Designing egalitarian weighted voting games

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Abstract

As an important form of collective decision-making, weighted voting games play a pivotal role in allocating social benefits. From an egalitarian perspective, an important pursuit is a decision rule that benefits everyone equally. The debate on egalitarian weighted voting games is primarily based on equality of opportunity. However, equality of outcome, an equally essential ingredient of equality, has received less attention. This paper investigates the problem of designing egalitarian weighted voting games, with a particular focus on equality of outcome. In some special circumstances, it is proved that a weighted majority rule can minimize outcome inequality. In general, designing an egalitarian weighted voting game can be transformed into an optimization problem. Heuristic algorithms for approximating optimal solutions are proposed.

1 Introduction

A collective decision could affect the distribution of welfare in society. As an important form of collective decision-making, weighted voting games have widespread applications. A non-negligible issue regarding these applications is how the voting weights should be distributed. From an egalitarian perspective, the weights should ensure that everyone will benefit equally. The majority rule widely adopted in modern democracies is usually regarded as egalitarian in that it guarantees everyone the same opportunity to win for the benefit, as opposed to the others in history, such as the dictatorial rule which would always favor one individual. In a more complex setting such as representative democracy, a rule is argued to be egalitarian if the voting weight of a constituency is roughly proportional to the square root of its population [5].

However, all the above arguments neglect another ingredient of equality, equality of outcome. Rather than focusing on whether the expected benefits are the same for everyone, equality of outcome is primarily concerned with whether everyone receives the same amount of benefits. Indeed, no rule would always yield equal an outcome if the interests of voters diverge. Therefore, the goal is to identify a rule such that the outcome inequality is minimized. Bruner[7] argues that the majority rule ensures outcome equality in that it always selects the alternative with the lower Gini coefficient. However, the conclusion is limited to when the equal intensity assumption (EI) holds.

This paper investigates the question of designing egalitarian weighted voting games, with a particular focus on outcome equality. Starting with an extension to Bruner's argument, I show that under a more general setting, the weighted majority rule rather than the majority rule ensures outcome equality. Further analysis suggests the conclusion is still limited as in general, it is not feasible to always select the more equal alternative. Therefore, a new approach that minimizes inequality in an average sense is adopted. Designing an egalitarian weighted voting rule is therefore an optimization problem. Equality of opportunity and equality of outcome are well coordinated in this framework. Iterative algorithms can efficiently solve the general optimization problem.

The related literature is as follows. The optimal weighted voting games have been discussed extensively from both theoretical and algorithmic aspects. Theoretical analysis hinges on a distribution of voting weights or power indices that maximizes or minimizes certain functions, e.g., sum of the utilities [4, 12, 2], majority deficit [11], similarly with this work. Research from the algorithmic aspect focuses on the inverse power-index problem, that is designing a weighted voting game from a desirable distribution of power indices. For the inverse Banzhaf problem, iterative approximation algorithms [17, 1] and exact algorithms based on linear programming [15] have been proposed. Recently, Diakonikolas et al. [9] propose a general solution to the inverse power-index problem based on a variant of stochastic gradient descent.

The paper is organized as follows. Section 2 introduces basic notations and definitions regarding voting and inequality measurement. Section 3 first extends Bruner's work and then discusses the limitations in methodology. To overcome the limitations, Section 4 adopts a new approach, where designing an egalitarian weighted voting game is transformed into a general optimization problem. Section 5 discusses algorithms for solving the problem. Synthetic examples are used to test the algorithms. Section 6 summarizes the main conclusions.

2 Basic notations and definitions

Let $N = \{1, 2, \dots, n\}$ denote the set of voters and 2^N denote the power set of N. Each voter may cast either 'yes' or 'no' on a proposal. Let S denote the coalition of 'yes' voters, then every voting profile can be represented by a subset $S \subseteq N$.

A voting rule or a game fully specifies the outcome of each voting profile. For simple games, the outcome is either acceptance or rejection. S is called a winning coalition if the proposal is accepted under S. A simple game is monotonic, that is a winning coalition still wins if any other voter joins.

Definition 1. *A simple game G is denoted by a pair (N, W), where W is the set of the winning coalitions satisfying if $S \in W$ and $S \subseteq T$, then $T \in W$. Alternatively, it can be denoted by a pair (N, v), where v is the characteristic function satisfying $v(S) = 1$ if $S \in W$; $v(S) = 0$ if $S \notin W$; if $S \subseteq T$, then $v(S) \leq v(T)$. The set of all simple games is denoted by $\mathcal{G}$.*

The weighted voting game is a common type of simple game. In a weighted voting game, every voter is assigned a voting weight, and a proposal is accepted if and only if the proportion of the weights in S to the total weights is larger than the quota. For monotonicity, the weights should be non-negative.

Definition 2. *A weighted voting game is denoted by $G = [q; w_1, w_2, \dots, w_n]$ where $w_i \geq 0$ is the voting weight of voter i and q is the quota. A voting profile $S \in W$ if and only if $\sum_{i \in S} w_i \geq q \sum_{i=1}^{n} w_i$. The set of all weighted voting games is denoted by $\mathcal{G}_w$.*

Success is an important aspect in evaluating the role of a voter in a game. A voter is successful if and only if her vote coincides with the collective outcome. Let $\mathbf{x}$ record the success or failure of these voters, where $x_i = 1$ if i wins and $x_i = -1$ if the opposite.

