Ties in Multiwinner Approval Voting

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Abstract

We study the complexity of deciding whether there is a tie in a given approval-based multiwinner election, as well as the complexity of counting tied winning committees. We consider a family of Thiele rules, their greedy variants, Phragmén's sequential rule, and Method of Equal Shares. For most cases, our problems are computationally hard, but for sequential rules we find an FPT algorithm for discovering ties (parameterized by the committee size). We also show experimentally that in elections of moderate size ties are quite frequent.

1 Introduction

In an approval-based multiwinner election, a group of voters expresses their preferences about a set of candidates—i.e., each voter indicates which of them he or she approves—and then, using some prespecified rule, the organizer selects a winning committee (a fixed-size subset of the candidates). Multiwinner elections can be used to resolve very serious matters—such as choosing a country's parliament—or rather frivolous ones—such as choosing the tourist attractions that a group of friends would visit—or those positioned anywhere in between these two extremes—such as choosing a department's representation for the university senate. In large elections, one typically does not expect ties to occur (although surprisingly many such cases are known[1]), but for small and moderately sized ones the issue is unclear. While perhaps a group of friends may manage to not spoil their holidays upon discovery that they were as willing to visit one monument as another, a person not selected for a university senate due to a tie may be quite upset, especially if this tie is discovered after announcing the results. To address such possibilities, we study the following three issues:

1. We consider the complexity of detecting if two or more committees tie under a given voting rule. While for most rules this problem turns out to be intractable, for many settings we find practical solutions (in most cases it is either possible to use a natural integer linear programming trick or an FPT algorithm that we provide).

2. We consider the complexity of counting the number of winning committees. We do so, because being able to count winning committees would be helpful in sampling them uniformly. Unfortunately, in this case we mostly find hardness and hardness of approximation results.

3. We generate a number of elections, both synthetic and based on real-life data, and evaluate the frequency of ties. It turns out to be surprisingly high.

We consider a subfamily of Thiele rules [28, 1, 15] that includes the multiwinner approval rule (AV), the approval-based Chamberlin–Courant rule (CCAV), and the proportional approval voting rule (PAV), as well as their greedy variants. We also study satisfaction approval voting (SAV), the Phragmén rule, and Method of Equal Shares (MEqS). This set includes rules appropriate for selecting committees of individually excellent candidates (e.g., AV or SAV), diverse committees (e.g., CCAV or GreedyCCAV), or proportional ones (e.g., PAV, GreedyPAV, Phragmén, or MEqS); see the works of Elkind et al. [6] and Faliszewski et al. [9] for more details on classifying multiwinner rules with respect to their application. See also the textbook of Lackner and Skowron [16]. We summarize our results in Table 1.

[1] https://en.wikipedia.org/wiki/List_of_close_ election_results

Rule	Unique-Committee	#Winning-Committees
AV	P	P
SAV	P	P
CCAV	coNP-hard, coW[1]-h. (k)	#P-hard, #W[1]-hard (k)
PAV	coNP-hard, coW[1]-h. (k)	#P-hard, #W[1]-hard (k)
GreedyCCAV	coNP-com., FPT(k)	#P-hard, #W[1]-hard (k)
GreedyPAV	coNP-com., FPT(k)	#P-hard, #W[1]-hard (k)
Phragmén	coNP-com., FPT(k)	#P-hard, #W[1]-hard (k)
MEqS (Phase 1)	coNP-com., FPT(k)	#P-hard

Table 1: Summary of our complexity results.

The issue of ties and tie-breaking has already received quite some attention in the literature, although typically in the context of single-winner voting. For example, Obraztsova and Elkind [18] and Obraztsova et al. [19] consider how various tie-breaking mechanisms affect the complexity of manipulating elections, and recently Xia [30] has made a breakthrough in studying the probability that ties occur in large, randomly-generated single-winner elections. Xia [31] also developed a novel tie-breaking mechanism, which can be used for some multiwinner rules, but he did not deal with such approval rules as we study here. Finally, Conitzer et al. [4] have shown that deciding if a candidate is a tied winner in an STV election is NP-hard. While STV is not an approval-based rule and they focused on the single-winner setting, many of our results are in similar spirit. All omitted proofs are available in the appendix.

2 Preliminaries

By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For each integer t, we write $[t]$ to mean $\{1, \ldots, t\}$. We use the Iverson bracket notation, i.e., for a logical expression F, we interpret $[F]$ as 1 if F is true and as 0 if it is false. Given a graph G, we write $V(G)$ to denote its set of vertices and $E(G)$ to denote its set of edges. For a vertex v, by $d(v)$ we mean its degree (i.e., the number of edges that are incident to it).

An election $E = (C, V)$ consists of a set of candidates $C = \{c_1, \ldots, c_m\}$ and a collection of voters $V = (v_1, \ldots, v_n)$, where each voter v_i has a set $A(v_i) \subseteq C$ of candidates that he or she approves. We refer to this set as v_i's approval set or v_i's vote, interchangeably. Similarly, we denote the set of voters approving candidate c_j as $A(c_j)$. A multiwinner voting rule f is a function that given an election $E = (C, V)$ and committee size $k \in [|C|]$ outputs a family of size-k subsets of C, i.e., a family of winning committees. Below we describe the rules that we focus on.

Let $E = (C, V)$ be an election and let k be the committee size. Under the multiwinner approval rule (AV), each voter assigns a single point to each candidate that he or she approves and winning committees consist of subsets of k candidates with the highest scores. Satisfaction approval voting (SAV) proceeds analogously, except that each voter $v \in V$ assigns $1/|A(v)|$ points to each candidate he or she approves. In other words, under AV each voter can give a single point to each approved candidate, but under SAV he or she needs to split a single point equally among them.

Next we consider the class of Thiele rules, defined originally by Thiele [28] and discussed, e.g., by Lackner and Skowron [15] and Aziz et al. [1]. Given a nondecreasing weight function $w \colon \mathbb{N} \to \mathbb{R}_+$ such that $w(0) = 0$, we define the w-Thiele score (w-score) of a committee $S = \{s_1, \ldots, s_k\}$ in election E to be:

$$w\text{-score}_E(S) = \textstyle\sum_{v \in V} w(|A(v) \cap S|).$$

The w-Thiele rule outputs all committees with the highest w-score. We require that for each of our weight functions w, it is possible to compute each value $w(i)$ in polynomial time with respect to i. Additionally, we focus on functions such that $w(1) = 1$ and for each positive integer i it holds that

$w(i) - w(i-1) \geq w(i+1) - w(i)$. We refer to such functions, and the Thiele rules that they define, as 1-concave. Three best-known 1-concave Thiele rules include the already defined AV rule, which uses function $w_{\text{AV}}(t) = t$, the approval-based Chamberlin–Courant rule (CCAV), which uses function $w_{\text{CCAV}}(t) = [t \geq 1]$, and the proportional approval voting rule (PAV), which uses function $w_{\text{PAV}}(t) = \sum_{i=1}^{t} 1/i$.

While it is easy to compute some winning committee under the AV rule in polynomial time (out of possibly exponentially many), for the other Thiele rules, including CCAV and PAV, even deciding if a committee with at least a given score exists is NP-hard (see the works of Procaccia et al. [22] and Betzler et al. [2] for the case of CCAV, and the works of Aziz et al. [1] and Skowron et al. [24] for the general case). Hence, sometimes the following greedy variants of Thiele rules are used (E is the input election and k is the desired committee size):

> Let f be a w-Thiele rule. Its greedy variant, denoted Greedy-f, first sets $W_0 := \emptyset$ and then executes k iterations, where for each $i \in [k]$, in the i-th iteration it computes $W_i := W_{i-1} \cup \{c\}$ such that c is a candidate in $C \setminus W_{i-1}$ that maximizes the w-score of W_i. Finally, it outputs W_k. In case of internal ties, i.e., if at some iteration there is more than one candidate that the algorithm may choose, the algorithm outputs all committees that can be obtained for some way of resolving each of these ties. In other words, we use the parallel-universes tie-breaking model [4].

