Relaxed Notions of Condorcet-Consistency and Efficiency for Strategyproof Social Decision Schemes

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Abstract

Social decision schemes (SDSs) map the preferences of a group of voters over some set of m alternatives to a probability distribution over the alternatives. A seminal characterization of strategyproof SDSs by Gibbard (1977) implies that there are no strategyproof Condorcet extensions and that only random dictatorships satisfy *ex post* efficiency and strategyproofness. The latter is known as the *random dictatorship theorem*. We relax Condorcet-consistency and *ex post* efficiency by introducing a lower bound on the probability of Condorcet winners and an upper bound on the probability of Pareto-dominated alternatives, respectively. We then show that *(i)* strategyproof SDSs can guarantee the Condorcet winner a probability of at most $2/m$, *(ii)* the SDS that assigns probabilities proportional to Copeland scores is the only anonymous, neutral, and strategyproof SDS that achieves this bound, and *(iii)* every strategyproof SDS that assigns Pareto-dominated alternatives less than $1/m$ probability will be dictatorial with positive probability. The last statement can be formulated as a continuous strengthening of Gibbard's random dictatorship theorem: the less probability we put on Pareto-dominated alternatives, the closer to random dictatorship is the resulting SDS.

1 Introduction

A pervasive phenomenon when making collective decisions is strategic manipulation: voters may be better off by lying about their preferences than reporting them truthfully. This is problematic for a number of reasons. For one, spending resources on finding out other voters' preferences and identifying beneficial manipulations is rewarded. These resources are typically not spread evenly across society. Perhaps more importantly, when a voting rule is manipulable, all of its desirable properties are in doubt because they were shown to hold under the assumption that all voters submit their preferences truthfully.

Unfortunately, Gibbard (1973) and Satterthwaite (1975) have shown independently that the only non-imposing social choice functions that are immune to strategic manipulation are dictatorships. Here, social choice functions are defined as mappings from ordinal preferences of the voters over some set of alternatives to a single collectively most-preferred alternative. Dictatorial social choice functions invariably return the most preferred alternative of one fixed voter, the dictator, and are therefore unacceptable for most applications. A natural question is whether more positive results can be obtained when allowing for randomization. Gibbard (1977) hence introduced *social decision schemes* (SDSs), which map ordinal preferences of the voters to a lottery over the alternatives and defined SDSs to be *strategyproof* if no voter can obtain more expected utility for any utility representation that is consistent with his ordinal preference relation. He then gave a complete characterization of strategyproof SDSs in terms of convex combinations of two types of restricted SDSs, so-called unilaterals and duples. An important consequence of Gibbard's characterization, known as the *random dictatorship theorem*, is that all *ex post* efficient and strategyproof SDSs are *random dictatorships*. Random dictatorships are convex combinations of dictatorships, i.e., each voter is selected with some fixed probability and the top choice of this voter is implemented as the collective decision.

While Gibbard's result may seem like an extension of the Gibbard-Satterthwaite theorem to the randomized context, it is in fact much more positive. In contrast to deterministic dictatorships, the uniform random dictatorship, in which every agent is picked with the same probability, enjoys a high degree of fairness and is in fact used in many subdomains of social choice that are concerned with the fair assignment of objects to agents (see, e.g., Abdulkadiroğlu and Sönmez, 1998; Che and Kojima, 2010). Nevertheless, random dictatorships are rather restricted. Since they only depend on the top choice of each voter, they cannot satisfy many desirable properties from the social choice literature such as *Condorcet-consistency*, which demands that an alternatives that beats every other alternative in pairwise majority comparisons should be selected with probability 1. Gibbard's theorem has been the point of departure for a lot of follow-up work on strategyproof SDSs. Apart from a number of alternative proofs of the theorem (e.g., Duggan, 1996; Nandeibam, 1997; Tanaka, 2003), there have been extensions with respect to manipulations by groups (Barberà, 1979a), cardinal preferences (e.g., Hylland, 1980; Dutta et al., 2007; Nandeibam, 2013), weaker notions of strategyproofness (e.g., Benoît, 2002; Sen, 2011; Aziz et al., 2018; Brandl et al., 2018), and restricted domains of preferences (e.g., Dutta et al., 2002; Chatterji et al., 2014).