Definition 3. *Voter i is successful and only if the group decision coincides with voter i's vote, i.e., $(i \in S \in W)$ or $(i \notin S \notin W)$.*

Voters receive different levels of satisfaction from the two alternatives. Suppose they vote sincerely, the satisfaction level would be higher if they win. Satisfaction may also vary across voters, depending on how the two alternatives are specified in the ballot. To denote the distribution of voter satisfaction, the utility vector is defined as below. It is assumed to have full interpersonal comparability. Under EI, everyone receives the same level of satisfaction from both the preferred option and the less preferred one, or u_i the same for all.

Definition 4. *The utility vector $\mathbf{u} = (u_1(x_1), \dots, u_n(x_n))^\top$, where $u_i : \{-1, 1\} \to \mathbb{R}$ and satisfies $u_i(1) \geq u_i(-1)$ for all i.*

An inequality index measures how equal a distribution is across population sizes. Let $D^n \subseteq \mathbb{R}^n$ denote the set of all possible distributions given n. Then $D = \bigcup_{n=2}^{+\infty} D^n$ [1] is the set of all possible distributions for varying populations. For normalization, the inequality is zero if everyone is equal.

Definition 5. *An inequality index is a function $I : D \to \mathbb{R}_+$ which satisfies $I(\mathbf{u}) = 0$ if and only if $\mathbf{u} = c\mathbf{1}$ [2] for some $c \in \mathbb{R}$.*

3 Further analysis of Bruner's conclusion

3.1 An extension to Bruner's proposition

This subsection first introduces Bruner's argument in favor of the majority rule in that it ensures outcome equality. Following that, I provide an extension to his proposition. It suggests that the weighted majority rule, rather than the majority rule ensures outcome equality under more general assumptions.

Bruner's argument is based on Proposition 1. It suggests that if under EI, then the majority rule always selects the alternative with the lower Gini coefficient and the relative standard deviation. Hence, the majority rule outperforms the others in terms of minimizing outcome inequality.

Proposition 1. *When EI holds, the majority rule minimizes the Gini coefficient[3] and the relative standard deviation[4] for every $S \subseteq N$.*

However, as Bruner himself acknowledges, the conclusion is limited to when EI is valid, a condition that rarely holds in practice. For generality, I relax the assumptions in two aspects: the utilities of voters and the inequality measurement.

Following Definition 4, let $k_i = (u_i(1) - u_i(-1))/2$ and $b_i = (u_i(1) + u_i(-1))/2$, the utility of voter i can be expressed as

$$u_i = k_i x_i + b_i. \tag{1}$$

The coefficient k_i, or the sensitivity of voter i, could reflect how intensely voter i would be affected by the proposal. Voter i is indifferent towards the two alternatives if $k_i = 0$. The coefficient b_i is equal to the arithmetic mean of the utility of winning and the utility of losing for voter i. For simplicity, I will refer to b_i as the average satisfaction of voter i. Differences in average satisfaction may originate from the fact that the two alternatives are not designed to be fair. For instance, some voters may at least benefit while others will at most not be harmed in a vote. Alternatively, if $\mathbf{u}$ measures the current welfare instead of the benefits from the vote, then differences in original status may also contribute. Without further notice, I will refer to $\mathbf{u}$ as the utilities from a single vote. A vote is unbiased (UB)

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[1] The concept of inequality is void for a single voter.

[2] $\mathbf{1}$ denotes a vector with all elements being 1.

[3] The Gini coefficient is given by

$$I(\mathbf{u}) = \sum_{1 \le i,j \le n} \frac{|u_i - u_j|}{2n^2 \bar{u}},$$

where $\bar{u} = \sum_{i=1}^{n} u_i/n$.

[4] The relative standard deviation is given by

$$I(\mathbf{u}) = \frac{1}{\bar{u}} \left(\frac{1}{n} \sum_{i=1}^{n} (u_i - \bar{u})^2 \right)^{\frac{1}{2}}.$$

if $b_i = b_j$ for all $i, j \in N$. To cope with the domain restrictions of the inequality index, I assume $u_i > 0$ for all.

The inequality indices can be generalized with a few properties. There are two important types of inequality index, the relative index and the absolute index. The discrepancy lies in different notions of inequality equivalence. The absolute index is invariant with respect to equal absolute changes, that is

$$I(\mathbf{u} + c\mathbf{1}) = I(\mathbf{u}). \tag{2}$$

whereas the relative index is invariant under the equal proportional changes to the utility vector, or

$$I(\lambda \mathbf{u}) = I(\mathbf{u}),\ \lambda > 0. \tag{3}$$

For relative indices, the domain $D^n \subseteq \mathbb{R}^n_{++}$ [5].

An absolute could be reflective (R), or the inequality remains unchanged under negation, i.e.,

$$I(-\mathbf{u}) = I(\mathbf{u}). \tag{4}$$

If an index is R, then the inequalities are the same for opposing things, such as happiness and unhappiness. If absolute inequality changes proportionally under proportional changes in distribution, or

$$I(\lambda \mathbf{u}) = \lambda I(\mathbf{u}),\ \lambda > 0, \tag{5}$$

then it is homogeneous (H). If an absolute index I_a satisfies H, its relative form could be obtained through

$$I_r = \frac{nI_a}{\sum_{i=1}^n u_i}. \tag{6}$$

Reversely, if the absolute form of a relative index $I_a = \bar{u} I_r$ exists, then it is H. Indices with the two properties are not uncommon, for instance, the absolute Gini index and the standard deviation.

Proposition 2 attempts a more general argument based on the above assumptions. It asserts that if the inequality is measured by a relative index whose absolute form is R and UB holds, then the weighted majority rule with weights proportional to sensitivities would always select the alternative with lower inequality. Proposition 1 is thus a special circumstance when EI holds and the inequality index is Gini or relative standard deviation.