When we discuss the operation of some Greedy-f rule on election E and we discuss the situation after its i-th iteration, where, so far, subcommittee W_i was selected, then by the score of a (not-yet-selected) candidate c we mean the value $w\text{-score}_E(W_i \cup \{c\}) - w\text{-score}_E(W_i)$, i.e., the marginal increase of the w-score that would result from selecting c. We refer to the greedy variants of CCAV and PAV as GreedyCCAV and GreedyPAV (in the literature, these rules are also sometimes called *sequential* variants of CCAV and PAV, see, e.g., the book of Lackner and Skowron [16]). Given a greedy variant of a 1-concave Thiele rule, it is always possible to compute at least one of its winning committees in polynomial time by breaking internal ties arbitrarily. Further, it is well-known that the w-score of this committee is at least a $1 - 1/e \approx 0.63$ fraction of the highest possible w-score; this follows from the classic result of Nemhauser et al. [17] and the fact that w-score is monotone and submodular.

The Phragmén (sequential) rule proceeds as follows (see, e.g., the work of Sánchez-Fernández et al. [23]):

> Let $E = (C, V)$ be an election and let k be the committee size. Each candidate costs a unit of currency. The voters start with no money, but they receive it continuously at a constant rate. As soon as there is a group of voters who approve a certain not-yet-selected candidate and who together have a unit of currency, these voters "buy" this candidate (i.e., they give away all their money and the candidate is included in the committee). The process stops as soon as k candidates are selected. For internal ties, we use the parallel-universes tie-breaking.

Method of Equal Shares (MEqS), introduced by Peters and Skowron [20] and Peters et al. [21], is similar in spirit, but gives the voters their "money" up front (we use the same notation as above):

> Initially, each voter has budget equal to $k/|V|$. The rule starts with an empty committee and executes up to k iterations as follows (for each voter v, let $b(v)$ denote v's budget in the current iteration): For each not-yet-selected candidate c we check if the voters that approve c have at least a unit of currency (i.e., $\sum_{v \in A(c)} b(v) \geq 1$). If so, then we compute value ρ_c such that $\sum_{v \in A(c)} \min(b(v), \rho_c) = 1$, which we call the per-voter cost of c. We extend the committee with this candidate c', whose per-voter cost $\rho_{c'}$ is lowest; the voters approving c' "pay" for him or her (i.e., each voter $v \in A(c')$ gives away $\min(b(v), \rho_{c'})$ of his or her budget). In case of internal ties, we use the parallel-universes tie-breaking. The process stops as soon as no candidate can be selected.

The above process, referred to as Phase 1 of MEqS, often selects fewer than k candidates. To deal with this, we extend the committee with candidates selected by Phragmén (started off with the budgets that the voters had at the end of Phase 1). We jointly refer to the greedy rules, Phragmén, MEqS, and Phase 1 of MEqS as sequential rules.

We assume that the reader is familiar with basic classes of computational complexity such as P, NP, and coNP. $\#\mathrm{P}$ is the class of functions that can be expressed as counting accepting paths of nondeterministic polynomial-time Turing machines. Additionally, we consider pararameterized complexity classes such as FPT and $\mathrm{W}[1]$. $\#\mathrm{W}[1]$ is a parameterized counting class which relates to $\mathrm{W}[1]$ in the same way as $\#\mathrm{P}$ relates to NP [11]. When discussing counting problems, it is standard to use Turing reductions: A counting problem $\#A$ reduces to a counting problem $\#B$ if there is a polynomial time algorithm that solves $\#A$ in polynomial time, provided it has oracle access to $\#B$ (i.e., it can solve $\#B$ in constant time).[2]

3 Unique Winning Committee

In this section we consider the problem of deciding if a given multiwinner rule outputs a unique committee in a given election. Formally, we are interested in the following problem.

Definition 3.1. *Let f be a multiwinner voting rule. In the f-*UNIQUE-COMMITTEE *problem we are given an election E and a committee size k, and we ask if $|f(E,k)| = 1$.*

It is a folk result that for AV and SAV this problem is in P (see beginning of Section 4 for an argument). For Thiele rules other than AV, the situation is more intriguing. In particular, already the problem of deciding if a given committee is winning under the CCAV rule is coNP-complete [25]. We show that for 1-concave Thiele rules other than AV the UNIQUE-COMMITTEE problem is coNP-hard (and we conjecture that the problem is not in coNP).

Proposition 3.1. *Let f be a 1-concave w-Thiele rule other than AV. Then f-*UNIQUE-COMMITTEE *is coNP-hard.*

Proof. Let $x = w(2) - w(1)$ and assume, for now, that $x < 1$. We give a reduction from INDEPENDENT-SET to the complement of f-UNIQUE-COMMITTEE. An instance of INDEPENDENT-SET consists of a graph G and integer k, and we ask if there are k vertices neither of which is connected with the others. Let G' be a graph obtained from G by adding k vertices such that each of the new vertices is connected to each of the old ones (but the new vertices are not connected to each other). If G does not have a size-k independent set, then G' has a unique one, and if G has at least one size-k independent set, then G' has at least two. Let us denote the vertices of G' as $V(G') = \{v_1, \dots, v_n\}$ and its edges as $E(G') = \{e_1, \dots, e_m\}$. Let δ be the highest degree of a vertex in $V(G')$. We fix the committee size to be k and we form an election E with candidate set $V(G')$ and with the following voters:

1. For each edge $e_\ell = \{v_i, v_j\}$ there is a single voter who approves v_i and v_j.
2. For each vertex v_i there are $\delta - d(v_i)$ voters approving v_i.

Consider a set of k vertices from $V(G')$. If this set is an independent set, then interpreted as a committee in election E, it has w-score equal to δk. On the other hand, if S is not an independent set, then its score is at most $(\delta k - 1) + x < \delta k$. We know that G' has an independent set of size k. If G also has one, then our election has at least two winning committees and, otherwise, the winning committee is unique.

Let us now consider the case that $x = 1$. Since f is not AV, there certainly is an integer t such that $w(t) - w(t-1) = 1$ and $w(t+1) - w(t) < 1$. In this case, we modify the reduction by adding $t-1$ candidates approved by every voter and changing the committee size to be $t + k - 1$. ☐

[2]For $\#\mathrm{W}[1]$, the running time can even be larger, but our $\#\mathrm{W}[1]$-hardness proofs use polynomial-time reductions.

For greedy variants of Thiele rules (with the natural exception of AV) and for the Phragmén rule, deciding if the winning committee is unique is coNP-complete. Our proof for the greedy variants of Thiele rules is inspired by a complexity-of-robustness proof for GreedyPAV, provided by Faliszewski et al. [10]. For Phragmén, somewhat surprisingly, their robustness proof directly implies our desired result. We also get analogous results for Method of Equal Shares and its Phase 1.

Theorem 3.2. *Let f be a 1-concave w-Thiele rule, $f \neq$ AV. Greedy-f-*UNIQUE-COMMITTEE *is* coNP*-complete.*

Corollary 3.3. UNIQUE-COMMITTEE *is* coNP*-complete for GreedyCCAV, GreedyPAV, and* PHRAGMÉN.

Theorem 3.4. UNIQUE-COMMITTEE *is* coNP*-complete for Phase 1 of MEqS.*

Proof. The following nondeterministic algorithm shows membership in coNP: First, we deterministically compute the output of Phase 1, breaking internal ties in some arbitrary way. This way we obtain some committee W. Next we rerun Phase 1, at each internal tie nondeterministically trying all possibilities. We accept on computation paths that output W and we reject on those outputting some other committee. This algorithm accepts on all computation paths if and only if the rule has a unique winning committee.

Next, we give a reduction from the complement of the classic NP-complete problem, X3C. An instance of X3C consists of a universe set $U = \{u_1, \dots, u_{3n}\}$ and a family $\mathcal{S} = \{S_1, \dots, S_{3n}\}$ of size-3 subsets of U. We ask if there are n sets from $\mathcal{S}$ whose union is U (we refer to such a family as an exact cover of U; note that the sets in such a cover must be disjoint). Without loss of generality, we assume that each member of U belongs to exactly three sets from $\mathcal{S}$ [14] and that n is even.

