Multi-Dimensional Social Choice under Frugal Information: The Tukey Median as Condorcet Winner *Ex Ante*

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This is an abbreviated version of our working paper [24]. All proofs, detailed discussion as well as additional examples and results can be found there.

Abstract

We study a voting model with partial information in which the evaluation of social welfare must be based on information about agents' top choices plus qualitative background conditions on preferences. The former is elicited individually, while the latter is not. The social evaluator is modeled as an imprecise Bayesian characterized by a set of priors over voters' complete ordinal preference profiles. We apply this 'frugal aggregation' model to multi-dimensional budget allocation problems and propose a solution concept of 'ex-ante' Condorcet winners. We show that if the social evaluator has symmetrically ignorant beliefs over profiles of quadratic preferences, the ex-ante Condorcet winners refine the set of Tukey medians [29].

1 Introduction

Many economic and political decisions involve the allocation of resources under a budget constraint. Examples are the allocation of public goods, the redistribution across classes of beneficiaries, the allocation of tax burden, the choice of intertemporal expenditure streams, or the macro-allocation between expenditure, tax receipts and net debt. Here we explore the possibility of taking these decisions collectively by voting. This will be done in a somewhat more general and abstract multi-dimensional setting in which alternatives are elements of a convex subset of a Euclidean space and preferences are convex. The budget allocation problem is the special case in which the set of alternatives is a budget hyperplane.

Standard approaches to preference aggregation and voting assume ordinal or even cardinal preference information as their input. Their application to budget allocation problems poses substantial difficulties. First, at the foundational level, except for the one-dimensional case with two public goods and single-peaked preferences [5, 1], one is faced with generic impossibility results under all reasonable domain restrictions [13, 17] just as in spatial voting models [25]. In particular, in higher dimensions there is no hope to generally find a Condorcet winner even if all agents have well-behaved preferences. Indeed, the indeterminacy of majority voting is generic and can be severe; for example, generically every alternative can be the outcome of a dynamic (non-strategic) majority vote for an appropriate agenda [20]. Thus, from the point of view of ordinal social choice theory, it is not even conceptually clear what allocations an optimal voting rule should aim at.

Second, at a pragmatic level, a basic problem already arises from the sheer number of alternatives which grows exponentially in the number of dimensions (i.e. alternative uses of the public resource). Articulating and communicating a complete ordering over the set of all alternatives for each agent (whether citizen or representative) is often simply infeasible. Clearly, much is to be said for making the task of the voter as easy as possible. Here, we take a minimalist approach by assuming that only voters' preference tops are individually elicited. As a collateral benefit, as detailed below, the parsimony of the informational basis entailed by our approach allows one to overcome the foundational indeterminacies of the classic preference aggregation model.

Tops-Only Information

We focus on the tops-only information assumption firstly because of its simplicity and prevalence in practice. Note that knowledge of voters' tops is indeed required to arrive at a reasonable decision in the most elementary instances of aggregation, namely those of unanimity.

Elicitation of tops is simple in that it requires every voter only to determine what she must in order to arrive at a choice all by herself. Reliance on tops-only elicitation thus addresses a fundamental tension in the standard ordinal aggregation framework, as the elicitation of a complete ordinal ranking requires much more cognitive effort on part of the individuals than would be required for solo decision making, while individual incentives to figure out ones own preferences are greatly reduced due the diluted impact of a voter on the final choice.[1]

The Social Evaluator as an Imprecise Bayesian

This paper aims at determining which of the feasible social choices (here: allocations) represent social welfare optima in the light of the available information. Due to the lack of knowledge of the profile of complete preferences underlying an elicited profile of tops, the social evaluator faces a decision problem under uncertainty. Besides the individually elicited tops, the social evaluation may also be based on background knowledge about the structure of voters preferences, such as preference convexity. Probabilistic judgments may play a role as well. So the social evaluator will be modeled as an 'imprecise Bayesian' whose epistemic state is described as a set of admissible probability measures ('priors') over profiles of ordinal preferences compatible with a given profile of top choices and the available background information.

Within this framework, one might want to postulate the evaluator to have precise probabilistic beliefs (i.e. unique priors). But this approach has limited appeal here. In particular, on what

[1] This theme of 'rational ignorance' goes back to Down's classic treatment [10, pp. 244-246, 266-271].

evidential basis is the evaluator to make the manifold subjective judgments required for a precise Bayesian approach? Are there any sound reference models to sensibly describe ignorance priors over a state space of profiles of ordinal preferences on a continuous, multi-dimensional domain? Indeed, *whose* subjective probability is supposed to serve as the basis of the evaluation? If the social evaluator was understood as a social planner ('bureaucrat'), one may think of the required judgmental input as reflecting the planner's expertise; but in a voting context, the social evaluator is naturally viewed as representing 'the group' at a constitutional stage at which individual preference profiles are unknown.