Proposition 2. *When UB holds, the weighted majority rule $G = [0.5; k_1, k_2, \ldots, k_n]$, where k_i is the sensitivity of voter i, would minimize the inequality for every $S \subseteq N$ if the relative index $I_r = I_a/\bar{u}$ with I_a be R.*

Proof. For every voting profile S, let $\mathbf{x}$ denote the success vector under G and let $\mathbf{x}'$ denote the success vector if the other alternative is chosen. Let $\mathbf{u}$ and $\mathbf{u}'$ denote the corresponding utility vectors.

I first show that for every S, $I_a(\mathbf{u}) = I_a(\mathbf{u}')$. If UB holds, then $\mathbf{u} = \mathbf{k} \odot \mathbf{x} + b\mathbf{1}$ [6]. By reflectivity,

$$I(\mathbf{u}) = I(\mathbf{k} \odot \mathbf{x} + b\mathbf{1}) = I(\mathbf{k} \odot \mathbf{x}) = I(-\mathbf{k} \odot \mathbf{x}) = I(\mathbf{k} \odot \mathbf{x}') = I(\mathbf{u}').$$

I now show that $\bar{u} \geq \bar{u}'$. Note that under G,

$$\sum_{i:x_i=1} k_i \geq \sum_{j:x_j=-1} k_j,$$

[5] $\mathbb{R}^n_{++} = \{\mathbf{x} \in \mathbb{R}^n | x_i > 0 \text{ for all } i\}$ and $\mathbb{R}^n_{+} = \{\mathbf{x} \in \mathbb{R}^n | x_i \geq 0 \text{ for all } i\}$.

[6] $\mathbf{a} \odot \mathbf{b}$ denotes the element-wise multiplication of $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{a} \odot \mathbf{b} = (a_1b_1, \ldots, a_nb_n)$.

which is equivalent to $\sum_{i=1}^{n} k_i x_i \geq 0$. Therefore,

$$\bar{u} - \bar{u}' = \frac{2}{n} \sum_{i=1}^{n} k_i x_i \geq 0.$$

Because $I_r = I_a/\bar{u}$, it easily follows that $I_r(\mathbf{u}) \leq I_r(\mathbf{u}')$ for every $S \subseteq N$, which concludes the proof. □

The proof of Proposition 2 relies on two facts. First, if an absolute index is R, then the inequality depends on the voting profile rather than the voting rule. Second, the relative index is obtained through $I_r = I_a/\bar{u}$. Thus, minimizing the outcome inequality is reduced to maximizing the sum of the utilities. In other words, because absolute inequality does not depend on which alternative is chosen, the rule should then be utilitarian to minimize the relative inequality. Several previous studies have already demonstrated that $G = [0.5; k_1, k_2, \ldots, k_n]$ is optimal [4, 12]. Azrieli and Kim [2] also proved a similar conclusion among all incentive compatible decision rules. Thus, Bruner's argument is akin to that of utilitarianism, but the latter does not require UB.

3.2 Limitations

From the previous subsection, it appears that the weighted majority rule with weights proportional to sensitivities would best satisfy the egalitarian ideal. This subsection shows the claim is flawed in several ways.

A first and the most apparent objection is that the conclusion may not hold for other inequality indices. If the generalized entropy index [8] is adopted instead, then a counterexample is as follows. The reason is that when $p < 0$, the generalized entropy is more sensitive to the disadvantaged, unlike the reflective indices. Therefore, when the alternative with lower $\bar{u}$ is selected, the gap between the worse off and the average is narrowed, contributing to a reduction in inequality.

Example 1. *Suppose $u_i(1) = 2$ and $u_i(-1) = 1$ for all voters. The generalized entropy index*[7] *is used to measure inequality. If $p = -2$, then the majority rule always selects the alternative with higher inequality.*

Proof. Let n_1 denote the percentage of voters who vote for 'yes', without loss of generality, $n_1 \geq 0.5$. Let $\mathbf{u}$ and $\mathbf{u}'$ denote the utility vectors when the proposal is accepted and rejected, respectively. Then $f(\mathbf{u}) - f(\mathbf{u}') = -0.125 n_1 (2n_1^2 - 3n_1 + 1) \geq 0$, which concludes the proof. □

Even if the above only applies to a small portion of indices, there is a second objection that such a rule is inconsistent with equality of opportunity. Suppose UB holds, the expected payoff is

$$k_i(2\Omega_i - 1) + b, \tag{7}$$

where Ω_i denotes the probability for voter i to be successful. Equality of opportunity involves equalizing the expected payoff, which entails that Ω_i should decrease as k_i increases. Note that for monotonic games, Ω_i should be non-decreasing as the weight w_i increases. Hence, equalizing the expected payoff makes it possible for w_i to be negatively correlated with k_i,

[7] The generalized entropy index is given by,

$$f_0(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{\bar{u}}{u_i}; \quad f_1(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}}; \quad f_p(\mathbf{u}) = \frac{1}{np(p-1)} \sum_{i=1}^{n} \left(\frac{u_i^p}{\bar{u}^p} - 1 \right), \ p \neq 0, 1.$$

contradicting the outcome egalitarian rule. Only under EI can the majority rule ensure equality at both the opportunity and the outcome levels.

The approach to identifying an egalitarian rule is also limited. The above discussion has not taken into account the cases when UB does not hold. The following counterexample shows that in the absence of UB, there may not exist a simple game that can always choose the less unequal alternative. It is then not feasible to minimize the inequality for every voting profile.