Now we describe our election. Ideally, we would like to distribute different amounts of budget between different voters, but as MEqS splits the budget evenly, we design the election in such a way that in the initial iterations the respective voters spend appropriate amounts of money on the candidates that otherwise are not crucial for the construction. We form the following groups of voters (we reassure the reader that the analysis is more pleasant than the following two enumerations may suggest):

1. Group B, which contains $144n^3 - 12n$ voters.
2. Group B_U, which contains $54n^3 + 9n^2$ voters.
3. Group U', which models the elements of the universe set U. For each $u_i \in U$, there is a single corresponding voter in U'. We have $|U'| = 3n$.
4. Group U'', which serves a similar purpose as U', but contains more voters. Specifically, for each $u_i \in U$, there are $6n$ corresponding voters in U''; $|U''| = 18n^2$.
5. Group V_{pd}, which contains $12n$ voters.
6. Group V_S, which contains $9n$ voters.
7. Two voters, d_1 and d_2.

In total, there are $198n^3 + 27n^2 + 12n + 2$ voters. Further, we have the following groups of candidates:

1. Group C_B of $144n^3 - 12n^2$ candidates approved by the $144n^3$ voters from $B \cup V_{pd}$.
2. Group C_U of $54n^3 + 24n^2 + {}^{5n}/_{2}$ candidates approved by the $54n^3 + 27n^2 + 3n$ voters from $B_U \cup U' \cup U''$.
3. Candidate p approved by the $12n$ voters from V_{pd}.

4. Candidate d approved by the $15n$ voters from $V_{pd} \cup U'$.
5. Candidates c_1 and c_2, both approved by d_1 and d_2.
6. Group D of $15n^2 + \frac{45n}{2} + 5$ candidates approved by d_1.
7. For each set $S_\ell \in \mathcal{S}$ such that $S_\ell = \{u_i, u_j, u_t\}$ we have a corresponding candidate s_ℓ approved by: (a) three unique voters from V_S, (b) the voters from U' and U'' that correspond to the elements u_i, u_j, u_t. We write S to denote this group of candidates and we refer to its members as the S-candidates. Each S-candidate is approved by $3 + 3 + 3 \cdot 6n = 18n + 6$ voters.

We have $198n^3 + 27n^2 + 28n + 9$ candidates in total. We set the committee size k to be equal to the number of voters, i.e., $k = 198n^3 + 27n^2 + 12n + 2$. Let us consider the following two committees (note that each of them contains fewer than k candidates; indeed, Phase 1 of MEqS sometimes chooses committees smaller than requested):

$$W_d = C_B \cup C_U \cup S \cup \{c_1, c_2\} \cup \{d\},$$
$$W_p = C_B \cup C_U \cup S \cup \{c_1, c_2\} \cup \{p\}.$$

We claim that Phase 1 of MEqS always outputs committee W_d, and if $(U, \mathcal{S})$ is a *yes*-instance then it also outputs W_p.

Let us analyze how Phase 1 of MEqS proceeds on our election. Since the committee size is equal to the number of voters, initially each voter receives budget equal to 1.

At first, we will select all candidates from C_B. Indeed, there are $144n^3 - 12n^2$ candidates in this group, each approved by $144n^3$ voters (from $B \cup V_{pd}$). Each of these voters pays $1/144n^3$ for each of the candidates (this is the lowest per-voter candidate cost at this point). After these purchases, each voter from $B \cup V_{pd}$ will be left with budget equal to $1 - (144n^3 - 12n^2) \cdot (1/144n^3) = 1/12n$.

Next, we will select all candidates from C_U. Indeed, this set contains $54n^3 + 24n^2 + 5n/2$ candidates approved by $54n^3 + 27n^2 + 3n$ voters (from $B_U \cup U' \cup U''$) who have not spent any part of their budget yet. All candidates in C_U will be purchased at the same pre-voter cost of $1/(54n^3+27n^2+3n)$ (the lowest one at this point). Each voter in $B_U \cup U' \cup U''$ will be left with budget equal to $1 - (54n^3 + 24n^2 + 5n/2) \cdot 1/(54n^3+27n^2+3n) = \frac{3n^2+n/2}{54n^3+27n^2+3n} = \frac{6n+1}{108n^2+54n+6} = \frac{6n+1}{(6n+1)\cdot(18n+6)} = 1/(18n+6)$.

Next, we consider the S-candidates who, at this point, have the highest approval score among the yet unselected candidates. As each S-candidate is approved by exactly $18n + 6$ voters and each voter still has budget higher or equal to $1/(18n+6)$, we keep selecting the S-candidates at the per-voter cost of $1/(18n+6)$ as long as there is at least one such candidate whose all voters still have budget of at least $1/(18n+6)$.

Upon selecting a given S-candidate, corresponding to set S_ℓ, all the voters who approve him or her pay $1/(18n+6)$. This includes the three unique voters from V_S and the voters from U' and U'' who correspond to the members of S_ℓ. Prior to this payment, the voters from U' and U'' have budget equal to $1/(18n+6)$, so they end up with 0 afterward (and we say that they are *covered* by this S-candidate). Consequently, the S-candidates that we buy at the per-voter cost of $1/(18n+6)$ correspond to disjoint sets.

Now let us consider what happens when there is no S-candidate left who can be purchased at the per-voter cost of $1/(18n+6)$. This means that for each unselected S candidate, at least $6n + 1$ voters approving him have already been covered and have no budget left. Hence, for a given S-candidate there are at least $6n + 1$ voters (from U' and U'') whose budget is 0, at most $12n + 2$ voters (from U' and U'') who each have budget of $1/(18n+6)$, and three voters (from V_S) who each have budget equal to 1. To buy this S candidate, the voters from U' and U'' would have to use up their whole budget, and the voters from V_S would have to pay at least:

$$\frac{1}{3}(1 - (12n + 2) \cdot \frac{1}{18n+6}) = \frac{18n+6-(12n+2)}{3\cdot(18n+6)} = \frac{6n+4}{54n+18}$$

each. However, at this point there are two candidates that can be purchased at lower per-voter cost.

Indeed, candidate p could be purchased by the $12n$ voters from V_{pd} at the per-voter cost of $1/12n$ (after buying the candidates from C_B, they still have exactly this amount of budget left). Since candidate d also is approved by all the voters from V_{pd}, and also by the voters from U', candidate d would either have the same per-voter cost as p (in case all the members of U' were already covered) or would have an even lower per-voter cost. The only other remaining candidates are c_1, c_2, and the candidates from D, but their per-voter costs are greater or equal to $1/2$. Hence, at this point, MEqS either selects p or d. The former is possible exactly if the already selected S-candidates form an exact cover of U' (and, hence, correspond to an exact cover for our input instance of X3C).

If we select p, then the $12n$ voters from V_{pd} use up all their budget. The remaining voters who approve d, those in U', have total budget equal to at most $3n \cdot \frac{1}{18n+6} < 1$, so d cannot be selected in any of the following iterations (within Phase 1). On the other hand, if we select d, then all the voters from U' would have to pay all they had left (that is, either 0 or $\frac{1}{18n+6}$, each) and voters from V_{pd} would split the remaining cost. That is, each voter from V_{pd} would have to pay at least:

$$\frac{1-3n\cdot\frac{1}{18n+6}}{12n} = \frac{18n+6-3n}{12n\cdot(18n+6)} = \frac{15n+6}{12n\cdot(18n+6)}.$$

Consequently, each voter from V_{pd} would be left with at most:

$$\frac{1}{12n} - \frac{15n+6}{12n\cdot(18n+6)} = \frac{18n+6-(15n+6)}{12n\cdot(18n+6)} = \frac{1}{72n+24}.$$

This would not suffice to purchase p, as $12n \cdot \frac{1}{72n+24} < 1$. Thus either we select d (and not p) or we select p (and not d; where this is possible only if we previously purchased S-candidates that cover all members of U').

