \mathrm{EW}(\mathbf{x})$, we have $\mathrm{M_p}(\mathbf{x}^*) \geqslant 1 - k/\ell \geqslant 1 - \epsilon/d$.

Putting the two bounds together, we have $\mathrm{D}^{\mathrm{M_p}}_{n,k} \geqslant d - \epsilon$. □

B.3 Nash and $\mathrm{M}_{\mathbf{p}\in(\mathbf{0},\infty)}$ Welfares

Example 2. Suppose each nonsecretive agent values each item uniformly at $1/m$.

In the next three lemmas, we use Example 2 with $m \geqslant kd/\epsilon$ items, where d is the proposed distortion value for a fixed welfare function defined in the lemma statement.

Set $\mathbf{v}_{\mathtt{sec}}$ according to Appendix B.1. Then, by (4) and that the nonsecretive agents have uniform utilities,

$$\sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i(\mathbf{x}) = \frac{1}{m}\sum_{j\in\mathcal{M}} p_j \leqslant \lambda. \tag{6}$$

We use this inequality in the proof of the following theorems.

Theorem 7. *For $k < n$ and $\epsilon > 0$, there exists an example such that any allocation incurs a distortion value with the Nash welfare of at least $d - \epsilon$ where $d = n(\frac{1}{n-k})^{\frac{n-k}{n}}$.*

Proof. Bounding $\mathrm{M_p}(\mathbf{x})$. By (6) and the concavity of NW, $\prod_{i\in\mathcal{N}_{\text{nonsec}}} v_i$ is bounded by $(\frac{\lambda}{n-k})^{n-k}$ when v_i's are equal to $\frac{\lambda}{n-k}$ for all $i \in \mathcal{N}_{\mathtt{nonsec}}$. Similarly, by (5) and the concavity of NW, $\prod_{i\in\mathcal{N}_{\text{sec}}} v_i$ is bounded by $(\frac{1-\lambda}{k})^k$ when v_i's are equal to $(1-\lambda)/k$. Hence,

$$\mathrm{NW}(\mathbf{x}) \leqslant \left(\left(\frac{\lambda}{n-k}\right)^{n-k}\left(\frac{1-\lambda}{k}\right)^k\right)^{\frac{1}{n}}.$$

The maximum is achieved at $\lambda = \frac{n-k}{n}$. Hence, $\mathrm{NW}(\mathbf{x}) \leqslant \frac{1}{n}$.

Bounding $\mathrm{NW}(\mathbf{x}^*)$. For the adversary, assign each secretive agent i her matched item $m_{t^\star,i}$, hence $v_i = 1$. Allocate the other $m-k$ items uniformly among the nonsecretive agents. This way

$v_i = \frac{1}{n-k} \cdot \frac{m-k}{m}$, and we have

$$\mathrm{NW}(\mathbf{x}^*) \geqslant \left(\left(\frac{m-k}{m} \cdot \frac{1}{n-k} \right)^{n-k} \cdot 1^k \right)^{\frac{1}{n}}$$
$$\geqslant \frac{m-k}{m} \left(\frac{1}{n-k} \right)^{\frac{n-k}{n}} = \left(1 - \frac{k}{m} \right) \frac{d}{n}.$$

Putting the two bounds together, we have

$$\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant \left(1 - \frac{k}{m} \right) d = d - \frac{dk}{m} \geqslant d - \epsilon. \tag{7}$$

□

Theorem 8. *For $k < n$, $p \in (0,1)$, and $\epsilon > 0$ there exists an example such that any allocation rule incurs $\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant d - \epsilon$ for $d = n \left(\frac{(n-k)^{1-p}+k}{n} \right)^{1/p}$.*