Example 2. *Let $b = [2, 2, 2, 2, 10]$, $k = [1, 1, 1, 1, 10]$. The inequality is measured by the Gini index. If $S = \{2, 3, 4\}$, then if $S \in W$, $f(\mathbf{u}) = 0.32$ and if $S \notin W$, $f(\mathbf{u}) = 0.615$. It should be $S \in W$ to minimize the inequality. Similarly if $S' = \{2, 3, 4, 5\}$, then if $S' \in W$, $f(\mathbf{u}) = 0.507$ and if $S' \notin W$, $f(\mathbf{u}) = 0.4$. It implies $S' \notin W$, which contradicts monotonicity.*

4 A new approach

This section discusses an alternative approach to searching for egalitarian weighted voting games. The inequality is minimized in an average sense, instead of for every voting profile. Equality of opportunity and equality of outcome can also be well integrated. Besides, it can be applied to an arbitrary inequality index.

Before proceeding, I first discuss the trade-off between different voting profiles introduced by minimizing in an average sense. A general solution is to equip the voting profiles with a voting behavior [16], which fully specifies how likely a voting profile is to occur.

Definition 6. *A voting behavior is a function $p : 2^N \to [0, 1]$ that satisfies $\sum_{S \subseteq N} p(S) = 1$.*

The voting behavior is impartial (IP) if and only if $p_i = 0.5$ for all voters, where p_i denotes the probability for i to vote 'yes', i.e.,

$$p_i = \sum_{S:i \in S} p(S). \tag{8}$$

The voting behavior is independent (IND) if everyone votes independently from the remaining voters. Under independence,

$$p(S) = \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j). \tag{9}$$

If everyone is impartial and independent, then $\forall S \subseteq N$, $p(S) = 1/2^n$.

The expectation of a random variable y is given by

$$\mathbf{E}[y] = \sum_{S} p(S) y(S). \tag{10}$$

Under IND and IP, it can be shown that [10, 18]

$$\mathbf{E}[\mathbf{x}] = \boldsymbol{\beta}', \tag{11}$$

where $\boldsymbol{\beta}' = (\beta'_1, \ldots, \beta'_n)^\top$ is the distribution of Banzhaf power indices [3] [8]. Also, under

[8] The Banzhaf power index of voter i is given by

$$\beta'_i = \sum_{\substack{S:i \in S \in W \\ S \setminus i \notin W}} \frac{1}{2^{n-1}} \left(v(S) - v(S - \{i\}) \right),$$

which can be interpreted as the probability for a voter to be decisive. A voter is decisive if she casts the pivotal vote, i.e.,

$$(i \in S \in W \text{ and } S \setminus i \notin W) \text{ or } (i \notin S \notin W \text{ and } S \cup i \in W).$$

IND and IP,

$$\mathbf{E}\left[\mathbf{x}\mathbf{x}^\top\right] = \mathbf{I}, \tag{12}$$

where $\mathbf{I}$ is the identity matrix, because for $i \neq j$, $x_i x_j = 1$ if they vote for the same alternative, and $x_i x_j = -1$ if they vote oppositely.

With the above definitions, two functionals that are closely related to two notions of equality are introduced below. The major difference lies in when the expectation is taken [19].

Definition 7. *Let f be an inequality index and $\mathbf{u} = (u_1, \ldots, u_n)^\top$ be a random vector. The ex-ante inequality is defined as $f(\mathbf{E}[\mathbf{u}])$ and the ex-post inequality is defined as $\mathbf{E}[f(\mathbf{u})]$.*

The ex-ante inequality measures the inequality in expected utilities and is related to equality of opportunity. Equality of opportunity is achieved if $\mathbf{E}[\mathbf{u}] = c\mathbf{1}$, or equivalently, $f(\mathbf{E}[\mathbf{u}]) = 0$. In this voting scenario, equality of opportunity is primarily concerned with whether the expected payoff from the vote is the same for everyone. Alternatively, if $\mathbf{u}$ measures the current welfare, then equality of opportunity probes if a voting rule can compensate for the pre-existing inequalities to ensure everyone has the same expected welfare level after this vote. Equality of opportunity may or may not be possible, subject to assumptions on voter satisfaction. For instance, if under IND, IP, and UB, then equality of opportunity is achieved if $\beta'_i \propto 1/k_i$ following Eq. (7) and (11). However, if b_i varies significantly across voters, then equality of opportunity is unattainable as suggested in Proposition 3. A viable alternate approach is to minimize $f(\mathbf{E}[\mathbf{u}])$.

Proposition 3. *If $\mathbf{Var}[b] \geq k_{\max}^2/n > 0$, where $k_{\max} = \max\{k_1, \ldots, k_n\}$, then $\forall G \in \mathcal{G}$, $f(\mathbf{E}[\mathbf{u}]) > 0$ under IND and IP.*

Proof. Suppose $f(\mathbf{E}[\mathbf{u}]) = 0$, then there exists $m > 0$ such that $k_i\beta'_i + b_i = m$ for all i. This leads to

$$\sum_{i=1}^{n}(m - b_i)^2 = \sum_{i=1}^{n}(k_i\beta'_i)^2. \tag{13}$$

For the right-hand side, it has been demonstrated that the Banzhaf power index is l_2-bounded [14], i.e.,

$$\sum_{i=1}^{n}(\beta'_i)^2 \leq 1. \ \forall G \in \mathcal{G},$$

Then let $c = \sum_{i=1}^{n}(k_i\beta'_i)^2$, we have $c \leq k_{\max}^2$. Note that

$$nc - n\sum_{i=1}^{n} b_i^2 + \left(\sum_{i=1}^{n} b_i\right)^2 = nc - n^2\mathbf{Var}[b] \leq 0,$$

the only possible solution for Eq. (13) is $m = \bar{b}$. However, because for all i, $k_i\beta'_i \geq 0$, it entails $b_i = m$ for all i, which contradicts. Therefore $f(\mathbf{E}[\mathbf{u}]) > 0$. □