In the following iterations, we purchase all remaining S-candidates (because each of them is approved by three unique voters from V_S), as well as candidates c_1 and c_2 (voters d_1 and d_2 buy them with per-voter cost of $1/2$ for each). This uses up the budget of d_1 and, so, no candidate from D is selected. All in all, if there is no exact cover for the input X3C instance, then W_d is the unique winning committee, but otherwise W_d and W_p tie. This finishes the proof. □

UNIQUE-COMMITTEE is also coNP-complete for the full version of MEqS. To see this, it suffices to note that after adding enough voters with empty votes (for example, m^2n^2 when m is the number of candidates and n is the current number of voters), MEqS becomes equivalent to Phragmén (because per-voter budget is so low that Phase 1 becomes vacuous) and inherits its hardness.

On the positive side, for sequential rules we can solve UNIQUE-COMMITTEE in FPT time with respect to the committee size: In essence, we first compute some winning committee and then we try all ways of breaking internal ties to find a different one. For small values of k, such as, e.g., $k \leq 10$, the algorithm is fast enough to be practical.

Theorem 3.5. *Let f be MEqS, Phase 1 of MEqS, Phragmén, or a greedy variant of a 1-concave Thiele rule. There is an* FPT *algorithm for f-*UNIQUE-COMMITTEE *parameterized by the committee size.*

Proof. Let E be the input election and let k be the committee size. First, we compute some committee W in $f(E,k)$, by running the algorithm for f and breaking the internal ties arbitrarily. Next, we rerun the algorithm, but whenever it is about to add a candidate into the constructed committee, we do as follows (let T be the set of candidates that the algorithm can insert into the committee): If T contains some candidate c that does not belong to W, then we halt and indicate that there are at least two winning committees (W and those that include c). If T is a subset of W, then we recursively try each way of breaking the tie. If the algorithm completes without halting, we report that there is a unique winning committee. The correctness is immediate. The running time is equal to $O(k!)$ times the running time of the rule's algorithm (for the case where each tie is broken in a given way). Indeed, at the first internal tie we may need to recurse over at most k different candidates, then over at most $k-1$, and so on. □

For 1-concave Thiele rules other than AV, UNIQUE-COMMITTEE is $\mathrm{coW}[1]$-hard when parameterized by the committee size (this follows from the proof of Proposition 3.1 as INDEPENDENT-SET is $\mathrm{W}[1]$-hard for parameter k). To solve the problem in practice, we note that for each 1-concave Thiele rule there is an integer linear program (ILP) whose solution corresponds to the winning committee. We can either use the ability of some ILP solvers to output several solutions (which only succeeds in case of a tie), or we can use the following strategy: First, we compute some winning committee using the basic ILP formulation. Then, we extend the formulation with a constraint that requires the committee to be different from the previous one and compute a new one. If both committees have the same score, then there is a tie.

4 Counting Winning Committees

Let us now consider the problem of counting the winning committees. Formally, our problem is as follows.

Definition 4.1. *Let f be a multiwinner voting rule. In the f-#WINNING-COMMITTEES problem we are given an election and a committee size k; we ask for $|f(E,k)|$.*

There are polynomial-time algorithms for computing the number of winning committees for AV and SAV. For an election E with committee size k, we first sort the candidates with respect to their scores in a non-increasing order and we let x be the score of the k-th candidate. Then, we let S be the number of candidates whose score is greater than x, and we let T be the number of candidates with score equal to x. There are $\binom{T}{k-S}$ winning committees.

Proposition 4.1. *$\{AV, SAV\}$-#WINNING-COMMITTEES $\in \mathrm{P}$*

On the other hand, whenever f-UNIQUE-COMMITTEE is intractable, so is f-#WINNING-COMMITTEES. Indeed, it immediately follows that there is no polynomial-time $(2-\varepsilon)$-approximation algorithm for f-#WINNING-COMMITTEES for any $\varepsilon > 0$ (if such an algorithm existed then it could solve f-UNIQUE-COMMITTEE in polynomial time as for an election with a single winning committee it would have to output 1, and for an election with 2 winning committees or more, it would have to output an integer greater or equal at least $\frac{2}{2-\varepsilon} > 1$, so we could distinguish these cases[3]). However, for all our rules a much stronger result holds.

Proposition 4.2. *Let f be a 1-concave Thiele rule (different from AV), its greedy variant, Phragmén, MEqS or Phase 1 of MEqS. Unless $\mathrm{P} = \mathrm{NP}$, there is no polynomial-time approximation algorithm for f-#WINNING-COMMITTEES with polynomially-bounded approximation ratio.*

Proof. For Phase 1 of MEqS, it suffices to use the proof of Theorem 3.4 with candidate p replaced by polynomially many copies, each approved by the same voters. Either we get a unique winning committee or polynomially many tied ones. The same trick works with the greedy variants of 1-concave Thiele rules and Theorem 3.2, and Phragmén and Corollary 3.3.

For the case of 1-concave Thiele rules, we use the following strategy. Let p be some positive integer and let (G, k) be an instance of INDEPENDENT-SET, where G is a graph and k is an integer. We form a graph G^p whose vertex set is: $V(G^p) = \{v^i \mid v \in V(G), i \in [p]\}$ and where two vertices, u^i and v^j, are connected by an edge either if $i \neq j$ or if $i = j$ and u and v are connected by an edge in G. Consequently, if G has x independent sets of size k, then G^p has px such sets (each independent set of G^p is a copy of an independent set of G, using only vertices with the same superscript). Hence, if in the proof of Propostion 3.1 we replace graph G with graph G^p,

[3]We assume here that if a solution for a counting problem is $x \in \mathbb{N}$, then an α-approximation algorithm, with $\alpha \geq 1$, has to output an integer between x/α and αx. If we allowed rational values on output, the inapproximability bound would drop to $\sqrt{2} - \varepsilon$.

where p is some polynomial function of the input size, then we obtain an election that either has a unique winning committee (if the input graph did not have an independent set of a required size) or an election that has polynomially many winning committees (if the graph had at least one such independent set). □

We note that the construction given in the proof of Proposition 3.1 also shows that for each 1-concave Thiele rule $f \neq$ AV, f-#WINNING-COMMITTEES is both $\#\mathrm{P}$-hard and $\#\mathrm{W}[1]$-hard for parameterization by the committee size (because this reduction produces elections that have one more winning committee than the number of size-k independent sets in the input graph, and counting independent sets is both $\#\mathrm{P}$-complete and $\#\mathrm{W}[1]$-complete for parameterization by k [29, 11]). For greedy variants of 1-concave Thiele rules and Phragmén, the situation is more interesting because UNIQUE-COMMITTEE is in FPT (for the parameterization by the committee size). Yet, #WINNING-COMMITTEES is also hard.

Theorem 4.3. *Let f be Phragmén or a greedy variant of a 1-concave Thiele rule (different from AV). f-#WINNING-COMMITTEES is $\#\mathrm{P}$-hard and $\#\mathrm{W}[1]$-hard (for the parameterization by the committee size).*

Proof. We first consider greedy variants of 1-concave Thiele rules. Let w be the weight function used by f. Let $x = w(2) - w(1)$. We have $w(1) = 1$ and we assume that $x < 1$ (we will consider the other case later). We show a reduction from the #MATCHING problem, where we are given a graph G, an integer k, and we ask for the number of size-k matchings (i.e., the number of size-k sets of edges such that no two edges in the set share a vertex). #MATCHING is $\#\mathrm{W}[1]$-hard for parameterization by k [5].

Let G and k be our input. We form an election E where the edges of G are the candidates and the vertices are the voters. For each edge $e = \{u, v\}$, the corresponding edge candidate is approved by the vertex voters corresponding to u and v. We also form an election E_p, equal to E except that it has two extra voters who both approve a single new candidate, p.

We note that every candidate in both E and E_p is approved by exactly two voters. Hence, the greedy procedure first keeps on choosing candidates whose score is 2 (i.e., edges that jointly form a matching, or candidate p in E_p). It selects the candidates with lower scores (i.e., edges that break a matching) only when score-2 candidates disappear.

Let W be some size-k f-winning committee for election E_p. We consider two cases:

1. If p does not belong to W, then the edge candidates in W form a matching. If it were not the case, then before including an edge candidate with score lower than 2, the greedy algorithm would have included p in the committee.