The question we pursue in this paper is which strategyproof SDSs satisfy relaxations of two classic conditions: Condorcet-consistency and *ex post* efficiency. To this end, we say that an SDS is α-*Condorcet-consistent* if a Condorcet winner always receives a probability of at least α and β-ex post *efficient* if a Pareto-dominated alternative always receives a probability of at most β. Furthermore, we define that an SDS is γ-*randomly dictatorial* if it can be represented as a convex combination of two strategyproof SDSs, one of which is a random dictatorship that will be selected with probability γ. All of these axioms are discussed in more detail in Section 2.2. Building on an alternative characterization of strategyproof SDSs by Barberà (1979b), we then show the following results (m is the number of alternatives and n the number of voters):

- Let $n \geq 3$ and $\alpha > \frac{2}{m}$. Then, there is no strategyproof SDS that satisfies α-Condorcet-consistency.
- The *randomized Copeland rule*, which assigns probabilities proportional to Copeland scores, is the only strategyproof SDS that satisfies anonymity, neutrality, and $\frac{2}{m}$-Condorcet-consistency when $m, n \geq 3$.
- Let $m \geq 3$ and $\beta < \frac{1}{m}$. Then, there is no strategyproof SDS that is β-*ex post* efficient and 0-randomly dictatorial.
- Let $0 \leq \epsilon \leq 1$. Every strategyproof SDS that is $\frac{1-\epsilon}{m}$-*ex post* efficient is γ-randomly dictatorial for $\gamma \geq \epsilon$.

The first two statement characterize the randomized Copeland rule as the "most" Condorcet-consistent strategyproof SDS. The third statement highlights the robustness of random dictatorships: even when relaxing *ex post* efficiency, we are still stuck with SDSs that have a "random dictatorship component". In other words, all SDSs that have *no* random dictatorship component are as "inefficient" as the SDS that always returns a uniform lottery over all alternatives. The last statement, which is a corollary of the third statement, can be interpreted as a continuous strengthening of Gibbard's random dictatorship theorem: the less probability we put on Pareto-dominated alternatives, the more randomly dictatorial is the resulting SDS.

2 The model

Let $N = \{1, 2, \dots, n\}$ be a finite set of voters and let $A = \{a, b, \dots\}$ be a finite set of m alternatives. Every voter i has a *(strict) preference relation* $\succ_i$, which is an anti-symmetric, complete, and transitive binary relation on A. We write $x \succsim_i y$ if voter i prefers x weakly to y and $x \succ_i y$ if voter i prefers x strictly to y. The set of all preference relations is denoted by $\mathcal{R}$. A *preference profile* $R \in \mathcal{R}^n$ is a n-tuple that contains the preference relation of each voter $i \in N$. We represent preference profiles as tables in which each column represents a preference relation and the number above the columns indicates the number of voters who report the corresponding preference relation (see Figure 1 for example). Furthermore, let $n_{xy}(R) = |\{i \in N : x \succ_i y\}|$ denote the *supporting size* for x against y in the preference profile R.

Given a preference profile, we are interested in the winning chance of each alternative. We therefore analyze social decision schemes (SDSs), which map each preference profile to a lottery over the alternatives. A *lottery* p is a probability distribution over the set of alternatives A, i.e., it assigns each alternative x a probability $p(x) \geq 0$ such that $\sum_{x \in A} p(x) = 1$. The set of all lotteries over A is denoted by $\Delta(A)$. Formally, a *social decision scheme (SDS)* is a function $f : \mathcal{R}^n \to \Delta(A)$. We denote with $f(R, x)$ the probability assigned to alternative x by f at the preference profile R.

Since there is a huge number of SDSs, we now discuss axioms formalizing desirable properties of SDSs. Two basic fairness conditions are anonymity and neutrality. Anonymity requires that every voter is treated equally. Formally, an SDS f is *anonymous* if $f(R) = f(\pi(R))$ for all preference profiles R and permutations $\pi : N \to N$. Note that $R' = \pi(R)$ denotes the profile with $R'_{\pi(i)} = R_i$ for all voters $i \in N$. Neutrality is a fairness axiom that guarantees that alternatives are treated equally. This idea can again be formalized using permutations: an SDS f is *neutral* if $f(R, x) = f(\tau(R), \tau(x))$ for all preference profiles R and permutations $\tau : A \to A$. Here, $R' = \tau(R)$ is the profile derived by permuting the alternatives in R according to τ, i.e, $\tau(x) \succsim'_i \tau(y)$ if and only if $x \succsim_i y$ for all alternatives $x, y \in A$ and voters $i \in N$.