The ex-post inequality measures the expected inequality of all possible voting profiles and is related to equality of outcome. Equality of outcome is reached if $\mathbf{E}[f(\mathbf{u})] = 0$. This would require $\mathbf{u} = c\mathbf{1}$ for all voting profiles with non-zero probability, which also entails equality of opportunity. In this voting situation, equality of outcome requires the payoffs from the vote are always the same for everyone. Evidently, equality of outcome is hardly attainable as there are at most two voting profiles that can satisfy $\mathbf{u} = c\mathbf{1}$ simultaneously. Similarly, the ideal of equality of outcome requires minimizing $\mathbf{E}[f(\mathbf{u})]$ instead, where minimizing $f(\mathbf{u})$ for every voting profile is a special circumstance.

The ex-ante and ex-post inequality can be further generalized [6]. The ex-ante inequality is an instance of the functional $I = I_n \circ I_p$, i.e.,

$$I(\mathbf{u}) = I_n(I_p(u_1), \ldots, I_p(u_n)), \tag{14}$$

where I_n operates on voters and I_p operates on voting profiles. Similarly, the ex-post inequality is an instance of the functional $I = I_p \circ I_n$, or

$$I(\mathbf{u}) = I_p(I_n(\mathbf{u}(S_1)), \ldots, I_n(\mathbf{u}(S_{2^n}))). \tag{15}$$

For weighted voting games, the utility vector $\mathbf{u}$ depends on the weights $\mathbf{w}$ and the quota q. Therefore, designing an egalitarian weighted voting game can be translated into the following problem in general,

$$\underset{\mathbf{w}\in\mathbb{R}^n_+,\ q\in[0,1]}{\text{minimize}}\ I(\mathbf{u}(\mathbf{w}, q)), \tag{16}$$

where I could be $\mathbf{E} \circ f$, $f \circ \mathbf{E}$, or functionals that balance equality of opportunity and equality of outcome [9], for instance, $\alpha\mathbf{E} \circ f + (1 - \alpha) f \circ \mathbf{E}$.

It is worth mentioning that the approach can have extensive applications other than designing egalitarian weighted voting games. In fact, f can be an arbitrary social welfare function. For instance, let f be the utilitarian social welfare function and $I = \mathbf{E} \circ f$ or $I = f \circ \mathbf{E}$, i.e., $I(\mathbf{u}) = \sum_{i=1}^n \mathbf{E}[u_i]$, a utilitarian voting rule can be designed via maximizing I. Following Eq. (11), the inverse Banzhaf problem can also be solved via minimizing $f(\mathbf{E}[\mathbf{x}], \boldsymbol{\beta}'_0)$, where $\boldsymbol{\beta}'_0$ is the target distribution of power indices and $f(\mathbf{y}, \mathbf{z})$ measures the difference between $\mathbf{y}$ and $\mathbf{z}$, e.g., $f(\mathbf{y}, \mathbf{z}) = \sum_{i=1}^n (y_i - z_i)^2$.

5 An algorithmic solution

5.1 Algorithms

This subsection presents heuristic algorithms to solve the optimization problem. As the inverse Banzhaf problem is a special instance, an exact but polynomial-time algorithm may not be viable. Difficulty first arises in that the evaluation of $I(\mathbf{u}(\mathbf{w}, q))$ takes $O(n2^n)$. Besides, the parameter space also grows exponentially with n.

To evaluate $I(\mathbf{u}(\mathbf{w}, q))$, a computationally feasible approach is to randomly sample several voting profiles for approximation. With m samples, the evaluation of $I(\mathbf{u}(\mathbf{w}, q))$ takes $O(nm)$. With a huge parameter space, the optimal parameter can be approximated iteratively with a hill climbing approach, analogous to the inverse Banzhaf problem. Algorithm 1 is presented below. It is primarily based on stochastic gradient descent.

A few technical issues are discussed. The gradient can be approximated with a forward finite difference, i.e.,

$$\nabla_{w_i} I(\mathbf{w}, q) = \frac{I(w_1, \ldots, w_i + h, \ldots, w_n, q) - I(w_1, \ldots, w_i, \ldots, w_n, q)}{h}, \tag{17}$$

where $h > 0$ is the step size. Errors in gradient calculation could arise from both finite difference approximation and random sampling. To mitigate the random errors from sampling, the voting profiles are only sampled once at each iteration. The voting weights $\mathbf{w}$ and the quota q are updated separately, where $\mathbf{w}$ is updated with stochastic gradient descent and q is searched globally. The main reason is that searching for q globally is more accurate. Also, searching for q globally is not computationally expensive. Gradient calculation with respect

[9] For axiomatic characterizations of these functionals, see [6, 13, 20].

Algorithm 1: A general algorithm searching for optimal weights and quota to minimize $I(\mathbf{u}(\mathbf{w}, q))$.

Input: Initial weights $\mathbf{w}_0$, initial quota q_0, maximum iteration K, step sizes α_k at each iteration.