2. If p belongs to W then $W \setminus \{p\}$ is an f-winning committee of size $k - 1$ for election E. Indeed, if we take the run of the greedy algorithm that computes W and remove the iteration where p is selected, we get a correct run of the algorithm for election E and committee size $k - 1$. Further, for every size-$(k - 1)$ committee winning in E, $S \cup \{p\}$ is a size-k winning committee in E_p (because we can always select p in the first iteration).

So, to compute the number of size-k matchings in G, it suffices to count the number of winning size-k committees in E_p and subtract from it the number of winning size-$(k - 1)$ committees in E. If $x = 1$, then we find the smallest value t such that $w(t) - w(t - 1) = 1$ and $w(t + 1) - w(t) < 1$ and use the same construction as above, except that there are $t - 1$ dummy candidates approved by every voter.

Regarding Phragmén, it turns out that the same construction as for the greedy variants of 1-concave Thiele rules still works. In time $t = 1/2$, each voter has $1/2$ budget and each candidate (including p) can be purchased (because each candidate is approved by exactly two voters and their total budget is 1). Hence, if W is a winning committee for E_p but W does not include p, then all its members were purchased at time $1/2$. It means that these candidates were approved by disjoints sets of voters, whose corresponding edges edges form a size-k matching. On the other hand, if p belongs to W then $W \setminus p$ is a winning size-$(k-1)$ committee for E, as in the above proof. $\square$

Corollary 4.4. *#*WINNING-COMMITTEES *is* $\#\mathrm{P}$*-hard and* $\#\mathrm{W}[1]$*-hard (for the parameterization by the committee size) for GreedyCCAV, GreedyPAV, Phragmén, and MEqS.*

The above result holds for MEqS because of its relation to Phragmén. For Phase 1 of MEqS, we have $\#\mathrm{P}$-hardness, but $\#\mathrm{W}[1]$-hardness so far remains elusive.

Theorem 4.5. *#*WINNING-COMMITTEES *is* $\#\mathrm{P}$*-hard for Phase 1 of MEqS.*

Proof. We give a reduction from #X3C, i.e., a counting version of the problem used in the proof of Theorem 3.4. Let E_{pd} be the same election as constructed in that proof, except for the following change: Group B_U contains $9n$ voters fewer and the $9n$ voters from V_S additionally approve the candidates from C_U. Consequently, the committee size decreases by $9n$ (because we maintain that the committee size is equal to the number of voters). Because of this change, when selecting the candidates from C_U, the budget of the voters from V_S drops to $1/(18n+6)$. Then, after the iterations where S-candidates are selected at per-voter cost of $1/(18n+6)$, no further S-candidates are selected (because the voters approving them have total budget lower than 1). As a consequence, Phase 1 of MEqS applied to election E_{pd} chooses all committees of the following forms:

1. Committees consisting of all candidates from $C_B \cup C_U \cup \{c_1, c_2\} \cup \{d\}$ and a subset of S-candidates such that all other S-candidates include at least one covered voter from $U' \cup U''$.
2. Committees consisting of all candidates from $C_B \cup C_U \cup \{c_1, c_2\} \cup \{p\}$ and a subset of S-candidates that correspond to an exact cover of U.

Next, we form election E_d identical to E_{pd} except that it does not include candidate p. For E_d, Phase 1 of MEqS selects all the committees of the first type above. Hence, to compute the number of solutions for our instance of #X3C, it suffices to subtract the number of committees selected by Phase 1 of MEqS for E_d from the number of committees selected by Phase 1 of MEqS for E_{pd}. This completes the proof. $\square$

5 Experiments

A'priori, it is not clear how frequent are ties in multiwinner elections. In this section we present experiments that show that they, indeed, are quite common, at least if one considers elections of moderate size.

5.1 Statistical Cultures and the Basic Experiment

Below we describe the statistical cultures that we use to generate elections (namely, the resampling model, the interval model, and PabuLib data) and how we perform our basic experiments.

Resampling Model [27]. We have two parameters, p and ϕ, both between 0 and 1. To generate an election with candidate set $C = \{c_1, \dots, c_m\}$ and with n voters, we first choose uniformly at random a central vote u approving exactly $\lfloor pm \rfloor$ candidates. Then, we generate the votes, for each considering the candidates independently, one by one. For a vote v and candidate c, with probability $1-\phi$ we copy c's approval status from u to v (i.e., if u approves c, then so does v; if u does not approve c then neither does v), and with probability ϕ we "resample" the approval status of c, i.e., we let v approve c with probability p (and disapprove it with probability $1-p$). On average, each voter approves about pm candidates.

Interval Model. In the Interval model, each voter and each candidate is a point on a $[0, 1]$ interval, chosen uniformly at random. Additionally, each candidate c has radius r_c and a voter v approves canidate c if the distance between their points is at most r_c. Intuitively, the larger the radius, the more appealing is a given candidate. We generate the radii of the candidates by taking a base radius r as input and, then, choosing each candidates' radius from the normal distribution with mean r and standard deviation $r/2$. Such spatial models are discussed in detail, e.g., by Enelow and Hinich [7, 8]. In the approval setting, they were recently considered, e.g., by Bredereck et al. [3] and Godziszewski et al. [13].

PabuLib Data. PabuLib is a library of real-life participatory budgeting (PB) instances, mostly from Polish cities [26]. A PB instance is a multiwinner election where the candidates (referred to as projects) have costs and the goal is to choose a "committee" of at most a given total cost. We restrict our attention to instances from Warsaw, which use approval voting, and we disregard the cost information (while this makes our data less realistic, we are not aware of other sources of real-life data for approval elections that would include sufficiently large candidate and voter sets). To generate an election with m candidates and n voters, we randomly select a Warsaw PB instance, remove all but m candidates with the highest approval score, and randomly draw n voters (with repetition, restricting our attention only to voters who approve at least one of the remaining candidates). We consider 120 PB instances from Warsaw that include at least 30 candidates (each of them includes at least one thousand votes, usually a few thousand).

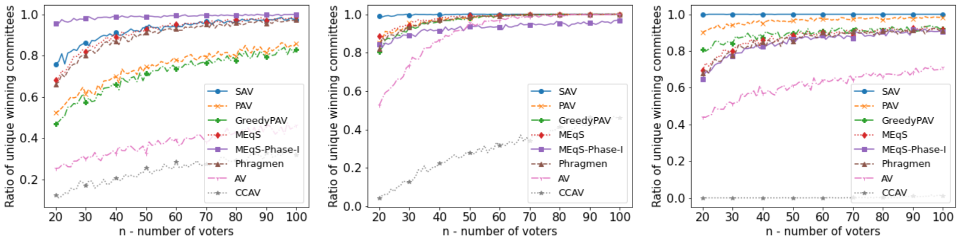
Basic Experiment. In a basic experiment we fix the number of candidates m, the committee size k, and a statistical culture. Then, for each number n of voters between 20 and 100 (with a step of 1) we generate 1000 elections with m candidates and n voters, and for each of them compute whether our rules have a unique winning committee (we omit GreedyCCAV). Then we present a figure that on the x axis has the number of voters and on the y axis has the fraction of elections that had a unique winning committee for a given rule. For AV and SAV, we use the algorithm from the beginning of Section 4, for sequential rules we use the FPT algorithm from Theorem 3.5, and for CCAV and PAV we use the ILP-based approach, with a solver that provides multiple solutions.

5.2 Results

All our experiments regard 30 candidates and committee size 5 (in the appendix we also consider 50 and 100 candidates, and committee size 10; general results are analogous). First, we performed three basic experiments for the resampling model with the parameter p (approval probability) set so that, on average, each voter approved either $k/2$, k, or $2k$ candidates. We used $\phi = 0.75$ (according to the results of Szufa et al. [27], this value gives elections that resemble the real-life ones). We present the results in the top row of Figure 1. Next, we also performed two basic experiments for the Interval model (with the base radius selected so that, on average, each voter approved either $k/2$ or k candidates), and with the PabuLib data (see the second row of Figure 1). These experiments support the following general conclusions.