Instead of assuming a precise Bayesian prior, alluding to the notion of 'fast and frugal heuristics' due to [11], in our 'frugal' approach we rely on a qualitative specification of the social evaluator's beliefs reflecting minimally demanding informational assumptions. Our aim is to show that even from these minimalist premises, attractive and credible choice implications can be derived.

The Ex-Ante Condorcet Approach

To determine 'ex-ante' optimal social choices, we propose a novel ex-ante Condorcet (EAC) approach. The EAC approach relies on ex-ante comparisons between arbitrary pairs of alternatives. These comparisons are based on the interval of expected majority counts consistent with the evaluator's imprecise set of priors. A simple yet fundamental observation shows that the pairwise comparisons can be made in canonical manner independently of subjective attitudes of pessimism vs. optimism, or ambiguity aversion vs. ambiguity proneness. The EAC approach then uses this ex-ante majority relation to select an ex-ante Condorcet winner if it exists, and settles for some Condorcet extension rule – left unspecified here – if not. Remarkably, in the models at the center of this paper, ex-ante Condorcet winners do exist and can be characterized explicitly.

The Plain Convex Model

An obvious starting point in the context of public resource allocation is to assume knowledge of preference convexity (together with knowledge of the tops), and complete ignorance about anything else. We shall refer to this as the *plain convex model* of the evaluator's beliefs. The plain convex model is very successful in the one-dimensional (two goods) case in which convexity is tantamount to single-peakedness of ex-post preferences. As the ex-post Condorcet winner is the median of voters' tops, it is known ex-ante and equal to the ex-ante Condorcet winner.[2]

But in the multi-dimensional case (at least three competing uses of resources), convexity by itself loses much of its bite. In particular, with tops in general position, convexity does not permit any significant novel inferences about preferences beyond those available from knowledge the tops; by consequence, all tops are ex-ante Condorcet winners (Proposition 3). This appears quite counterintuitive and unsatisfactory, since any notion of centrality of the ex-ante Condorcet winner is lost, in stark contrast to the one-dimensional case.

Looking more closely, this negative result indeed hinges on extreme cases involving special ex-post profiles which appear unlikely a priori. Heuristically, one would want to rule out such cases and obtain more plausible majority intervals by assuming that preferences over pairs depend on the preference tops in a regular manner.

Symmetric Quadratic Models

To execute this formally, we assume a parametric form of convex preferences, namely quadratic preferences. A particular quadratic form $\mathcal{Q}$ describes the substitution-complementation structure of a quadratic preference ordering in terms of the cross-partials of the utility function. Notably, assuming quadraticity does not help by itself to overcome the counterintuitive implications of

[2] We use the ex-ante vs. ex-post metaphor purely for conceptual purposes in order to describe the epistemic state of the social evaluator, without any assumption of an ex-post stage in real time at which the actual profile of ('ex-post') preferences is observed.

the plain convex model, for the expected majority counts remain the same as the plain convex model (Fact 4.1).

Yet things change significantly once it is assumed that the evaluator's beliefs are symmetric in the sense that, for each admissible prior, the marginal distribution over quadratic forms is the same across voters irrespective of their top. Heuristically, symmetry expresses the idea that the evaluator lacks any grounds a priori to form different probabilistic beliefs about the unknown quadratic preference structure of different voters; in particular, the knowledge of voters' tops does not form such a ground. In addition, we also assume that the social evaluator is completely ignorant about the preference structure for each voter in isolation, just as in the plain convex model. These assumptions define the class of *symmetrically ignorant quadratic (s.i.q.) models* of the evaluator's beliefs. The main result of the paper, Theorem 1, shows that in any s.i.q. model ex-ante Condorcet winners exist and coincide, when unique, with the classical Tukey median [29]. When not unique, the EAC winners coincide with a well-defined refinement of the set of Tukey medians. The Tukey median is a well studied coordinate-free generalization of ordinary medians to multiple dimensions, see [28] for a classic survey and [26] for comprehensive treatment.

Related Literature

To the best of our knowledge, the present EAC approach and its application to the 'frugal aggregation' model of budget allocation are new to the literature. But there are, of course, related approaches in the literature. Indirectly, the Tukey median has been studied in the social choice literature inasmuch as it is equivalent to the outcome of the minimax voting rule in standard spatial voting with Euclidean preferences [15, 9, 7]. This model can be viewed as a degenerate frugal model in which voters preferences conditional on their top are known. But with this additional, sub-top preference information, the Tukey median is no longer welfare optimal as we shall argue in Section 6.2.