2.1 Stochastic Dominance and Strategyproofness

In this paper, we investigate strategyproof SDSs, i.e., social decision schemes in which voters cannot benefit by misrepresenting their preferences. Unfortunately, it is not obvious in which situations a voter benefits from misrepresenting his preferences because he only reports his ordinal preferences over alternatives while the outcome of the SDS is a lottery. We use the concept of stochastic dominance to let agents compare two lotteries with each other: a voter i (weakly) prefers a lottery p to another lottery q, written as $p \succsim_i q$, if $\sum_{y \in A : y \succ_i x} p(y) \geq \sum_{y \in A : y \succ_i x} q(y)$ for every alternative $x \in A$. Less formally, a voter prefers a lottery p weakly to a lottery q if, for every alternative $x \in A$, p returns a better alternative than x with as least as much probability as q. Note that stochastic dominance is no complete order on the set of lotteries, i.e., there are lotteries p and q such that a voter i neither prefers p to q nor q to p.

Based on stochastic dominance, we can now formalize strategyproofness. An SDS f is *strategyproof* if $f(R) \succsim_i f(R')$ for all preference profiles R and R' and voters $i \in N$ such that $R_j = R'_j$ for all $j \in N \setminus \{i\}$. This means that an SDS is strategyproof if every voter prefers the lottery obtained by voting truthfully to any lottery that he could obtain by voting dishonestly. Conversely, we call an SDS f *manipulable* if it is not strategyproof.

While there are other ways to compare lotteries with each other, stochastic dominance is the most common one (see, e.g, Gibbard, 1977; Barberà, 1979b; Bogomolnaia and Moulin, 2001; Ehlers et al., 2002; Aziz et al., 2018). This is mainly due to the following reason: if

$p \succsim_i q$, the expected utility of p is at least as high as the expected utility of q for every vNM utility function that is ordinally consistent with voter i's preferences. Hence, if an SDS is strategyproof, no voter can manipulate regardless of his exact utility function (see, e.g., Sen, 2011; Brandl et al., 2018). This observation immediately implies that the *convex combination* $h = \lambda f + (1 - \lambda)g$ (for some $\lambda \in [0, 1]$) of two strategyproof SDSs f and g is again strategyproof: if a manipulator obtains more expected utility for $h(R')$ than for $h(R)$, then he has to prefer $f(R')$ to $f(R)$, or $g(R')$ to $g(R)$.

Two particularly important results about strategyproof SDSs are due to Gibbard (1977) and Barberà (1979b). Gibbard (1977) shows that every strategyproof SDS can be represented as a convex combination of unilaterals and duples.[1] The terms "unilaterals" and "duples" refer here to special classes of SDSs: a *unilateral* is a strategyproof SDS that only depends on the preferences of a single voter i, i.e., $f(R) = f(R')$ for all preference profiles R and R' such that $R_i = R'_i$. A *duple*, on other hand, is a strategyproof SDS that can only choose between two alternatives x and y, i.e., $f(R, z) = 0$ for all preference profiles R and alternatives $z \in A \setminus \{x, y\}$.

Theorem 1 (Gibbard, 1977)**.** *An SDS is strategyproof if and only if it can be represented as a convex combination of unilaterals and duples.*

Note that we define duples and unilaterals as strategyproof SDSs. Thus, Theorem 1 only states that strategyproof SDSs can be decomposed into a mixture of strategyproof SDSs, each of which must be of a special type. In order to circumvent this restriction, Gibbard proves another characterization of strategyproof SDSs.

Theorem 2 (Gibbard, 1977)**.** *An SDS is strategyproof if and only if it is non-perverse and localized.*

Non-perversity and localizedness are two axioms describing the behavior of an SDS. In more detail, non-perversity—which is now often referred to as monotonicity—requires that reinforcing an alternative should not reduce its probability. For a formal definition, we denote with $R^{i:yx}$ the profile derived from R by only reinforcing y against x. Note that this requires that $x \succ_i y$ and that there is no alternative $z \in A$ such that $x \succ_i z \succ_i y$ in R. Then, an SDS f is *non-perverse* if $f(R^{i:yx}, y) \geq f(R, y)$ for all preference profiles R, voters $i \in N$, and alternatives $x, y \in A$. Moreover, an SDS is localized if changes in the preferences of voters only affect the probabilities of the alternatives that are involved in these changes. Formally, an SDS f is called *localized* if $f(R^{i:yx}, z) = f(R, z)$ for all preference profiles R, voters $i \in N$, and distinct alternatives $x, y, z \in A$. Together, Theorem 1 and Theorem 2 show that each strategyproof SDS can be represented as a mixture of unilaterals and duples, each of which is non-perverse and localized.