Output: Approximate optimal weights $\mathbf{w}_K$, approximate optimal quota q_K.

for $k = 0, 1, 2, \dots, K-1$ **do**

$\mathbf{g}_k \leftarrow \nabla_{\mathbf{w}} I(\mathbf{w}_k, q_k)$

$\mathbf{w}_{k+1} \leftarrow \mathbf{w}_k - \alpha_k \mathbf{g}_k$

$q_{k+1} \leftarrow \underset{q \in [0,1]}{\text{minimize}}\ I(\mathbf{w}_{k+1}, q)$

end

to $\mathbf{w}$ takes $O(n^2 m)$ and searching for q takes $O(nm/\Delta q)$ globally, where Δq is the search interval. The initial step size can be trialed by a backtrack line search. The subsequent step sizes decay according to an exponential law.

Under IND and IP, some functionals can be simplified as $I = I(\beta'_1, \dots, \beta'_n)$, e.g., I is the ex-ante inequality. As the Banzhaf power index of integer weights can be computed in $O(nQ)$ via dynamic programming [21], where $Q = q \sum_{i=1}^n w_i$, I can also be computed efficiently. Based on this, a more efficient Algorithm 2 is designed as below.

Algorithm 2: An algorithm searching for optimal weights and quota to minimize $I(\mathbf{u}(\mathbf{w}, q)) = I(\boldsymbol{\beta}'(\mathbf{w}, q))$.

Input: Initial weights $\mathbf{w}_0$, initial quota $q_0 = 0.5$, maximum iteration K, threshold for rounding error ε.

Output: Approximate optimal weights $\mathbf{w}_K$, approximate optimal quota q_K.

for $k = 0, 1, 2, \dots, K-1$ **do**

$\mathbf{g}_k \leftarrow \nabla_{\mathbf{w}} I(\mathbf{w}_k, q_k)$

$\alpha_k \leftarrow \underset{\alpha \geq 0}{\text{minimize}}\ I(\boldsymbol{\beta}')$, where $\boldsymbol{\beta}' = \boldsymbol{\beta}'(R(\mathbf{w_k} - \alpha \mathbf{g_k}, \varepsilon), q_k)$ with R denoting the rounding operation.

$\mathbf{w}_{k+1} \leftarrow R(\mathbf{w_k} - \alpha_{\mathbf{k}} \mathbf{g_k}, \varepsilon)$

$q_{k+1} \leftarrow \underset{q \in [0.5,1]}{\text{minimize}}\ I(\mathbf{w}_{k+1}, q)$

end

There are three major differences from Algorithm 1. The first is that input weights $\mathbf{w}$ for I should be rounded off to integers. The rounding error is given by

$$r(\mathbf{w}) = \frac{\sum_{i=1}^n (w_i - R_0(w_i))^2}{\sum_{i=1}^n w_i^2}, \tag{18}$$

where R_0 rounds a number to the nearest integer. Note that naive rounding could introduce large errors. To reduce the error, the weights can be multiplied by a coefficient $\lambda > 1$ before rounding as two weighted voting games $G' = [q; \lambda \mathbf{w}]$ and $G = [q; \mathbf{w}]$ are equivalent. A proper λ is selected when $r(\lambda \mathbf{w}) \leq \varepsilon$, the threshold for rounding error.

The second difference is that algorithm 2 uses steepest descent instead to update $\mathbf{w}$. The optimal step size at each iteration α_k can be determined via a backtrack line search algorithm. The last is that in Algorithm 2, the parameter space for q is $[0.5, 1]$ instead of $[0, 1]$. The reason is that a weighted voting game and its dual game[10] can have identical

[10]The dual of $G = (N, v)$ is defined as $G' = (N, v')$, where $v'(S) = v(N) - v(N - S)$. For weighted voting games, $G' = [1 - q; \mathbf{w}]$.

inequalities as Proposition 4 suggests, as it is not hard to check the expectation operator $\mathbf{E}$ is symmetric and $\forall S \subseteq N$, $p(S) = p(N - S) = 1/2^n$.

Proposition 4 can be related to some existing results. For instance, let $I = \mathbf{E} \circ P_i$,[11] then it entails $\mathbf{E}[u_i] = \mathbf{E}[u'_i]$, Proposition 1 in Beisbart and Bovens [5]. If further under EI, IND, and IP, then $\boldsymbol{\beta}'(G) = \boldsymbol{\beta}'(G')$, Theorem 5 in Dubey and Shapley [10].

Proposition 4. *Let $G = [q; \mathbf{w}]$ and $G' = [1 - q; \mathbf{w}]$. Suppose I_p is symmetric, i.e., $I_p(\pi u; \pi p) = I_p(u; p)$ and $\forall S \subseteq N$, $p(N - S) = p(S)$. Let $\mathbf{u}$ and $\mathbf{u}'$ denote the utility vectors under G and G', respectively. Then $I(\mathbf{u}) = I(\mathbf{u}')$ if $I = I_p \circ I_n$ or $I = I_n \circ I_p$.*

Proof. Let $\mathbf{y}$ denote the vote vector, where $y_i = 1$ if i vote 'yes' and $y_i = -1$ if i vote 'no'. Then for all S, $\mathbf{y}(S) = -\mathbf{y}(N - S)$.

Suppose $v(S) = 1$. Then $\sum_{i \in S} w_i \geq q \sum_{i=1}^n w_i$ and $\sum_{i \in N-S} w_i < (1 - q) \sum_{i=1}^n w_i$. Therefore, $v'(N - S) = 0$. Similarly if $v(S) = 0$, then $v'(N - S) = 1$. To sum, for all S, $v(S) = 1 - v'(N - S)$.