First, for most scenarios and for most of our rules, there is a nonnegligible probability of having a tie (where, depending on the rule and the number of voters, this probability may be as low as 5% or as high as nearly 100%). This justifies why one needs to be ready to detect and handle ties in moderately sized multiwinner elections.

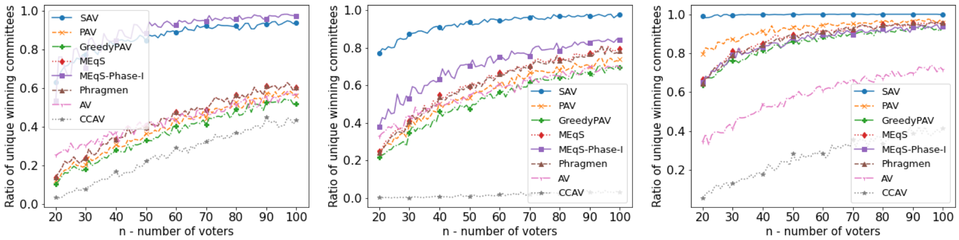
Second, we see that SAV generally leads to fewest ties, CCAV leads to most, and AV often holds a strong second position in this category (in the sense that it also leads to a high probability of having a tie in many settings). The other rules are in between. Phase 1 of MEqS often has significantly fewer ties than the other rules, but full version of MEqS does not stand out. PAV occasionally leads to fewer ties (in particular, on PabuLib data and on the resampling model with $2k$ approvals per vote).



(a) $m = 30$, $k = 5$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$

(b) $m = 30$, $k = 5$,
k approvals/vote
resampling model, $\phi = 0.75$

(c) $m = 30$, $k = 5$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$



(d) $m = 30$, $k = 5$,
$k/2$ approvals/vote
Interval

(e) $m = 30$, $k = 5$,
k approvals/vote
Interval

(f) $m = 30$, $k = 5$,
PabuLib (Warsaw)

Figure 1: Results of our experiments. By "$k/2$ approvals/vote" we mean that on average a single vote contains approximately $k/2$ approvals (the meaning of k and $2k$ is analogous).

6 Summary

We have shown that, in general, detecting ties in multiwinner elections is intractable, but doing so for moderately-sized ones is perfectly possible. Our experiments show that ties in such elections are a realistic possibility and one should be ready to handle them. Intractability of counting winning committees suggests that tie-breaking by sampling committees may not be feasible. Looking for fair tie-breaking mechanisms is a natural follow-up research direction.

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A The Proof of Theorem 3.2

Membership in coNP is clear: Given an election and committee size, we run the greedy algorithm breaking the ties arbitrarily, and we compute some winning committee W. Then, we rerun the same algorithm nondeterministically, at each internal tie trying each possible choice; if a given computation completes with a committee different than W then it rejects (and the whole computation rejects; indeed, we found two different winning committees) and otherwise it accepts (if all paths accept, then the whole computation accepts; indeed, all ways of handling the internal ties lead to the same final committee). In the following, we focus on showing coNP-hardness.

Let $\delta_1 = w(1) - w(0)$, $\delta_2 = w(2) - w(1)$, and $\delta_3 = w(3) - w(2)$. For example, for w_{PAV} we would have $\delta_1 = 1$, $\delta_2 = \frac{1}{2}$ and $\delta_3 = \frac{1}{3}$. By our assumptions on weight functions, we know that (a) these numbers are rational, (b) $\delta_1 = 1$ (but we will not use this), and that (c) $\delta_1 \geq \delta_2 \geq \delta_3$. We additionally assume that $\delta_1 - \delta_2 > \delta_2 - \delta_3$, but later we will show how to relax this assumption.

We give a reduction from INDEPENDENT SET to the complement of Greedy-f-UNIQUE-COMMITTEE. Our input consists of a graph G, where $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$, and an integer k. The question is if there are k vertices in $V(G)$ that are not connected by an edge. Without loss of generality, we assume that G is 3-regular, i.e., each vertex touches exactly three edges [12]. Let α be a positive integer such that $\alpha\delta_1$ and $\alpha\frac{\delta_1-\delta_2}{\delta_1}$ are integers and $\alpha(\delta_1 - \delta_2) > \delta_1$. We fix values $t = \alpha(nmk)^3$, $T = 10\alpha(nmk)^6$, and $D = \beta T^{10}$, where β is the smallest positive integer greater than $\frac{\delta_1}{\delta_1-\delta_2}$; while we could choose smaller ones, these suffice.

We form an election where the candidate set is $V(G) \cup \{p, d\}$ and we have the following three groups of voters:

1. For each edge $e_\ell = \{v_i, v_j\}$, we have t voters with approval set $\{v_i, v_j, d\}$.
2. For each candidate v_i we have $D + T^3 + ((m-1)n+1)T - 3t$ voters with approval set $\{v_i\}$, and for each pair of distinct candidates v_i and v_j we have T voters with approval set $\{v_i, v_j\}$.
3. We have $D + T^3 + nmT + 1 - \frac{(\delta_1-\delta_2)}{\delta_1}kT - mt$ voters with approval set $\{p, d\}$, $\frac{3(\delta_1-\delta_2)}{\delta_1}kt$ voters who approve d, and mt voters who approve p.

We let the committee size be $n + 1$. We claim that if G contains an independent set of size k then there are two Greedy-f winning committees, $V(G) \cup \{d\}$ and $V(G) \cup \{p\}$, and otherwise there is only one, $V(G) \cup \{d\}$. The proof follows.

Let $X = \delta_1 D + \delta_1 T^3 + \delta_1 nmT$. Prior to the first iteration of Greedy-f, each candidate v_i has score X, candidate d has score: $X - (\delta_1 - \delta_2)kT + 3(\delta_1 - \delta_2)kt + \delta_1$, and candidate p has score: $X - (\delta_1 - \delta_2)kT + \delta_1$. During the first k iterations, Greedy-f selects some k candidates from $V(G)$. This is so, because whenever some candidate v_i is selected, the scores of the remaining members of $V(G)$ decrease by $(\delta_1 - \delta_2)T$ due to the voters in the second group, and by at most $(\delta_1 - \delta_2)t$, due to the voters in the first group. Hence, after the first $k - 1$ iterations each remaining candidate from $V(G)$ has score at least $X - (\delta_1 - \delta_2)(k-1)T - (\delta_1 - \delta_2)(k-1)t$, which—by our choices of α, t, and T—is larger than the scores that both p and d had even prior to the first iteration (note that the scores of the candidates cannot increase between iterations). On the other hand, after the k-th iteration, each remaining member of $V(G)$ has score at most $X - (\delta_1 - \delta_2)kT$, which is less than p has (since p is only approved by voters who do not approve members of $V(G)$, at this point his or her score is the same as prior to the first iteration). As a consequence, in the $(k+1)$-st iteration Greedy-f either chooses p or d. Let us now analyze which one of them.

Let S be the set of candidates from $V(G)$ selected in the first k iterations. If S forms an independent set, then prior to the $(k+1)$-st iteration, the score of d is $X - (\delta_1 - \delta_2)kT + \delta_1$. This is so, because for each candidate v_i in S, d loses exactly $3(\delta_1 - \delta_2)t$ points due to the voters in the first group that correspond to the three edges that include v_i (since S is an independent set, for each member of S these are different three edges). In this case, Greedy-f is free to choose either among p and d. However, if S is not an independent set, then the score of d drops by at most $(3k-1)(\delta_1 - \delta_2)t + (\delta_2 - \delta_3)t$. This is so, because S contains at least two candidates v_i and v_j that are connected by an edge; when the second one of them is included in the committee, then the score of d drops by at most $2(\delta_1 - \delta_2) + (\delta_2 - \delta_3)$. In this case Greedy-$f$ is forced to select d in the $(k+1)$-st iteration. In the following $n-k$ iterations, f selects the remaining members of $V(G)$ (after either p or d is selected in the $(k+1)$-st iteration, the score of the other one drops so much that he or she cannot be selected; this is due to the D voters who approve $\{p, d\}$).