Most work of theoretical interest in the problem of incomplete information as studied here has come from the computer science literature, see [6] for an overview.[3] One strand explores the implications of partial knowledge of complete (ex-post) preference profiles for inferences about the outcome of standard social choice rules and criteria, e.g. via the notions of 'possible' vs. 'necessary' winners [14]. A rather small strand in the literature adopts a decision-theoretic ex-ante approach as the present paper does. Some papers seek solutions that maximize expected welfare based on some utilitarian welfare criterion and a probability distribution over profiles, frequently uniform. Others argue for the modeling of the social evaluator's epistemic state in terms of a set of possible profiles, as we do, and propose to apply classical criteria of decision making under ignorance such as maximin or minimax regret [18]. In the highly complex state spaces associated with the epistemic models studied here, it may be very difficult to execute these approaches if that is possible at all. Significantly, the two quoted strands share the major conceptual limitation of having to rely on an interprofile-comparable standard of aggregate welfare ex post. Thus, they in fact suppose that the Arrovian problems of coherent aggregation and interpersonal non-comparability have been solved or assumed away, e.g. by assuming strong forms of utilitarian aggregation ex post.

By contrast, the EAC approach introduced here rests on an evaluation of decisions in pairs of alternatives taking the full state space (set of possible profiles) into account. In such pairwise comparisons, the majority criterion carries over naturally to the ex-ante stage, without raising new issues of interpersonal comparison, and allowing a tractable characterization in many cases. These pairwise comparisons need then be put together to obtain a coherent rationale for an ex-ante evaluation of complex choices such as budget allocations. At this juncture, Arrovian style issues of coherent aggregation might arise in principle. It is a rather remarkable finding of this paper that, in the models studied here, these problems do not arise.

With respect to the focal application to the allocation of public budgets, there is also an important recent literature on 'participatory budgeting' with intended application to cities and local communities [27]. Participatory budgeting schemes have been put into practice at various scales in many places around the world. The ballots are typically very parsimonious, often

[3]We thank Jérôme Lang who pointed us to the pertinent literature.

taking the form of a set of projects approved.[4] Again, most of the theoretical contributions come from the computer science community, with a focus on indivisibilities and on 'proportionality' considerations to ensure that the interest of different local subcommunities are fairly represented [4]. By contrast, our focus is on continuous divisible budgets, and on finding allocations that best satisfy the aggregate interest (in parallel with most of standard voting theory).

2 Condorcet Winners, Ex-Ante

We envisage a social evaluator who has to choose from a universe of alternatives X on behalf of a group of $n \in \mathbb{N}$ voters under uncertainty about their preferences. The social evaluator is modeled as an 'imprecise' Bayesian decision maker, i.e. his epistemic state is described by a set of probability distributions over 'admissible' profiles $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ of true ('ex-post') preferences.

Concretely, denote by π a probability measure over profiles $(\succcurlyeq_1, ..., \succcurlyeq_n)$ of complete preference orderings over X, and by Π a non-empty set of admissible such priors.[5] The social evaluator is completely ignorant as to which probability distribution in Π is the most appropriate and therefore needs to take into account all of them.

Often one will be interested in cases in which the priors in Π satisfy specific additional properties. For instance, an important special case in the following will involve $X \subseteq \mathbb{R}^L$ and the assumption that all priors are concentrated over profiles of *convex* preferences.

For all distinct $x, y \in X$, a prior $\pi \in \Pi$ induces an expected support count $m_\pi(x, y)$ of votes for x against y, i.e.

$$m_\pi(x, y) \ := \ E_\pi \left[\#\{i : x \succ_i y\}\right],$$

where E_π denotes the expectation operator with respect to the probability distribution π. Thus, a set of priors induces an interval $m_\Pi(x, y)$ of expected support counts in the vote of x against y,

$$m_\Pi(x, y) \ := \ \left[m_\Pi^-(x, y)\, , \, m_\Pi^+(x, y)\right],$$

where

$$m_\Pi^-(x, y) \quad := \quad \inf_{\pi \in \Pi} m_\pi(x, y), \tag{2.1}$$

$$m_\Pi^+(x, y) \quad := \quad \sup_{\pi \in \Pi} m_\pi(x, y). \tag{2.2}$$

The family of these intervals will be what matters in our analysis. In deciding ex-ante on a hypothetical choice between x and y, it is natural to base this choice on a comparison of the intervals $m_\Pi(x, y)$ and $m_\Pi(y, x)$. Due to the imprecision of priors, the intervals $m_\Pi(x, y)$ and $m_\Pi(y, x)$ may well overlap in general. But due to the additivity of the complementary vote counts for x against y and for y against x, a comparison of the lower and upper expected counts must yield the same result. This evidently holds if preferences are known to be strict ex-post. To guarantee it more generally, the following regularity condition is needed which ensures that possible indifferences play a negligible role; this condition is satisfied in all applications considered in the following, and we maintain it throughout. Say that a set of priors Π is *regular* if for all priors $\pi \in \Pi$ and all pairs $x, y \in X$ of distinct alternatives, there exists a prior π' such that $\pi'(x \sim_i y) = 0$ for all $i = 1, ..., n$, and $m_{\pi'}(x, y) \leq m_\pi(x, y)$. Thus, regularity guarantees that, for any pair $x, y \in X$, the minimal/infimal expected support for x against y is realized by a prior for which all indifferences between x and y have zero probability.