Let $\mathbf{x}$ and $\mathbf{x}'$ denote the success vectors under G and G'. For every voting profile S, we have $\mathbf{x}(S) = (2v(S) - 1)\mathbf{y}(S) = (1 - 2v'(N - S))(-\mathbf{y}(N - S)) = \mathbf{x}'(N - S)$. This also leads to $\mathbf{u}(S) = \mathbf{u}'(N - S)$ for all S.

I first show that $I(\mathbf{u}) = I(\mathbf{u}')$ for $I = I_n \circ I_p$. Let $\pi(S) = N - S$, then for all i,

$$\begin{aligned} I_p(u_i; p) &= I_p(\pi u_i; \pi p) = I(u_i(\pi(S_1)), \ldots, u_i(\pi(S_{2^n})); p(\pi(S_1)), \ldots, p(\pi(S_{2^n}))) \\ &= I_p(u_i(N - S_1), \ldots, u_i(N - S_{2^n}); p(N - S_1), \ldots, p(N - S_{2^n})) \\ &= I_p(u'_i(S_1), \ldots, u'_i(S_{2^n}); p(S_1), \ldots, p(S_{2^n})) = I_p(u'_i; p). \end{aligned} \tag{19}$$

Consequently, $I(\mathbf{u}) = I_n(I_p(u_1), \ldots, I_p(u_n)) = I_n(I_p(u'_1), \ldots, I_p(u'_n)) = I(\mathbf{u}')$.

For $I = I_p \circ I_n$, first for all S, $z(S) = I_n(\mathbf{u}(S)) = I_n(\mathbf{u}'(N - S)) = z'(N - S)$. In light of Eq. (19), $I(\mathbf{u}) = I_p(z) = I_p(z') = I(\mathbf{u}')$ immediately. □

5.2 Application to synthetic models

This subsection uses synthetic models to test the validity of the algorithms. The approximate optimal rules agree well with the optimal one in theory.

A simple model where $k_i = 0.1i + 0.9$ and $b_i = 0.2i + 9.8$ $(1 \leq i \leq 30)$ is first employed to test the validity of the algorithms. Voters are independent and impartial. The initial weights are uniform. The approximate optimal weights are displayed in Figure 1(a). The optimal quota is 0.5.

Two phenomena are noteworthy. First, for $1 \leq i \leq 15$, the weights can be fitted nicely with a quadratic curve. Second, the rest voters are dummy players. As the sensitivities and the average satisfaction are distributed linearly, it may be hypothesized that the optimal weights may either be $b_i(\bar{k} - k_i)$ or $k_i(\bar{b} - b_i)$, with negative weights being 0. In fact, only $k_i(\bar{b} - b_i)$ is possible because if UB holds, then w_i should not rely on k_i as Proposition 2 suggests.

Proposition 5. *Let f be the variance. Under IND and IP, the weighted majority rule with $w_i = \max\{k_i(\bar{b} - b_i), 0\}$ minimizes $\mathbf{E}[f(\mathbf{u})]$.*

[11] P_i denotes the projection, i.e., $P_i(\mathbf{x}) = x_i$.

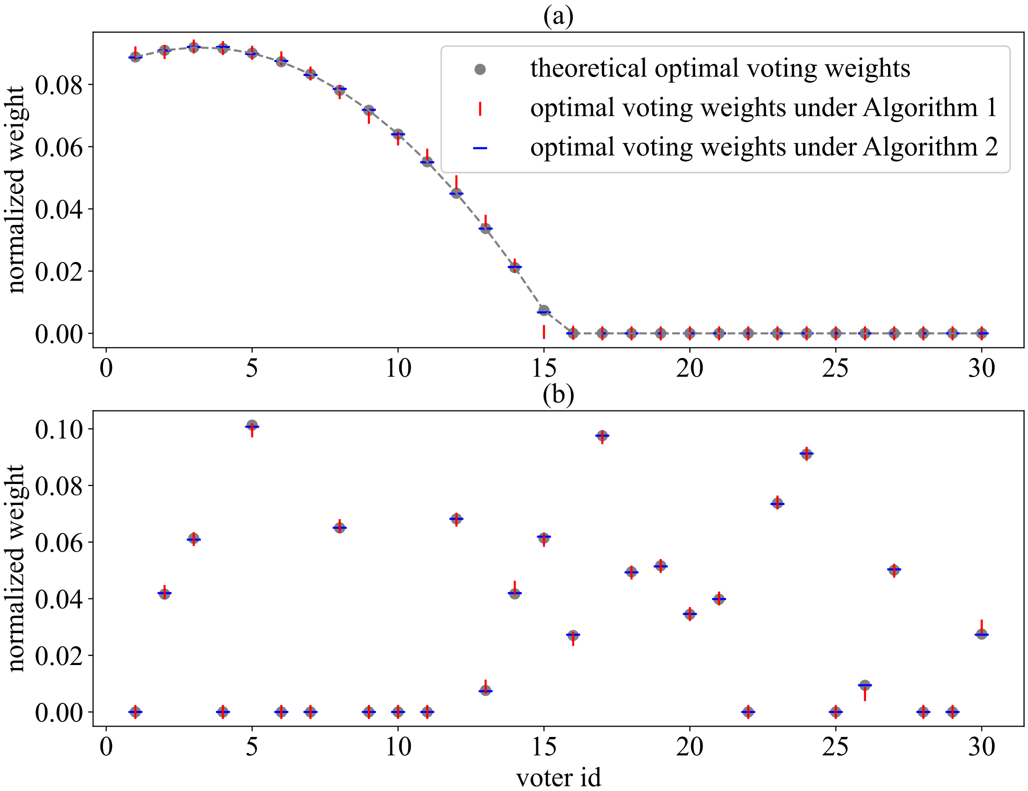

Figure 1: (a) Approximate optimal voting weights and expected voting weights for the linear model. (b) Approximate optimal voting weights and expected voting weights for the stochastic model. The weights are normalized by the sum.