Proof. Bounding $\mathrm{M_p}(\mathbf{x})$. Following the same analysis for the case of Nash welfare in Theorem 7, to obtain an upper bound on $\mathrm{M_p}(\mathbf{x})$, we may assume for all $i \in \mathcal{N}_{\texttt{nonsec}}$, $v_i = \frac{\lambda}{n-k}$ and for all $i \in \mathcal{N}_{\texttt{sec}}$, $v_i = \frac{1-\lambda}{k}$. Hence,

$$\mathrm{M_p}(\mathbf{x}) \leqslant \left(\frac{1}{n} \left(\left(\frac{\lambda}{n-k} \right)^p (n-k) + \left(\frac{1-\lambda}{k} \right)^p k \right) \right)^{\frac{1}{p}}.$$

The maximum is achieved at $\lambda = \frac{n-k}{n}$. Hence, $\mathrm{M_p}(\mathbf{x}) \leqslant \frac{1}{n}$.

Bounding $\mathrm{M_p}(\mathbf{x}^*)$. For the adversary, assign each secretive agent i her matched item $m_{t^\star,i}$, hence $v_i = 1$. Allocate the other $m-k$ items uniformly among the nonsecretive agents. This way $v_i = \frac{1}{n-k} \cdot \frac{m-k}{m}$, and we have

$$\mathrm{M_p}(\mathbf{x}^*) \geqslant \left(\frac{1}{n} \left(\left(\frac{m-k}{m} \cdot \frac{1}{n-k} \right)^p (n-k) + k \right) \right)^{\frac{1}{p}}$$
$$\geqslant \left(\frac{1}{n} \left(\left(\frac{m-k}{m} \right)^p (n-k)^{1-p} + \left(\frac{m-k}{m} \right)^p k \right) \right)^{\frac{1}{p}} \quad \left(\text{since } \frac{m-k}{m} \leqslant 1 \text{ and } p > 0.\right)$$
$$\geqslant \frac{m-k}{m} \left(\frac{(n-k)^{1-p}+k}{n} \right)^{\frac{1}{p}} = \left(1 - \frac{k}{m} \right) \frac{d}{n}.$$

Putting the bounds together, we arrive at the same expression as in (7). Hence, $\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant d - \epsilon$. □

Theorem 9. *For $k < n$, $p \in (1, \infty)$, and $\epsilon > 0$ there exists an example such that any allocation rule incurs $\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant d - \epsilon$, where $d = (k+1)^{1/p}$.*

Proof. Bounding $\mathrm{M_p}(\mathbf{x})$. For the nonsecretive agents, by (6) and the convexity of $\mathrm{M_p}$, $\sum_{i \in \mathcal{N}_{\texttt{nonsec}}} v_i^p$ is bounded by λ^p when $v_i = \lambda$ for one agent and $v_{i'} = 0$ for other nonsecretive agents $i' \neq i$. Similarly, for the secretive agents, by (5) and the convexity of $\mathrm{M_p}$, $\sum_{i \in \mathcal{N}_{\texttt{sec}}} v_i^p$ is bounded by $(1-\lambda)^p$ when $v_i = 1 - \lambda$ for one secretive agent i and the rest have 0 utility. Hence,

$$\mathrm{M_p}(\mathbf{x}) \leqslant \left(\frac{\lambda^p + (1-\lambda)^p}{n} \right)^{\frac{1}{p}} \leqslant \left(\frac{1}{n} \right)^{\frac{1}{p}},$$

because the maximum is achieved at $\lambda \in \{0, 1\}$.

Bounding $\mathrm{M_p}(\mathbf{x}^*)$. For the adversary, assign each secretive agent i her matched item $m_{t^\star,i}$, hence $v_i = 1$. Allocate the other $m-k$ items all to a single nonsecretive agent i. This way $v_i = \frac{m-k}{m}$ while the rest get 0 utility. Then,

$$\begin{aligned}
\mathrm{M_p}(\mathbf{x}^*) &\geqslant \left(\frac{(\frac{m-k}{m})^p + k}{n}\right)^{\frac{1}{p}} \\
&\geqslant \left(\frac{(\frac{m-k}{m})^p + (\frac{m-k}{m})^p k}{n}\right)^{\frac{1}{p}} && \left(\text{since } \frac{m-k}{m} \leqslant 1 \text{ and } p > 0.\right) \\
&= \frac{m-k}{m}\left(\frac{k+1}{n}\right)^{\frac{1}{p}}.
\end{aligned}$$