Proposition 1. *Let Π be regular. For all θ and all distinct $x, y \in X$,*

$$m_\Pi^-(x, y) \geq m_\Pi^-(y, x) \iff m_\Pi^+(x, y) \geq m_\Pi^+(y, x). \tag{2.3}$$

[4]See, for instance, the open source project 'Stanford Participatory Budgeting Platform' (https://pbstanford.org) which offers guidance and allows municipalities, cities and other institutions to run participatory budgeting elections online.

[5]To make this fully rigorous, one needs to specify a measure space on the set of profiles. For our purposes, the essential property is that, for each agent i and all alternatives x and y, the 'event' that agent i prefers x to y represents a measurable set.

By Proposition 1, an *unambiguous* balance of uncertainties ex-ante is possible; in contrast to the classical theory of decision making under ignorance [19], there is no need or even meaningful role for an evaluator's degree of pessimism vs. optimism (ambiguity aversion vs. ambiguity proneness in more modern terminology).[6]

The **ex-ante majority relation** R_Π (for regular Π) is now defined as follows. For all distinct $x, y \in X$,

$$\begin{aligned} xR_\Pi y \quad &:\Longleftrightarrow \quad m^-_\Pi(x,y) \geq m^-_\Pi(y,x) \\ &\Longleftrightarrow \quad m^+_\Pi(x,y) \geq m^+_\Pi(y,x). \end{aligned} \tag{2.4}$$

The maximal elements with respect to the ex-ante majority relation are referred to as the **ex-ante Condorcet winners**, i.e.

$$\mathrm{CW}(\Pi) \;:=\; \{x \in X \mid xR_\Pi y \text{ for all } y \in X\}.$$

An aggregation rule is called **ex-ante Condorcet consistent** if it selects all ex-ante Condorcet winners (if there are any).

In the following, we will refer to a set of priors Π as a **model** (of the evaluator's epistemic state). Moreover, we will say that two models are *equivalent* if they induce the same expected majority intervals. Note that, trivially, sets of priors with the same convex hull are equivalent, but the converse need not be true. Evidently, two equivalent models induces the same set of ex-ante Condorcet winners, i.e. $\mathrm{CW}(\Pi') = \mathrm{CW}(\Pi)$ whenever Π' and Π are equivalent.

3 The Plain Convex Model

In the rest of this paper, we will study the case in which X is a convex subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$, and all preferences in any profile are convex. For our purposes, the following notion of convex preference will be useful. A weak order $\succcurlyeq$ on $X \subseteq \mathbb{R}^L$ is *convex* if, (i) for all $x, y, z, w \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 \leq t \leq 1$, $x \succcurlyeq w$ and $z \succcurlyeq w$ jointly imply $y \succcurlyeq w$, and (ii) for all $x, y, z \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 < t < 1$, and $x \succ z$ jointly imply $y \succ z$.[7]

An important economic application is the *budget allocation problem* in which X takes the form of a budget hyperplane. Concretely, consider a group of agents that has to collectively decide on how to allocate a fixed budget, normalized to unity, to a number L of public goods. Assuming given prices, the problem is fully determined by specifying the expenditure shares. The corresponding allocation problem can thus be modeled as the choice of an element of the following $(L-1)$-dimensional polytope:

$$X := \left\{ x \in \mathbb{R}^L \mid \sum_{\ell=1}^{L} x^\ell = 1 \text{ and } x^\ell \geq 0 \text{ for all } \ell = 1, ..., L \right\}, \tag{3.1}$$

where $x = (x^1, ..., x^L)$. Convex preferences are entirely standard in this context.

Other applications include the spatial voting model in which the coordinates represent different issues and alternatives represent political positions on these issues [10], or the collective choice of design of projects positioned in a characteristics space in the sense of [16].

The model of all priors with convex preferences on X without any further restriction is referred to as the **plain convex model** and denoted by Π_{co}.

3.1 Certainty about Tops

To simplify the task of the social evaluator, we assume first that the evaluator knows the top choices of voters. Concretely, denote by $\theta = (\theta_1, ..., \theta_n)$ the profile of the voters' top alternatives

[6] Nor is there a conflict – possibly even threatening an Arrow-like impossibility – between axioms of choice consistency and of independence; see [21, 2, 22, 23].