Proof. From Eq. (11) and (12), $\mathbf{E}[f(\mathbf{u})]$ can be simplified as,

$$
\begin{aligned}
\mathbf{E}[f(\mathbf{u})] = \mathbf{E}\left[\frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u})^2\right] &= \frac{1}{n}\mathbf{E}\left[\sum_{i=1}^{n}\left(k_i x_i + b_i - \overline{kx} - \bar{b}\right)^2\right] \\
&= \frac{1}{n}\mathbf{E}\left[\sum_{i=1}^{n}\left(k_i x_i - \overline{kx}\right)^2 + 2\sum_{i=1}^{n}\left(k_i x_i - \overline{kx}\right)\left(b_i - \bar{b}\right) + \sum_{i=1}^{n}\left(b_i - \bar{b}\right)^2\right] \\
&= \frac{1}{n}\left[\left(1 - \frac{1}{n}\right)\sum_{i=1}^{n}k_i^2 + 2\sum_{i=1}^{n}\left(k_i\beta_i' - \overline{k\beta'}\right)\left(b_i - \bar{b}\right) + \sum_{i=1}^{n}\left(b_i - \bar{b}\right)^2\right] \\
&= \frac{1}{n}\left[2\sum_{i=1}^{n}k_i\left(b_i - \bar{b}\right)\beta_i' + \left(1 - \frac{1}{n}\right)\sum_{i=1}^{n}k_i^2 + \sum_{i=1}^{n}\left(b_i - \bar{b}\right)^2\right].
\end{aligned}
$$

Minimizing $\mathbf{E}[f(\mathbf{u})]$ is then equivalent to maximizing $\sum_{i=1}^{n} k_i(\bar{b} - b_i)\beta_i'$. Therefore, the optimal weight $w_i = \max\{k_i(\bar{b} - b_i), 0\}$ as a consequence of Lemma 6. □

Lemma 6. *Let $f(\beta_1', \ldots, \beta_n') = \sum_{i=1}^{s} z_i\beta_i' - \sum_{i=s+1}^{n} z_i\beta_i'$, where $z_i \geq 0$ for all $i \in N$, then $G = [0.5; z_1, \ldots, z_k, 0, \ldots, 0]$ maximizes f.*

Proof. First $G = [0.5; z_1, \ldots, z_k, 0, \ldots, 0]$ minimize $\sum_{i=s+1}^{n} z_i\beta_i'$ as $\beta_i' \geq 0$.

I now show that it also maximizes $\sum_{i=1}^{s} z_i\beta_i'$. Note that $\sum_{i=1}^{s} z_i\beta_i' = \mathbf{E}[\sum_{i=1}^{s} z_i x_i]$ under IND and IP. Similar to Proposition 2, G maximizes $\sum_{i=1}^{s} z_i x_i$ for every profile S. Hence G maximizes $\mathbf{E}[\sum_{i=1}^{s} z_i x_i]$, which concludes the proof. $\square$

Theoretical analysis from Proposition 5 validates the hypothesis. The theoretical optimal is also plotted in Figure 1(a). The optimal voting weights under algorithm 2 fit perfectly with the theoretical optimal. The optimal voting weights under algorithm 1 in general fit well, but the weight for voter $i = 15$ has some errors. A possible reason is that I is less sensitive to w_{15}. Figure 1(b) shows that the algorithms also work well on a stochastic model where utilities and sensitivities are generated stochastically.

The main difference between the optimal rule in Proposition 5 and Proposition 2 is that the former takes average satisfaction into account. The optimal weights are negatively correlated with average satisfaction, while still proportional to sensitivities. It implies that to better ensures outcome equality, those who are likely to benefit more from the proposal should have less power. If $\mathbf{u}$ measures current welfare, it then entails the disadvantaged should be compensated with additional voting weights. A controversial point is then the advantaged would get zero weight. However, such controversy might be mitigated by social welfare functions that combine utilitarianism and egalitarianism.

6 Conclusion

This paper analyses egalitarian weighted voting games from both theoretical and algorithmic aspects. An extension of Bruner's work suggests that the weighted majority rule with weights proportional to sensitivities would always minimize outcome inequality under two conditions. First, inequality is measured by a relative index whose absolute form is reflective, and second, the arithmetic mean of the utility of winning and the utility of losing is the same for all. However, it is not uncommon for one of these conditions to be unmet.

Hence, a more general approach is adopted. Designing an egalitarian weighted voting game is then transformed into an optimization problem. A rule that minimizes ex-ante inequality best satisfies the ideal of equality of opportunity and the ideal of equality of outcome is best served by a rule that minimizes ex-post inequality. Two efficient heuristic algorithms are therefore proposed. The first is a general algorithm based on stochastic gradient descent. The second applies to cases when the objective function has a simplified form. A synthetic experiment suggests both algorithms can approach the theoretically optimal weights, and the second performs better.

I conclude with a few unanswered questions. First, the theoretical results and algorithms can be further applied to practical voting scenarios. But a practical difficulty is how to quantify the satisfaction of voters. The sensitivity can be determined by the square root law in a two-tier voting system, whereas the average satisfaction remains unknown. Second, the theoretical guarantees regarding convergence rate and errors remain unexplored. There also remains a question of whether the algorithm can be further improved. Diakonikolas et al. [9] show a variant of stochastic gradient descent algorithm performs better for the inverse power-index problem. It is worth exploring whether it works in this case as well.

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