Putting the bounds together, we get the same result as in (7). □

C Distortion with Indivisible Items

In the divisible case, the player has the advantage of dividing items uniformly among the secretive agents, which allows, the player to guarantee a minimum welfare for each secretive agent. However, in the indivisible case, we will show in Appendix C.2 that the adversary can set $\mathbf{v_{sec}}$ in a way that all secretive agents have a utility of 0. The only exception is when we allocate all items to one secretive agent. That agent is guaranteed to have a utility of 1, while any other agent will have a utility of 0. In either case, if $k > 0$, the adversary can make the utility of at least one agent 0 leading to a Nash and egalitarian welfare of 0 and an unbounded $\mathrm{M_p}$ welfare for $p < 0$.

C.1 Upper Bounds

While the Nash, the egalitarian, and the $\mathrm{M_{p<0}}$ welfare functions have unbounded distortion (shown in Appendix C.2), for $p > 0$, an almost worst-case optimal strategy is as follows:

- *Case* $k = n$. Allocate all items to one (secretive) agent.
- *Case* $0 \leqslant k < n$. Return the allocation maximizing the $\mathrm{M_p}$ welfare for the non-secretive agents — similar to $\mathtt{A}_1$ in the divisible case.

Theorem 10. *For $p > 0$, the distortion value with $\mathrm{M_p}$ welfare in the case of indivisible items is upper bounded by* $\min\{(k+1)^{\frac{1}{p}}, n^{\frac{1}{p}}\}$.

Proof. For $k = n$, we allocate all items to one agent. That agent will have a utility of 1, while the rest will have a utility of 0. Hence, $\mathrm{M_p}(\mathbf{x}) = n^{-1/p}$. The mean welfare of adversary is at most 1, therefore the distortion is upper bounded by $n^{1/p}$.

For $0 \leqslant k < n$, suppose the welfare maximizing allocation for the non-secretive agents achieve $\sum_{i \in \mathcal{N}_{\text{nonsec}}} v_i^p = \beta$. The player returns this allocation. Therefore, $\mathrm{M_p}(\mathbf{x}) = (\beta/n)^{1/p}$. Similar to the proof of Lemma 3, the adversary cannot improve on β for the non-secretive agents. Moreover, they cannot obtain a utility of more than 1 for each secretive agent. Hence, $\mathrm{M_p}(\mathbf{x}^*) \leqslant ((\beta+k)/n)^{\frac{1}{p}}$.

Putting the bounds together, we have

$$\mathrm{D}_{n,k}^{\mathrm{M_p}} \leqslant \left(\frac{\beta+k}{\beta}\right)^{\frac{1}{p}} \leqslant (k+1)^{\frac{1}{p}},$$

where the last inequality follows from two facts. First, that $\frac{x+k}{x}$ is a decreasing function over $x > 0$, and second, that $\beta \geqslant 1$ as allocating all items to one agent ensures it for any instance. □

C.2 Lower Bounds

Note that in the case of $n = 1$ or $k = 0$ (no secretive agents), the distortion value is always 1 as the player has full information to compute the optimal allocation. The following theorem shows a lower bound for $n > 1$ and $k > 0$.

Theorem 11. *For $n > 1$, $k > 0$, and the Nash, egalitarian, and $\mathrm{M_{p<0}}$ welfare functions, the distortion value in the case of indivisible items is* unbounded*.*

Proof. Consider an instance with $m = n$ items where the non-secretive agents value all items at $\frac{1}{m}$. If one agent is given all items by the player, then, as $n > 1$, at least one agent has a utility of 0. Otherwise, if no agent is allocated all items, take any secretive agent i. At least one item j is not allocated to i. Let $v_{i,j} = 1$. This way, $v_i(\mathbf{x}) = 0$, and the Nash, egalitarian, and $\mathrm{M_{p\leqslant 0}}$ welfare of the player are all 0. However, the adversary can match items and agents in a way that each secretive agent gets a utility of 1 from their matched item and the non-secretive agents each get a utility of $\frac{1}{m}$. All the aforementioned welfare functions for the adversary are at least $\frac{1}{m}$, hence, the player incurs an unbounded distortion. □

The following lemma is useful in proving the lower bounds for the other welfare functions.