Since Gibbard's results can be quite difficult to work with, we now state another characterization of strategyproof SDSs due to Barberà (1979b). Barberà has shown that every strategyproof SDS that satisfies anonymity and neutrality can be represented as a convex combination of a supporting size SDS and a point voting SDS. A *point voting SDS* is defined by a scoring vector $(a_1, a_2, \dots, a_m)$ that satisfies $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ and $\sum_{i \in \{1,\dots,m\}} a_i = \frac{1}{n}$. The probability assigned to an alternative x by a point voting SDS f is $f(R, x) = \sum_{i \in N} a_{|\{y \in A : y \succsim_i x\}|}$. Furthermore, *supporting size SDSs* also rely on a scoring vector $(b_n, b_{n-1}, \dots, b_0)$ with $b_n \geq b_{n-1} \geq \dots \geq b_0 \geq 0$ and $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{0, \dots, n\}$ to compute the outcome. The probability assigned to an alternative x by a supporting size SDS f is then $f(R, x) = \sum_{y \in A \setminus \{x\}} b_{n_{xy}(R)}$. Note that point voting SDSs can be seen as a generalization of (deterministic) positional scoring rules and supporting size SDSs can be seen as a variant of Fishburn's C2 functions (Fishburn, 1977).

[1] In order to simplify the exposition, we slightly modified Gibbard's terminology by requiring that duples and unilaterals have to be strategyproof.

Theorem 3 (Barberà, 1979b). *An SDS is anonymous, neutral, and strategyproof if and only if it can be represented as a convex combination of a point voting SDS and a supporting size SDS.*

Many well-known SDSs can be represented as point voting SDSs or supporting size SDSs. For example, the *uniform random dictatorship* f_{RD}, which chooses one voter uniformly at random and returns his best alternative, can be formalized as the point voting SDS with the scoring vector $\left(\frac{1}{n}, 0, \dots, 0\right)$. An instance of a supporting size SDS is the *randomized Copeland rule* f_C, which assigns probabilities proportional to the Copeland scores $c(x, R) = |\{y \in A \setminus \{x\} \colon n_{xy}(R) > n_{yx}(R)\}| + \frac{1}{2}|\{y \in A \setminus \{x\} \colon n_{xy}(R) = n_{yx}(R)\}|$. This SDS is the supporting size SDS defined by the vector $b = (b_n, b_{n-1}, \dots, 0)$, where $b_i = \frac{2}{m(m-1)}$ if $i > \frac{n}{2}$, $b_i = \frac{1}{m(m-1)}$ if $i = \frac{n}{2}$, and $b_i = 0$ otherwise. Furthermore, there are SDSs that can be represented both as point voting SDSs and supporting size SDSs. An example is the *randomized Borda rule* f_B, which randomizes proportional to the Borda scores of the alternatives. This SDS is the point voting SDS defined by the scoring vector $\left(\frac{2(m-1)}{nm(m-1)}, \frac{2(m-2)}{nm(m-1)}, \cdots, \frac{2}{nm(m-1)}, 0\right)$ and equivalently the supporting size SDS defined by the scoring vector $\left(\frac{2n}{nm(m-1)}, \frac{2(n-1)}{nm(m-1)}, \cdots, \frac{2}{nm(m-1)}, 0\right)$. The randomized Copeland rule as well as the randomized Borda rule were rediscovered several times by authors who were apparently unaware of Barberà's work (see Heckelman, 2003; Conitzer and Sandholm, 2006; Procaccia, 2010; Heckelman and Chen, 2013).

2.2 Relaxing Classic Axioms

The goal of this paper is to analyze the effectiveness of strategyproof SDSs by relaxing classic axioms from social choice theory. In more detail, we investigate how much probability can be guaranteed to Condorcet winners and how little probability must be assigned to Pareto-dominated alternatives by strategyproof SDSs. These ideas are formalized using α-Condorcet-consistency and β-*ex post* efficiency, which are introduced in the sequel.

Let us first consider β-*ex post* efficiency, which is based on Pareto-dominance. An alternative x *Pareto-dominates* another alternative y in a preference profile R if $x \succ_i y$ for all $i \in N$. Clearly, a Pareto-dominated alternative y should have no chance of winning because every voter agrees that its dominator x is a better choice. This is formalized by *ex post efficiency*, which requires that $f(R, x) = 0$ for all preference profiles R and alternatives x such that x is Pareto-dominated in R. As first shown by Gibbard, random dictatorships are the only strategyproof SDSs that satisfy *ex post* efficiency. These SDSs choose each voter with a fixed probability and return his best alternative as winner. However, this result, which is often called random dictatorship theorem, breaks down once we allow that Pareto-dominated alternatives can have a non-zero chance of winning $\beta > 0$. This follows by considering two SDSs d and g such that d is a random dictatorship and g is another strategyproof SDS. Then, the SDS $f^* = (1 - \beta)d + \beta g$ is strategyproof for every $\beta \in (0; 1)$ and no random dictatorship, but assigns a probability of at most β to Pareto-dominated alternatives. We call the last property β-*ex post* efficiency: an SDS f is β-*ex post efficient* if $f(R, x) \leq \beta$ for all preference profiles R and alternatives x that are Pareto-dominated in R.