[7] Observe that (ii) is clearly implied by but significantly weaker than strict convexity. For instance, linear preferences satisfy both conditions (i) and (ii) but are not strictly convex.

which we assume to be unique. The epistemic state of the social evaluator will now be denoted by Π^θ_{co} to indicate the knowledge of θ. Here, the set Π^θ_{co} is assumed to consist only of priors π that are *compatible* with the top profile θ in the sense that every profile $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ in the support of π has $\theta = (\theta_1, ..., \theta_n)$ as the corresponding top profile.

3.2 The One-Dimensional Case: Median Voting

In the one-dimensional case, our notion of preference convexity is equivalent to the standard notion of single-peakedness, and the choice of the median top(s) constitutes the unique ex-ante Condorcet consistent aggregation rule; specifically, we have the following result. For every profile $\theta = (\theta_1, ..., \theta_n)$, denote by θ_{med} the unique median if n is odd, and by $[\theta_{\mathrm{med}^-}, \theta_{\mathrm{med}^+}]$ the median interval if the number of voters is even.

Proposition 2. *Suppose that $X \subseteq \mathbb{R}$, and let $\theta = (\theta_1, ..., \theta_n)$ be a profile of tops in X. Then,*

$$\mathrm{CW}(\Pi^\theta_{\mathrm{co}}) \;=\; \begin{cases} \{\theta_{\mathrm{med}}\} & \text{if } n \text{ is odd} \\ [\theta_{\mathrm{med}^-}, \theta_{\mathrm{med}^+}] & \text{if } n \text{ is even} \end{cases}.$$

Thus, in the one-dimensional case the ex-post and ex-ante Condorcet criterion give the same result under single-peakedness. The reason is, evidently, that under knowledge of single-peakedness any given top uniquely determines the preference on both sides of the top, and that is all what is needed to apply the Condorcet criterion.

3.3 The Multi-Dimensional Case: Generic Plurality Rule

In the multi-dimensional case, a result similar to Proposition 2 holds if the top profile is contained in a one-dimensional subspace; but in general, in the plain convex model the ex-ante Condorcet winners essentially coincide with the plurality winners.

In the following, we say that a set of points $Y \subseteq \mathbb{R}^L$ is *in general position* if no three elements of Y are collinear. The crucial observation for the plain convex model is that, if θ, x, y are not collinear, then there exist convex preferences $\succcurlyeq$ and $\succcurlyeq'$ with top θ such that $x \succ y$ and $y \succ' x$. This implies the following characterization of the ex-ante Condorcet winners in the plain convex model. For its formulation, it will be useful to identify profiles of individual tops with *type profiles* of tops with different counts. Specifically, we denote by $\theta = (\theta_1; p_1, ..., \theta_m; p_m)$ the anonymous profile in which the fraction p_i of all voters has top θ_i, where $0 < p_i \leq 1$ and $\sum_i p_i = 1$; in that context, we also refer to θ_i as the *type* of voter i and assume without of loss of generality that the θ_i are pairwise distinct.

Proposition 3. *Consider a type profile $(\theta_1; p_1, ..., \theta_m; p_m)$ such that $\{\theta_1, ..., \theta_m\} \subseteq X$ are in general position. If p_{i^*} is maximal among $\{p_1, ..., p_m\}$, then $\theta_{i^*} \in \mathrm{CW}(\Pi^\theta_{\mathrm{co}})$. Moreover, if p_{i^*} is uniquely maximal among $\{p_1, ..., p_m\}$, then*

$$\mathrm{CW}(\Pi^\theta_{\mathrm{co}}) \;=\; \{\theta_{i^*}\}.$$

This is somewhat paradoxical. Intuitively it would appear that preference convexity contains substantial information beyond knowledge of the tops but Proposition 3 appears to contradict this. What is amiss?

Example 1. *Consider a set of voters with pairwise distinct tops in a set U. In addition, suppose that two voters are concentrated at a point x outside U (see Figure 1). If all tops in U plus the point x are in general position then, according to Proposition 3, x is the unique ex-ante Condorcet winner. Indeed, for any point $z \neq x$, $m^-_{\Pi^\theta_{\mathrm{co}}}(x,z) = 2$ while $m^-_{\Pi^\theta_{\mathrm{co}}}(z,x) \leq 1$, or equivalently, $m^+_{\Pi^\theta_{\mathrm{co}}}(x,z) \geq n-1$. Note that the expected majority intervals are extremely wide, and the ex-ante Condorcet winner is left to 'grasp for straws' in picking the optimal alternative that happens to be the top of two voters rather than just of one. Nonetheless, if the epistemic state of the social evaluator is* literally *that of complete ignorance within Π^θ_{co}, then the ex-ante preference for x over any other alternative z seems defensible.*

However, this rationale is not very robust. Consider in particular the comparison of x to y where y is sufficiently close to x and 'between' x and U as shown in Fig. 1. Note that for x to be preferred to y by some voter with top θ_i in U, i's preference must be very special; for instance, geometrically, only rather special ellipses with center at θ_i that include x will not include y.