Lemma 7 (Matching Argument)**.** *Suppose there does not exist an agent that is given all items. Then, there exists a matching from the secretive agents to items $m \colon \mathcal{N}_{\mathtt{sec}} \to \mathcal{M}$ such that item m_i is not allocated to i.*

Proof. Take the bipartite graph from the secretive agents to all items where each item has an edge to the agent owning the item. For the complement of this graph, Hall's theorem condition is satisfied for the secretive agents. This holds because each secretive agent has at least one edge, and for all subsets S of agents of size at least two, every item j is allocated to at most one agent i, hence there exists at least one edge from $S \setminus \{i\}$ to the j. By Hall's theorem, there exists a complete matching in the complement graph from the secretive agents to items. □

First, we will resolve the case $k = n$ using the lemma above.

Lemma 8. *For $k = n$ and $p > 0$, the distortion value in the case of indivisible items is lower bounded by $n^{\frac{1}{p}}$.*

Proof. If all items are not allocated to one agent, by Lemma 7 there is matching such that agent i is not given m_i. The adversary can set $v_i(m_i) = 1$, leading to $\mathrm{M_p}(\mathbf{x}) = 0$. The adversary can allocate items according to the matching. Hence, $\mathrm{M_p}(\mathbf{x}^*) = 1$ and the distortion is unbounded.

Otherwise, $v_i = 1$ for an agent i and the rest have 0 utility. This way $\mathrm{M_p}(\mathbf{x}) = n^{-1/p}$. The adversary can again take an arbitrary matching and allocate accordingly to obtain $\mathrm{M_p}(\mathbf{x}^*) = 1$. Hence, $\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant n^{1/p}$. □

Next, we will show a lower bound matching the upper bound of $(k+1)^{1/p}$ for $p \geqslant 1$. However, for $p \in (0, 1)$, our bounds are not tight. We have two lower bounds for this case, one shown in Theorem 12 and another in the Theorem 13. We conclude that the distortion value for $p \in (0, 1)$ is more than the maximum of the two.

Theorem 12. *For $k < n$, $p \geqslant 1$, and $\epsilon > 0$, the distortion value in the case of indivisible items is lower bounded by $(k+1)^{\frac{1}{p}} - \epsilon$.*

Furthermore, for $p \in (0, 1)$, the distortion value is at least $\left(\frac{z+k}{z}\right)^{\frac{1}{p}} - \epsilon$ for $z = (n-k)^{1-p}$.

Proof. Let $m \geqslant k + k(n-k)d/\epsilon$ where $d = (k+1)^{1/p}$ and $m-k$ is divisible by $n-k$. By our choice of m, we have that $m/k \geqslant d/\epsilon \Rightarrow 1 - k/m \geqslant 1 - \epsilon/d$, which we use in the analysis. We break the analysis into two cases.

Case 1. Suppose the player allocates all items to one agent. Then $\mathrm{M_p}(\mathbf{x}) = n^{-1/p}$. For the adversary, take any matching from the first k items to the secretive agents. Each value their matched item at 1, hence their utility is 1. Give all of the other items to one of the non-secretive agents, that is her utility is $1 - k/m \geqslant 1 - \epsilon/d$. This way,

$$\begin{aligned}\mathrm{M_p}(\mathbf{x}^*) &\geqslant \left(\frac{1}{n}\left(k + \left(1 - \frac{\epsilon}{d}\right)^p\right)\right)^{1/p}\\ &\geqslant n^{-1/p} \cdot \left(1 - \frac{\epsilon}{d}\right) \cdot (k+1)^{1/p}\\ &= n^{-1/p} \cdot \left(1 - \frac{\epsilon}{(k+1)^{1/p}}\right)(k+1)^{1/p} = ((k+1)^{1/p} - \epsilon)n^{-1/p}.\end{aligned}$$

By combining the two bounds, we have $\mathrm{D}^{\mathrm{M_p}}_{n,k} \geqslant \frac{\mathrm{M_p}(\mathbf{x}^*)}{\mathrm{M_p}(\mathbf{x})} \geqslant (k+1)^{1/p} - \epsilon$.