A natural generalization of the random dictatorship theorem is to ask which strategyproof SDSs satisfy β-*ex post* efficiency for small values of β. If β is sufficiently small, β-*ex post* efficiency may be quite acceptable. As we show, the random dictatorship theorem is quite robust in the sense that all SDSs that satisfy β-*ex post* efficiency for $\beta < \frac{1}{m}$ are similar to random dictatorships. In order to formalize this observation, we introduce γ-randomly dictatorial SDSs: an SDS f is γ-*randomly dictatorial* if $\gamma \in [0, 1]$ is the maximal value such

1	1	1
a	b	c
c	c	a
b	a	b

R

1	1	1
a	b	c
b	c	a
c	a	b

R'

Figure 1: Condorcet-consistent SDSs violate strategyproofness when $m = n = 3$. Due to the symmetry of R', we may assume without loss of generality that $f(R', a) > 0$. Since f is Condorcet-consistent, it holds that $f(R, c) = 1$. Thus, voter 1 can manipulate by swapping c and b in R, resulting in profile R', because $f(R', a) = \sum_{x \succ_1 c} f(R', x) > \sum_{x \succ_1 c} f(R, x) = f(R, a) = 0$.

that f can be represented as $f = \gamma d + (1-\gamma) g$, where d is a random dictatorship and g is another strategyproof SDS. Note that this definition entails that γ-randomly dictatorial SDSs are strategyproof since they are convex combinations of strategyproof SDSs. In particular, we require that g is strategyproof as otherwise, SDSs that seem "non-dictatorial" are not 0-randomly dictatorial. For instance, the uniform lottery, which always assigns probability $\frac{1}{m}$ to all alternatives, is not 0-randomly dictatorial if g is not required to be strategyproof. Moreover, it should be mentioned that the maximality of γ implies that g is 0-randomly dictatorial. Otherwise, we could also represent g as a mixture of a random dictatorship and some other strategyproof SDS h, which means that f is γ'-randomly dictatorial for $\gamma' > \gamma$.

For a better understanding of γ-randomly dictatorial SDSs, we first provide a characterization of these SDSs. Recall for the following lemma that $R^{i:yx}$ denotes the profile derived from R by only reinforcing y against x in voter i's preference.

Lemma 1. *A strategyproof SDS f is γ-randomly dictatorial if and only if there are non-negative values $\gamma_1, \dots, \gamma_n$ that satisfy the following conditions:*

i) $\sum_{i \in N} \gamma_i = \gamma$.

ii) $f(R^{i:yx}, y) - f(R, y) \geq \gamma_i$ *for all alternatives $x, y \in A$, voters $i \in N$, and preference profiles R in which voter i prefers x the most and y the second most.*

iii) for every voter $i \in N$ there are alternatives $x, y \in A$ and a profile R such that voter i prefers x the most and y the second most in R, and $f(R^{i:yx}, y) - f(R, y) = \gamma_i$.

Due to space restrictions, we defer the proof of this lemma to the appendix. Note that Lemma 1 gives an intuitive interpretation of γ-randomly dictatorial SDSs: this axiom only requires that there are voters who always increase the winning probability of an alternative by at least γ_i if they reinforce it to the first place. Hence, for small values of γ, this axiom is desirable as it only formulates a variant of strict monotonicity. However, for larger of γ, γ-randomly dictatorial SDSs become more similar to random dictatorships. Furthermore, Lemma 1 implies that the decomposition of γ-randomly dictatorial SDSs is unique because it is completely determined by the values $\gamma_1, \dots, \gamma_n$.

Finally, we introduce α-Condorcet-consistency. To this end, we first define the notion of a Condorcet winner. A *Condorcet winner* is an alternative x that wins every majority comparison according to preference profile R, i.e., $n_{xy}(R) > n_{yx}(R)$ for all $y \in A \setminus \{x\}$. *Condorcet-consistency* demands that $f(R, x) = 1$ for all preference profiles R and alternatives x such that x is the Condorcet winner in R. Condorcet-consistent SDS are sometimes also called *Condorcet extensions.* Unfortunately, all Condorcet extensions violate strategyproofness, which can easily be derived from Gibbard's random dictatorship theorem. A

SDS	α-Condorcet -consistency	β-*ex post* efficiency	γ-random dictatorship
uniform random dictatorship	0	0	1
uniform lottery	$\frac{1}{m}$	$\frac{1}{m}$	0
randomized Borda rule	$\frac{1}{m} + \frac{t}{mn}$	$\frac{2(m-2)}{m(m-1)}$	$\frac{2}{m(m-1)}$
randomized Copeland rule	$\frac{2}{m}$	$\frac{2(m-2)}{m(m-1)}$	0