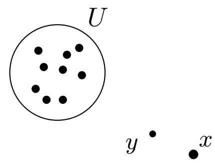

Figure 1: Illustration of Proposition 3

The conceivable convex preference for x against y of a voter with top in U, on which the conclusion in Example 1 hinges, seems very unlikely a priori. It would therefore be desirable to capture this intuition by an appropriate specification of somewhat more precise evaluator's beliefs. The challenge is to describe these beliefs in a qualitative manner that is weak enough to be acceptable on slim information while at the time sufficiently strong to have substantive implications. This task is at the heart of this paper and is taken on in the next section.

4 Symmetrically Ignorant Quadratic Models

Our proposal for modeling the evaluators beliefs in a more appropriate and specific manner involves two key features: First, we conceptually separate voters' tops from the substitution vs. complementation structure of preferences, and secondly, we assume that this substitution vs. complementation structure is 'ex-ante independent' of the tops. The first feature allows one to model preferences as quadratic; the second feature means that knowledge of the tops is not informative ex-ante for the substitution vs. complementation structure described by the quadratic preferences.

Specifically, say that a preference $\succcurlyeq$ on X is *quadratic* if it can be represented ordinally by a utility function of the form

$$u_{\theta_i}(x) \;=\; -(x-\theta_i)^T \cdot \mathcal{Q}_i \cdot (x-\theta_i), \tag{4.1}$$

for some $\theta_i \in X$ and a positive definite, symmetric $L \times L$ matrix $\mathcal{Q}_i$. Geometrically, the representation in (4.1) means that the indifference curves are ellipsoids generated from circles with center θ_i by a common affine transformation. The special case in which the quadratic form $\mathcal{Q}_i$ is the identity matrix $\mathcal{I}$ corresponds to the case of *Euclidean preferences* which has been extensively studied in the literature on spatial voting [3].

The cross-partial derivatives given by $\mathcal{Q}_i$ capture the specific pattern of complementarities and/or substitutabilities between different goods. Quadratic preferences can thus also be viewed as (second-order) Taylor approximations of arbitrary smooth convex preferences. Denote by $\Pi_{\text{quad}} \subseteq \Pi_{\text{co}}$ the model consisting of all sets of priors over profiles of quadratic preferences on X, the *plain quadratic model.* Evidently, for all tops $\theta_i \in X$ and all $x, y \in X$ such that θ_i, x, y are not collinear, there exist quadratic preferences $\succcurlyeq_i, \succcurlyeq_i'$ both with top θ_i such that $x \succ_i y$ and $y \succ_i' x$. By consequence, we have:

Fact 4.1. *The models Π_{quad} and Π_{co} are equivalent. In particular, the two models induce the same ex-ante majority relation and* $\text{CW}(\Pi_{\text{quad}}) = \text{CW}(\Pi_{\text{co}})$.

Thus, the plain quadratic model can be viewed as a parametrized version of the plain convex model. In particular the 'generic plurality' conundrum posed by Example 1 continues to apply to the plain quadratic model. But the great boon of the quadratic model is that it allows for a clear separation between the preference top and the preference *structure* (described by the quadratic from $\mathcal{Q}_i$). This will be the key in our proposed resolution of the puzzle posed by Example 1.

Specifically, the epistemic state of the evaluator is given by a set of priors Π with state space $X_1 \times ... \times X_n \times \boldsymbol{\mathcal{Q}}_1 \times ... \times \boldsymbol{\mathcal{Q}}_n$ where X_i is the set of possible tops for voter i and $\boldsymbol{\mathcal{Q}}_i$ the set of possible quadratic forms (symmetric and positive definite $L \times L$ matrices) for voter i. For every prior $\pi \in \Pi$ and all $i = 1, ..., n$, denote by π_{X_i} and $\pi_{\mathcal{Q}_1}$ the marginal distributions induced by π on X_i and $\boldsymbol{\mathcal{Q}}_i$, respectively.

In the remainder of this section, we will impose the following conditions on a model Π. For all $x \in X$, denote by $\delta(x)$ the degenerate probability distribution that puts unit mass on x; similarly, for all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}$, denote by $\delta(\mathcal{Q})$ the degenerate probability distribution that puts unit mass on $\mathcal{Q}$.

1. **Concentration on Quadratic Preferences.** $\Pi \subseteq \Pi_{\text{quad}}$.
2. **Tops Certainty.** For all $\pi \in \Pi$ and all i, $\pi_{X_i} = \delta(\theta_i)$ for some $\theta_i \in X$.
3. **Symmetry.** For all $\pi \in \Pi$ and all i, j, $\pi_{\mathcal{Q}_i} = \pi_{\mathcal{Q}_j}$.
4. **Complete Ignorance of Marginals.** For all i and all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}$, there exists $\pi \in \Pi$ such that $\pi_{\mathcal{Q}_i} = \delta(\mathcal{Q})$.