Case 2. No agent is allocated all items by the player.

Deciding on $\mathbf{v}_{\mathtt{sec}}$. Take the matching from Lemma 7, and for each secretive agent, suppose they value their matched item at 1 and the rest at 0. This way, $v_i(\mathbf{x}) = 0$ for all secretive agents i.

Bounding $\mathrm{M_p}(\mathbf{x})$. Let $\beta = \max_{\mathbf{x}} \sum_{i \in \mathcal{N}_{\mathrm{nonsec}}} v_i(\mathbf{x})^p$. By the way we set the utilities, all secretive agents get zero utility and $\sum_{i \in \mathcal{N}_{\mathrm{sec}}} v_i(\mathbf{x})^p = 0$, therefore, $\mathrm{M_p}(\mathbf{x}) \leqslant (\beta/n)^{1/p}$.

Bounding $\mathrm{M_p}(\mathbf{x}^*)$. Suppose the adversary assigns each secretive agent $i \in \mathcal{N}_{\mathtt{sec}}$ her matched item, hence $v_i(\mathbf{x}^*) = 1$ and $\sum_{i \in \mathcal{N}_{\mathrm{sec}}} v_i(\mathbf{x}^*)^p = k$. In the next two sub-cases, we show that the adversary can guarantee

$$\sum_{i \in \mathcal{N}_{\mathrm{nonsec}}} v_i(\mathbf{x}^*)^p \geqslant \beta \left(1 - \frac{\epsilon}{d}\right)^p \tag{8}$$

for $p \in (0,1)$ and $p \geqslant 1$. Assuming this, and together with that $\sum^p_{i \in \mathcal{N}_{\mathrm{sec}} v_i(x^*)} = k$, we have

$$\mathrm{M_p}(\mathbf{x}^*) \geqslant \left(\frac{k + ((1-\epsilon/d)\cdot\beta)^p}{n}\right)^{1/p} \geqslant \left(1 - \frac{\epsilon}{d}\right) \cdot \left(\frac{k+\beta}{n}\right)^{1/p}.$$

Bounding Distortion. Combining the two bounds above, we have

$$\mathrm{D}^{\mathrm{M_p}}_{n,k} \geqslant \left(1 - \frac{\epsilon}{d}\right)\left(\frac{\beta + k}{\beta}\right)^{1/p}. \tag{9}$$

Since the above is a decreasing function when β increases, to obtain a lower bound on the distortion, it suffices to show an upper bound on the value of β. In the following subcases, which are based on the ranges of values of p, we determine an upper bound on β and show that Equation (8) holds for both cases.

Case 2.1 (p $\geqslant$ 1). Since all non-secretive agents have identical valuations, $\sum_{i \in \mathcal{N}_{\mathrm{nonsec}}} v_i(\mathbf{x}^*) = 1$. By the convexity of $\mathrm{M_p}$, β is maximized when $v_i(\mathbf{x}^*) = 1$ for one agent i (i.e., all items are given to i) and the rest have 0 utility. Hence, $\beta \leqslant 1$.