Table 1: Values of α, β, and γ for which specific SDSs are α-Condorcet-consistent, β-*ex post* efficient, and γ-randomly dictatorial. Each row shows the values of α, β, and γ for which a specific SDS satisfies the corresponding axioms. The value of α for the randomized Borda rule depends on the parity of n: $t = 1$ if n is odd and $t = 2$ otherwise.

simple two-profile proof for this fact when $m = n = 3$ is given in Figure 1. To circumvent this impossibility, we relax Condorcet-consistency: instead of requiring that the Condorcet winner always obtains probability 1, we only require that it receives a probability of at least α. This idea leads to α-*Condorcet-consistency*: an SDS f satisfies this axiom if $f(R, x) \geq \alpha$ for all profiles R and alternatives $x \in A$ such that x is the Condorcet winner in R. For small values of α, this axiom is clearly compatible with strategyproofness and therefore, we are interested in the maximum value of α such that there are α-Condorcet-consistent and strategyproof SDSs.

For a better understanding of α-Condorcet-consistency, β-*ex post* efficiency, and γ-random dictatorships, we discuss some of the values in Table 1 as examples. The uniform random dictatorship is 1-randomly dictatorial and 0-*ex post* efficient by definition. Moreover, it is 0-Condorcet-consistent because a Condorcet winner may not be top-ranked by any voter. The randomized Borda rule is $\frac{2(m-2)}{m(m-1)}$-*ex post* efficient because it assigns this probability to an alternative that is second-ranked by every voter. Moreover, it is $\frac{2}{m(m-1)}$-randomly dictatorial as we can represent it as $\frac{2}{m(m-1)} f_{RD} + \left(1 - \frac{2}{m(m-1)}\right) g$, where f_{RD} is the uniform random dictatorship and g is the point voting SDS defined by the scoring vector $\left(\frac{2(m-2)}{n(m(m-1)-2)}, \frac{2(m-2)}{n(m(m-1)-2)}, \frac{2(m-3)}{n(m(m-1)-2)}, \dots, 0\right)$. Finally, the randomized Copeland rule is 0-randomly dictatorial because there is for every voter a profile in which he can swap his two best alternatives without affecting the outcome. Moreover, it is $\frac{2}{m}$-Condorcet-consistent because a Condorcet winner x satisfies that $n_{xy}(R) > \frac{n}{2}$ for all $y \in A \setminus \{x\}$ and hence, $f_C(R, x) = \sum_{y \in A \setminus \{x\}} b_{n_{xy}(R)} = (m-1)\frac{2}{m(m-1)} = \frac{2}{m}$. Note that Table 1 also contains a row corresponding to the uniform lottery. We consider this SDS as a threshold with respect to α-Condorcet-consistency and β-*ex post* efficiency because we can compute the uniform lottery without knowledge about the voters' preferences. Hence, if an SDS performs worse than the uniform lottery with respect to α-Condorcet-consistency or β-*ex post* efficiency and if we are only interested in these axioms, we could also dismiss the voters' preferences.

3 Results

In this section, we present our results about the α-Condorcet-consistency and the β-*ex post* efficiency of strategyproof SDSs. Our results are rather negative for both axioms. No strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$ and the randomized Copeland rule f_C is the only anonymous, neutral, and strategyproof SDS that satisfies α-

Condorcet-consistency for $\alpha = \frac{2}{m}$. Moreover, we show that no 0-randomly dictatorial SDS satisfies both strategyproofness and β-*ex post* efficiency for $\beta < \frac{1}{m}$. This means that 0-randomly dictatorial SDSs cannot be more *ex post* efficient than the uniform lottery. A corollary of this impossibility is that every $\frac{1-\epsilon}{m}$-*ex post* efficient and strategyproof SDS is γ-randomly dictatorial for $\gamma \geq \epsilon$. This corollary can be seen as a continuous generalization of the random dictatorship theorem.

We derive these results through a series of lemmas. Because of space restrictions, all these lemmas, as well as all other results for which the proof has been omitted, are proved in the appendix and we only present short proof sketches instead.

3.1 α-Condorcet-consistency

As discussed in Section 2.2, Condorcet-consistent SDSs violate strategyproofness. Therefore, we analyze the maximal α such that α-Condorcet-consistency and strategyproofness are compatible. Our results are rather negative: we prove that no strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$. This bound is tight as the randomized Copeland rule satisfies $\frac{2}{m}$-Condorcet-consistency. We use this observation to characterize the randomized Copeland rule as the only strategyproof SDS that satisfies $\frac{2}{m}$-Condorcet-consistency, neutrality, and anonymity.