It remains to observe that if G contains an independent set of size k, then Greedy-f can choose its members in the first k iterations. This is the case, because whenever Greedy-f chooses a member of the independent set, then the score of its other members never drops more than the score of the other remaining vertex candidates. Hence, if G has a size-k independent set, then, due to the parallel-universes tie-breaking, Greedy-f outputs two winning committees, $V(G) \cup \{p\}$ and $V(G) \cup \{d\}$. Otherwise we have a unique winning committee $V(G) \cup \{d\}$. This completes the proof for the case that $\delta_1 - \delta_2 > \delta_2 - \delta_3$.

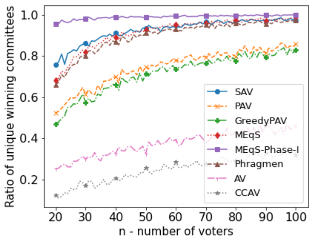
Let us now consider the case where $\delta_1 - \delta_2 \leq \delta_2 - \delta_3$. Let $\delta_4 = w(4) - w(3)$, $\delta_5 = w(5) - w(4)$, and so on. If there is some positive integer t such that $\delta_{t+1} - \delta_{t+2} > \delta_{t+2} - \delta_{t+3}$ then it suffices to use the same reduction as above, extended so that we have candidates $d_1, \ldots, d_t$ that are approved by every voter and the committee size is increased by t. Greedy-f will choose these t candidates in the first t iterations and then it will continue as described in the reduction, with $\delta_{t+1}, \delta_{t+2}$, and δ_{t+3} taking the roles of δ_1, δ_2, and δ_3. In fact, such a t must exist. Otherwise, if $\delta_{t+1} - \delta_{t+2} \leq \delta_{t+2} - \delta_{t+3}$ for every t then either f is AV (which we assumed not to be the case) or w is not nondecreasing, which is forbidden by definition.

B Explanation for Corollary 3.3

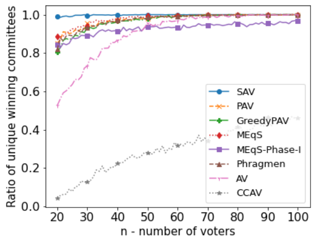
The results for GreedyCCAV and GreedyPAV follow directly from the preceding theorem. For Phragmén, Faliszewski et al. [10] have shown that the following problem, known as Phragmén-ADD-ROBUSTNESS-RADIUS, is NP-complete: Given an election E, committee size k, and number B, is it possible to add at most B approvals to the votes so that the winning committee under the resolute variant of the Phragmén rule (where all internal ties are resolved according to a given tie-breaking order) changes. Their proof works in such a way that adding approvals only affects how ties are broken. Hence, effectively, it also shows that UNIQUE-COMMITTEE is coNP-complete for the (non-resolute) variant of Phragmén.

C Additional Figures

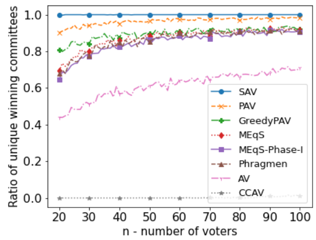
In this section we show several additional sets of basic experiments for the resampling model, the disjoint model, and the interval model.



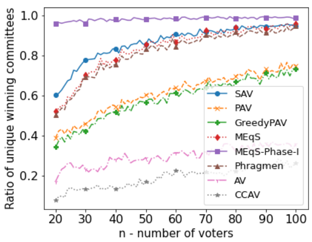
(a) $m = 30$, $k = 5$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$



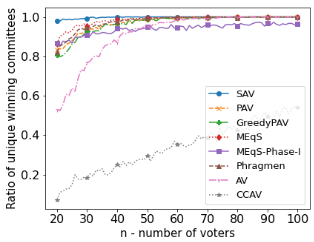
(b) $m = 30$, $k = 5$,
k approvals/vote
resampling model, $\phi = 0.75$



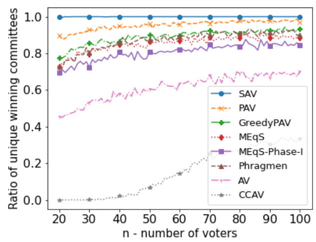
(c) $m = 30$, $k = 5$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$



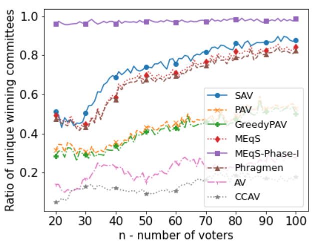
(d) $m = 50$, $k = 5$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$



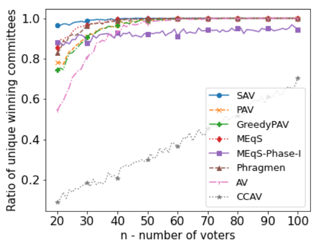
(e) $m = 50$, $k = 5$,
k approvals/vote
resampling model, $\phi = 0.75$



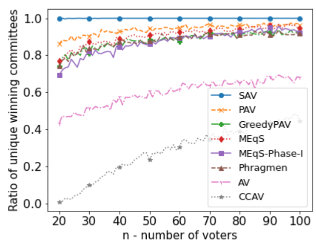
(f) $m = 50$, $k = 5$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$



(g) $m = 100$, $k = 5$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$

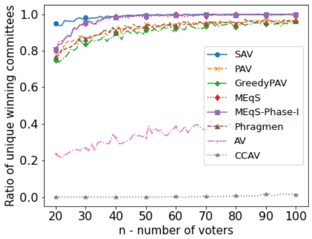


(h) $m = 100$, $k = 5$,
k approvals/vote
resampling model, $\phi = 0.75$

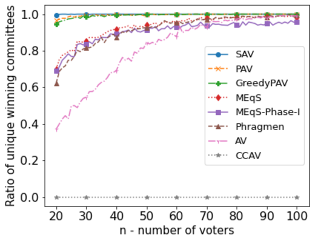


(i) $m = 100$, $k = 5$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$

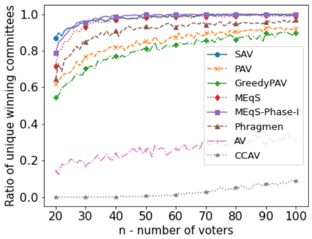
Figure 2: Results for the resampling model with committee size $k = 5$ and different numbers of candidates.



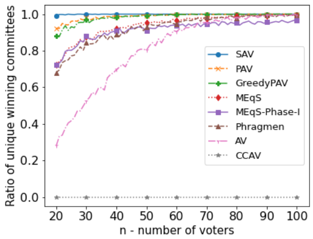
(a) $m = 30$, $k = 10$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$



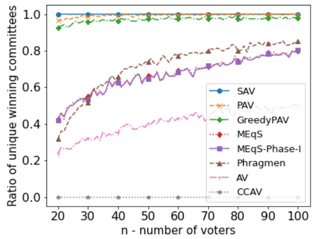
(b) $m = 30$, $k = 10$,
k approvals/vote
resampling model, $\phi = 0.75$



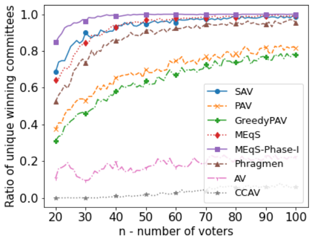
(d) $m = 50$, $k = 10$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$



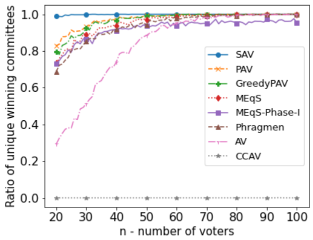
(e) $m = 50$, $k = 10$,
k approvals/vote
resampling model, $\phi = 0.75$



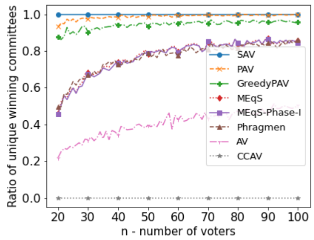
(f) $m = 50$, $k = 10$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$



(g) $m = 100$, $k = 10$,
$k/2$ approvals/vote,
resampling model, $\phi = 0.75$

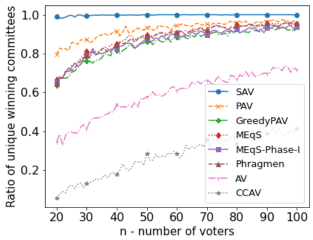


(h) $m = 100$, $k = 10$,
k approvals/vote
resampling model, $\phi = 0.75$

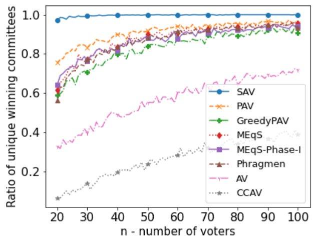


(i) $m = 100$, $k = 10$,
$2k$ approvals/vote
resampling model, $\phi = 0.75$

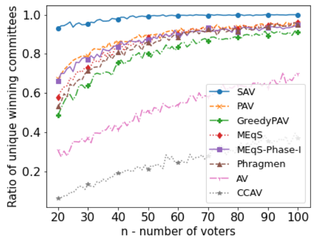
Figure 3: Results for the resampling model with committee size $k = 10$ and different numbers of candidates.