A model Π satisfying Assumptions 1 to 4 will be called **symmetrically ignorant quadratic**, or **s.i.q.** for short. Assumption 2 means that all voters' tops are known; therefore Assumptions 1 and 2 can be summarized as requiring $\Pi \subseteq \Pi^{\theta}_{\text{quad}}$ in our previous notation, where $\theta = (\theta_1, ..., \theta_n)$ is the known profile of voters tops. Assumption 3 means that an individual's top (or any other observable individual characteristic) does not contain any information on the distribution of the individual's preferences by itself. Finally, Assumption 4 assumes in effect complete ignorance about each agent's $\mathcal{Q}_i$.

The plain quadratic model satisfies all assumptions except Symmetry. The 'regularizing' effect of the Symmetry assumption can be illustrated in Example 1.

Example 1 (cont.) *Consider again the situation depicted in Fig. 1 above, but now suppose that the epistemic state of the social evaluator is described by a symmetric quadratic model* Π *rather than by the plain convex model. The minimal expected majority count for* x *against* y *is still* 2, *since it is evidently possible to find a symmetric prior such that all voters in* U *prefer* y *to* x, *i.e.* $m^-_{\Pi}(x, y) = 2$. *For example, one may take the prior that assumes with certainty that all preferences are Euclidean. What about* $m^-_{\Pi}(y, x)$? *As before, one can assign quadratic forms* $(\mathcal{Q}_1, ..., \mathcal{Q}_n)$ *to the tops such that all voters with top in* U *prefer* x *to* y. *But, as is evident from Fig. 1, these quadratic forms generally have to be distinct for different voters; the prior assuming this profile with certainty is therefore not symmetric. Any symmetric prior must thus be properly probabilistic; for example, a symmetric prior might assign equal probability* $1/n!$ *to each of the permutations of the profile* $(\mathcal{Q}_1, ..., \mathcal{Q}_n)$. *But for any such prior the expected majority count for* y *against* x *will be at least* 3, *i.e.* $m^-_{\Pi}(y, x) \geq 3$, *as a key argument in the proof of our main result shows; for the geometric intuition behind this argument, see Figure 2 below. Hence, for any symmetric quadratic model* Π *we obtain* $m^-_{\Pi}(y, x) > m^-_{\Pi}(x, y)$, *and thus* $y P_{\Pi} x$, *where* P_{Π} *denotes the asymmetric part of the ex-ante majority relation* R_{Π}; *in other words,* x *is not an ex-ante Condorcet winner.*

At one extreme, there exists a unique largest (most imprecise) s.i.q. model consisting of *all* symmetric priors. Note that it does not impose any additional knowledge, i.e. probability one restrictions, beyond the plain quadratic model; it can thus be viewed as a regularized version of that model.

At the other extreme, there is also a unique smallest (most precise) s.i.q. model, as follows. Call a prior *uniform* if it puts all mass on profiles of the form $(\mathcal{Q}, ..., \mathcal{Q})$ for some quadratic form $\mathcal{Q}$, and denote by Π_{unif} the **uniform (quadratic) model** consisting of all uniform priors; moreover, denote by Π_{exunif} the **extremal uniform model** consisting of all uniform priors of the form $\delta(\mathcal{Q}, ..., \mathcal{Q})$, i.e. all priors that put unit mass on some single profile of the form $(\mathcal{Q}, ..., \mathcal{Q})$. Evidently, the extremal uniform model satisfies Assumptions 1 to 4; conversely, combining Assumptions 3 and 4 also shows that any s.i.q. model contains the extremal uniform model.[8]

There is a wide range of intermediate specifications. For example, the quadratic forms $\mathcal{Q}_i$ can be assumed to be drawn i.i.d. from some unknown distribution. In the subjectivist tradition, this is captured (and slightly generalized in the finite case) by assuming that Π consists of all exchangeable priors in the sense of [8].[9] Finally, a s.i.q. model Π may also incorporate beliefs in possibly learnable correlations between the tops and the quadratic forms; it only excludes prior information about what these correlations are.