We first focus on the impossibility of strategyproof α-Condorcet-consistent SDSs for $\alpha > \frac{2}{m}$. As first step, we show in Lemma 2 that we can use a strategyproof and α-Condorcet-consistent SDS to construct another strategyproof SDS that satisfies anonymity, neutrality, and α-Condorcet-consistency for the same α. Hence, we can restrict our attention to anonymous, neutral and strategyproof SDSs, which means that we can use Theorem 3 to represent the SDS as a mixture of a point voting SDS and a supporting size SDS. Next, we prove bounds on the α-Condorcet consistency of point voting SDSs and supporting size SDSs in Lemmas 3 and 4, respectively. By combining all these results, we derive that no strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$.

Lemma 2. *If a strategyproof SDS satisfies α-Condorcet-consistency for some $\alpha \in [0, 1]$, there is also a strategyproof SDS that satisfies anonymity, neutrality, and α-Condorcet-consistency for the same α.*

The central idea in the proof of Lemma 2 is the following: if there is a strategyproof and α-Condorcet-consistent SDS f, then the SDS $f^{\pi\tau}(R, x) = f(\tau(\pi(R)), \tau(x))$ is also strategyproof and α-Condorcet-consistent for all permutations $\pi : N \to N$ and $\tau : A \to A$. Since mixtures of strategyproof and α-Condorcet-consistent SDSs are also strategyproof and α-Condorcet-consistent, it follows that $f^* = \frac{1}{m!n!} \sum_{\pi\in\Pi} \sum_{\tau\in\mathrm{T}} f^{\pi\tau}$ satisfies all requirements of the lemma, where Π denotes the set of all permutations on N and T the set of all permutations on A.

Remark 1. Lemma 2 can be applied to properties other than α-Condorcet-consistency, too. For example, given a strategyproof and β-*ex post* efficient SDS, we can construct another SDS that satisfies these axioms as well as neutrality and anonymity. In general, our construction maintains all axioms $\mathbb{P}$ that are simple (for given m and n, $\mathbb{P}$ can be described by a finite number of linear inequalities) and symmetric (if $\mathbb{P}$ entails inequalities for $f(R)$, it also entails the corresponding inequalities for $f(\tau(\pi(R))$ for all $\pi : N \to N$ and $\tau : A \to A$).

Next, we derive upper bounds for the α-Condorcet-consistency of point voting SDSs and supporting size SDSs. We first discuss point voting SDSs.

Lemma 3. *No point voting SDS is α-Condorcet-consistent for $\alpha \geq \frac{2}{m}$ if $n \geq 3$ and $m \geq 3$.*

The proof of this lemma relies on the observation that there can be $\lceil \frac{m}{2} \rceil$ Condorcet winner candidates, i.e., alternatives x that can be made into the Condorcet winner by keeping x at the same position in the preferences of every voter and only reordering the other alternatives. Since reordering the other alternatives does not affect the probability of x in a point voting SDS, it follows that every Condorcet winner candidate has a probability of at least α. Hence, we derive that $\alpha \leq \frac{1}{\lceil \frac{m}{2} \rceil} \leq \frac{2}{m}$ and a slightly more involved argument shows that the inequality is strict.

Remark 2. Point voting SDSs can be interpreted as positional scoring rules that randomize proportional to the assigned scores. A result by Smith (1973) shows that for large n, every scoring rule except Borda's rule can assign the Condorcet winner the lowest score. Hence, for every point voting SDS except the randomized Borda rule, there is a profile where the Condorcet winner receives less than $\frac{1}{m}$ probability. Moreover, the randomized Borda rule is $(\frac{1}{m} + \frac{t}{nm})$-Condorcet-consistent, where $t = 2 - (n \mod 2)$. This argument gives a more restrictive bound on the α-Condorcet-consistency of point voting SDSs when there is a large number of voters. In particular, the construction of Smith (1973) requires $\mathcal{O}(m!)$ voters.

The last ingredient of our proof is that no supporting size SDS can assign a probability of more than $\frac{2}{m}$ to any alternative. This immediately implies that no supporting size SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$.

Lemma 4. *No supporting size SDS can assign more than $\frac{2}{m}$ probability to an alternative.*

The proof of this lemma follows straightforwardly from the definition of supporting size SDSs. Each such SDS is defined by a scoring vector $(b_n, \dots, b_0)$ such that $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{0, \dots, n\}$ and $b_n \geq b_{n-1} \geq \dots \geq b_0 \geq 0$. The probability of an alternative x in a supporting size SDS f is therefore $f(R, x) = \sum_{y \in A \setminus \{x\}} b_{n_{xy}(R)} \leq (m-1)\frac{2}{m(m-1)} = \frac{2}{m}$.