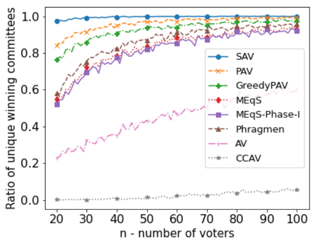
(a) $m = 30$, $k = 5$,
PabuLib (Warsaw)



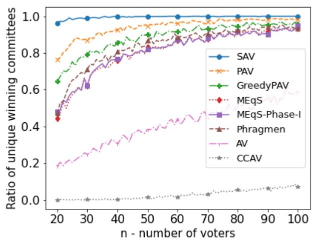
(b) $m = 50$, $k = 5$,
PabuLib (Warsaw)



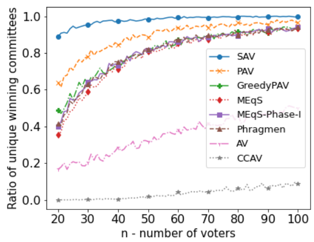
(c) $m = 100$, $k = 5$,
PabuLib (Warsaw)



(d) $m = 30$, $k = 10$,
PabuLib (Warsaw)

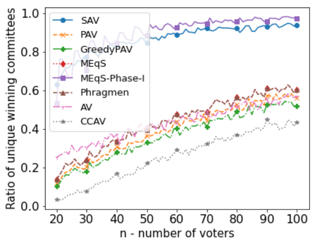


(e) $m = 50$, $k = 10$,
PabuLib (Warsaw)

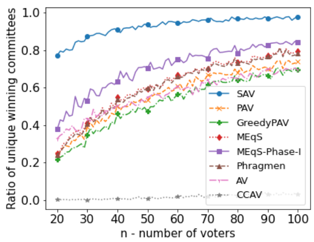


(f) $m = 100$, $k = 10$,
PabuLib (Warsaw)

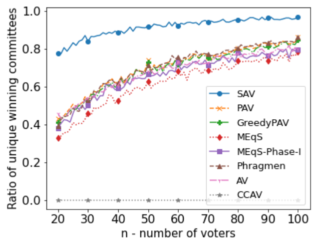
Figure 4: Results for PabuLib data with committee sizes $k = 5$ (top row) and $k = 10$ (bottom row), and different numbers of candidates.



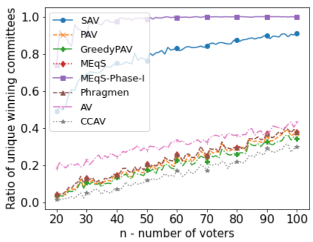
(a) $m = 30$, $k = 5$,
$k/2$ approvals/vote
Interval



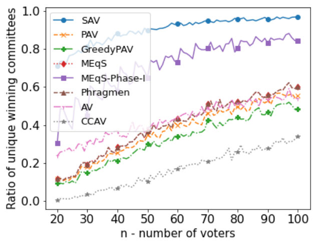
(b) $m = 30$, $k = 5$,
k approvals/vote
Interval



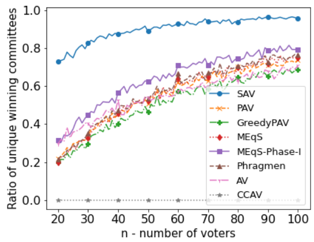
(c) $m = 30$, $k = 5$,
$2k$ approvals/vote
Interval



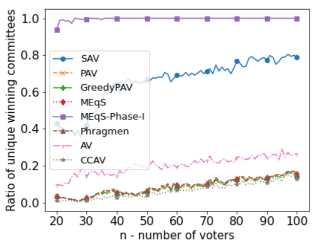
(d) $m = 50$, $k = 5$,
$k/2$ approvals/vote
Interval



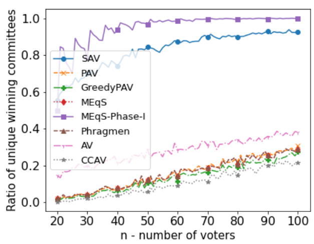
(e) $m = 50$, $k = 5$,
k approvals/vote
Interval



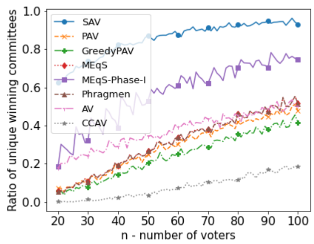
(f) $m = 50$, $k = 5$,
$2k$ approvals/vote
Interval



(g) $m = 100$, $k = 5$,
$k/2$ approvals/vote
Interval

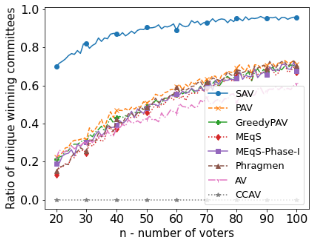


(h) $m = 100$, $k = 5$,
k approvals/vote
Interval

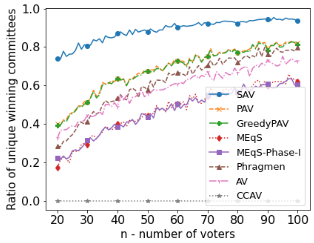


(i) $m = 100$, $k = 5$,
$2k$ approvals/vote
Interval

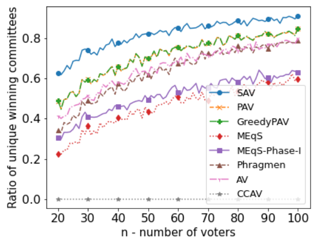
Figure 5: Results for the Interval model with committee size $k = 5$ and different numbers of candidates.



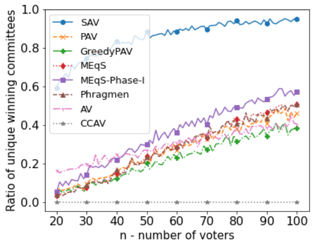
(a) $m = 30$, $k = 10$,
$k/2$ approvals/vote
Interval



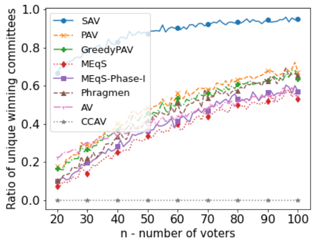
(b) $m = 30$, $k = 10$,
k approvals/vote
Interval



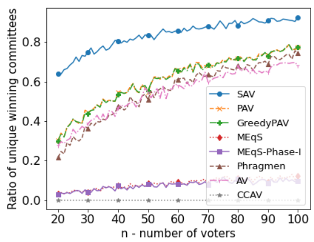
(c) $m = 30$, $k = 10$,
$2k$ approvals/vote
Interval



(d) $m = 50$, $k = 10$,
$k/2$ approvals/vote
Interval



(e) $m = 50$, $k = 10$,
k approvals/vote
Interval



(f) $m = 50$, $k = 10$,
$2k$ approvals/vote
Interval

Figure 6: Results for the Interval model with committee size $k = 10$ and different numbers of candidates.