It turns out that the ex-ante Condorcet winners in the s.i.q. models are *Tukey medians* [29] of a particular kind. For all $x \in X$, denote by $\mathcal{H}_x$ the family of all Euclidean half-spaces that contain x (i.e. the family of all sets of the form $\{y \in X : a \cdot y \geq a \cdot x\}$ for some non-zero vector $a \in \mathbb{R}^L$). For all profiles $\theta = (\theta_1, ..., \theta_n)$ and all half-spaces H, denote $\theta(H) := \#\{i : \theta_i \in H\}$, and define the *Tukey depth* of x at the profile θ by

$$\mathfrak{d}(x;\theta) := \min_{H \in \mathcal{H}_x} \theta(H).$$

Intuitively, the Tukey depth measures the 'centrality' of x with respect to the profile of tops: the larger $\mathfrak{d}(x;\theta)$ the more tops θ_i are guaranteed to lie in every direction viewed from x, and $\mathfrak{d}(x;\theta) = 0$ means that x can be separated from the entire set of tops θ by a hyperplane. Denote by $\mathfrak{d}^*(\theta) := \max_{x \in X} \mathfrak{d}(x;\theta)$ the maximal Tukey depth over X. The *Tukey median rule* selects, for every profile θ, the alternatives that attain this maximal depth:

$$T(\theta) \;:=\; \arg\max_{x \in X} \mathfrak{d}(x;\theta) \;=\; \{x \in X \mid \mathfrak{d}(x;\theta) = \mathfrak{d}^*(\theta)\}.$$

Our main result involves the following refinement. For all profiles θ and all x, denote by $\mathcal{H}^*_x := \{H \ni x : \theta(H) = \mathfrak{d}^*(\theta)\}$. A Tukey median $x \in T(\theta)$ is *strict* if, for no $y \in T(\theta)$, $\mathcal{H}^*_y \subsetneq \mathcal{H}^*_x$. The set of **strict Tukey medians** is denoted by $T^*(\theta)$.

Theorem 1. *For all profiles θ and every symmetrically ignorant quadratic model $\Pi \subseteq \Pi^\theta_{\text{quad}}$, $\mathrm{CW}(\Pi)$ is non-empty. Moreover,*

$$\mathrm{CW}(\Pi) \;=\; T^*(\theta).$$

The proof of Theorem 1 proceeds in a series of steps. First, it is shown that all s.i.q. models are equivalent. The argument relies crucially on both the symmetry assumption and the EAC solution concept. It allows to focus on the characterization of the analytically convenient uniform model. This simplifies matters greatly since the uniform model is characterized by strong ex-post restrictions on profiles. In particular, profiles of preferences with a common quadratic form are intermediate preferences in the sense of [12]. More specifically, for any two alternatives x and y, the tops in a profile of preferences with a common quadratic form $\mathcal{Q}$ that prefer x to y are separated from those preferring y to x by a hyperplane through the midpoint between x and y.

From this, one can show that the ex-ante majority relation of the uniform model coincides locally with the comparison of alternatives in terms of their *relative Tukey depth*: for all distinct $x, y \in X$, let

$$xR_{\mathfrak{d}}y \;:\Longleftrightarrow\; \min_{H \in \mathcal{H}_x,\, y \notin H} \theta(H) \geq \min_{H \in \mathcal{H}_y,\, x \notin H} \theta(H). \tag{4.2}$$

The ex-ante majority relation does not coincide globally with the relative Tukey depth relation (4.2) since the half-spaces separating the underlying tops in the quadratic model must go through the midpoint between x and y. Nonetheless, the set of local maxima of this relation is shown to coincide with the set of global maxima, which in turn coincides with the set of strict Tukey medians. Finally, the existence of strict Tukey medians is shown by an appeal to the Hausdorff maximal principle.

[8]By the preceding observation, the class of all s.i.q. models forms a bounded lattice partially ordered by set inclusion.

[9]Specifically, in our context a prior π is *exchangeable* if, for all events $E \subseteq \boldsymbol{\mathcal{Q}}_1 \times \ldots \times \boldsymbol{\mathcal{Q}}_n$ and all permutations σ of agents, $\pi(E) = \pi(\sigma(E))$, where $\sigma(E)$ is the event obtained from E by applying σ.

Example 1 (cont.) *In Example 1, the Tukey depth of x relative to y is evidently* $\min_{H\in\mathcal{H}_x,\, y\notin H}\theta(H) = 2$. *Conversely, the Tukey depth of y relative to x is obtained by looking at the straight line ∂H through x and y: the tops that support y against x must at least contain the tops in $U \cap H$, or the tops in $U \cap H^c$. As can be inferred from Fig. 2, we therefore have* $\min_{H\in\mathcal{H}_y,\, x\notin H}\theta(H) = 3$, *and hence $yP_{\mathfrak{d}}x$, where $P_{\mathfrak{d}}$ denotes the asymmetric part of $R_{\mathfrak{d}}$. It follows from the arguments provided in the proof of Theorem 1 in the appendix that we thus also obtain $yP_{\Pi}x$ for any s.i.q. model Π, as claimed above.*

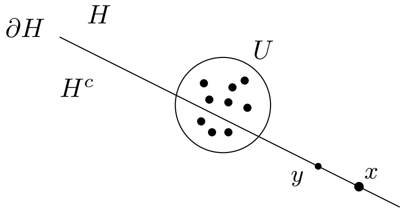

Figure 2: The Tukey depth of y relative to x is equal to 3

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