Suppose the adversary gives $m-k$ items to a non-secretive agent $i^* \in \mathcal{N}_{\mathtt{nonsec}}$. Then, $v_{i^*}(\mathbf{x}^*) = 1 - \frac{k}{m}$. Next,

$$\sum_{i \in \mathcal{N}_{\mathrm{nonsec}}} v_i(\mathbf{x}^*)^p = v_{i^*}(\mathbf{x}^*)^p = \left(1 - \frac{k}{m}\right)^p \overset{(1)}{\geqslant} \left(1 - \frac{\epsilon}{d}\right)^p \geqslant \beta\left(1 - \frac{\epsilon}{d}\right)^p,$$

where (1) follows from our choice of m and last inequality holds since $\beta \leqslant 1$. This establishes Equation (8), and by substituting $\beta \leftarrow 1$ in Equation (9) we have $\mathrm{D}^{\mathrm{M_p}}_{n,k} \geqslant (k+1)^{\frac{1}{p}} - \epsilon$.

Case 2.2 ($0 < p < 1$). Since all non-secretive agents have identical valuations, $\sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i(\mathbf{x}^*) = 1$. By the concavity of $\mathrm{M_p}$ for $p \in (0,1)$, β is maximized when all have equal utilities, i.e., $v_i(\mathbf{x}^*) = 1/(n-k)$ for all non-secretive agents i. Hence, $\beta = \sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i(\mathbf{x}^*)^p \leqslant (n-k)\cdot\left(\frac{1}{n-k}\right)^p = (n-k)^{1-p} = z$.

Similarly, in our example, suppose the adversary divides the $m-k$ items equally among the $n-k$ non-secretive agents. This is feasible by our choice of m. Therefore, for $i \in \mathcal{N}_{\texttt{nonsec}}$ we have $v_i(\mathbf{x}^*) = \frac{m-k}{m}\cdot\frac{1}{n-k} \geqslant \left(1-\frac{k}{m}\right)\cdot\frac{1}{n-k} \geqslant \left(1-\frac{\epsilon}{d}\right)\cdot\frac{1}{n-k}$. Then,

$$\sum_{i\in\mathcal{N}_{\text{nonsec}}} v_i(\mathbf{x}^*)^p \geqslant \left(\left(1-\frac{\epsilon}{d}\right)\cdot\frac{1}{n-k}\right)^p (n-k)$$
$$= (n-k)^{1-p}\cdot\left(1-\frac{\epsilon}{d}\right)^p = z\cdot\left(1-\frac{\epsilon}{d}\right)^p \geqslant \beta\cdot\left(1-\frac{\epsilon}{d}\right)^p,$$

which establishes Equation (8). By substituting $\beta \leftarrow z$ in (9) we have

$$\mathrm{D}_{n,k}^{\mathrm{M_p}} \geqslant \left(\frac{z+k}{z}\right)^{1/p} - \epsilon\cdot\frac{1}{d}\cdot\left(\frac{z+k}{z}\right)^{1/p} \geqslant \left(\frac{z+k}{z}\right)^{1/p} - \epsilon,$$

where the inequality holds since $\left(\frac{z+k}{z}\right)^{1/p} \leqslant d = \left(\frac{1+k}{1}\right)^{1/p}$ and that $z > 1$. □

Note that $\lim_{p\to 0}(1+\frac{k}{(n-k)^{1-p}})^{1/p} = \infty$ as expected from Theorem 11, and for $p = 1$ the bound is $k+1$, which matches Theorem 12 and implies tightness when p is close to 1. Next, we will show another lower bound of $k^{1/p}$. This lower bound is better than the former for most values of $p \in (0,1)$ but not when p is very close to 1, e.g., for $p = 1$, it evaluates to k which is less than the former lower bound of $k+1$.

Theorem 13. *For $k < n$, $p \in (0,1)$, the distortion value in the case of indivisible items is lower bounded by* $\max\left\{k^{\frac{1}{p}}, \left(\frac{z+k}{z}\right)^{\frac{1}{p}} - \epsilon\right\}$ *for $z = (n-k)^{1-p}$ and any $\epsilon > 0$.*

Proof. In Theorem 12, we showed that the distortion is lower bounded by $\left(\frac{z+k}{z}\right)^{\frac{1}{p}} - \epsilon$. Now, we show it is also lower bounded by $k^{\frac{1}{p}}$.

Take an instance with $m = k+1$ items, where all non-secretive agents value item j_1 at 1 and the rest at 0.

Case 1. Suppose there is a secretive agent i such that the player has allocated all items in $\mathcal{M}\setminus\{j_1\}$ to i. As the adversary, set $v_{i,j_1} = 1$ and 0 for the other items. Take any $k-1$ items from $\mathcal{M}\setminus\{j_1\}$ and match it arbitrarily to the other $k-1$ secretive agents. Suppose they value their matched item at 1.

The adversary can match $\mathcal{N}_{\texttt{sec}}\setminus\{i\}$ to their matched item, and allocate j_1 to a non-secretive agent. This way, $\mathrm{M_p}(\mathbf{x}^*) \geqslant (k/n)^{1/p}$. However, the player will obtain 0 utility for secretive agents other than i, and at most one non-secretive agent or agent i is allocated j_1 and has a utility of 1. Hence, $\mathrm{M_p}(\mathbf{x}) \leqslant (1/n)^{1/p}$. Combining the bounds, we conclude that the distortion value is at least $k^{1/p}$.

Case 2. Now, suppose no secretive agent is allocated all items from $\mathcal{M}\setminus\{j_1\}$. Then, according to Lemma 7, match each secretive agent i with an item m_i not allocated to them. Set $v_{i,m_i} = 1$. Then, the player obtains $v_i(\mathbf{x}) = 0$ for all secretive agents. Similar to the former case, at most one non-secretive agent is allocated item j_1 and has a utility of 1. Therefore, $\mathrm{M_p}(\mathbf{x}) \leqslant (1/n)^{1/p}$. The adversary can allocate each secretive agent her matched item and obtain a utility of 1, while allocation j_1 to one of the non-secretive agents. Therefore, $\mathrm{M_p}(\mathbf{x}^*) \geqslant ((k+1)/n)^{1/p}$. Putting all together, we achieve a lower bound of $(k+1)^{1/p}$ for this case. □

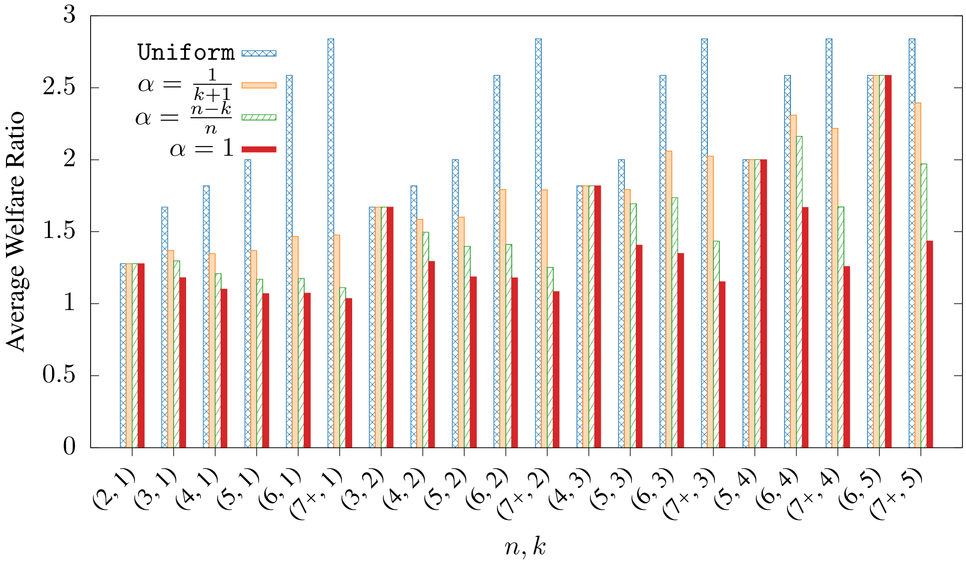

Figure 5: Average welfare ratio by different strategies on the Spliddit data. The x-axis is sorted by k and then n.

D Additional Experiment Plots

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