Finally, we have all necessary lemmas for the proof of our first theorem: no strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$ if $n \geq 3$.

Theorem 4. *No strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$ if $n \geq 3$.*

Proof. The theorem is trivially true if $m \leq 2$ because α-Condorcet consistency for $\alpha > 1$ is impossible. Hence, let f denote a strategyproof SDS for $m \geq 3$ alternatives. We show in the sequel that f cannot satisfy α-Condorcet-consistency for $\alpha > \frac{2}{m}$. As a first step, we use Lemma 2 to construct a strategyproof SDS f^* that satisfies anonymity, neutrality, and α-Condorcet-consistency for the same α as f. It suffices to show that f^* violates α-Condorcet-consistency for any $\alpha > \frac{2}{m}$ as the contraposition of Lemma 2 entails in this case that f cannot satisfy this axiom either. Since f^* is anonymous, neutral, and strategyproof, it follows from Theorem 3 that f^* can be represented as a mixture of a point voting SDS f_{point} and a supporting size SDS f_{sup}, i.e., $f^* = \lambda f_{point} + (1-\lambda) f_{sup}$ for some $\lambda \in [0, 1]$.

Next, we consider f_{point} and f_{sup} separately. Lemma 3 implies for f_{point} that there is a profile R with a Condorcet winner a such that $f_{point}(R, a) < \frac{2}{m}$. Moreover, Lemma 4 shows that no supporting size SDS f_{sup} can assign more than $\frac{2}{m}$ probability to an alternative. Hence, it holds also that $f_{sup}(R, a) \leq \frac{2}{m}$ and we derive the following inequality.

$$\alpha \leq f^*(R, a) = \lambda f_{point}(R, a) + (1-\lambda) f_{sup}(R, a) \leq \lambda \frac{2}{m} + (1-\lambda)\frac{2}{m} = \frac{2}{m}$$

This equation shows that f^* fails α-Condorcet-consistency for $\alpha > \frac{2}{m}$. We thus conclude that no strategyproof SDS satisfies α-Condorcet-consistency for $\alpha > \frac{2}{m}$ when $n \geq 3$. ☐

Remark 3. The theorem does not hold when there are only $n = 2$ voters because random dictatorships are strategyproof and Condorcet-consistent in this case since a Condorcet winner needs to be the most preferred alternative of both voters. If there are $m = 2$ alternatives, the randomized Copeland rule is strategyproof and Condorcet-consistent.

Theorem 4 shows that strategyproofness does not allow for any meaningful notion of Condorcet-consistency. The reason for this is that strategyproof SDSs can guarantee the Condorcet winner a probability of at most $\frac{2}{m}$, which is only twice the amount compared to the uniform lottery. Therefore, our result significantly strengthens the incompatibility of Condorcet-consistency and strategyproofness.

A natural follow-up question is to ask which strategyproof SDSs satisfy $\frac{2}{m}$-Condorcet-consistency. As already mentioned in Section 2.2, the randomized Copeland rule f_C satisfies this axiom and it is even the only strategyproof SDS that satisfies $\frac{2}{m}$-Condorcet consistency, neutrality, and anonymity. Recall that f_C is the supporting size SDS defined by the scoring vector b with $b_i = \frac{2}{m(m-1)}$ if $b_i > \frac{n}{2}$, $b_i = \frac{1}{m(m-1)}$ if $b_i = \frac{n}{2}$ and $b_i = 0$ otherwise.

Theorem 5. *The randomized Copeland rule is the only strategyproof SDS that satisfies anonymity, neutrality, and* $\frac{2}{m}$*-Condorcet-consistency if* $m \geq 3$ *and* $n \geq 3$*.*

The theorem consists of two claims: on the one side, we show that the randomized Copeland rule is anonymous, neutral, strategyproof, and $\frac{2}{m}$-Condorcet-consistent and, on the other side, that no other SDS satisfies all these axioms. As the randomized Copeland rule is a supporting size SDS, it satisfies by definition anonymity, neutrality and strategyproofness. Moreover, it is also $\frac{2}{m}$-Condorcet consistent because the Condorcet winner x wins every majority comparison in R, i.e, $n_{xy}(R) > \frac{n}{2}$ for all alternatives $y \in A\setminus\{x\}$. Hence, the probability of a Condorcet winner x is always $f_C(R,x) = \sum_{y\in A\setminus\{x\}} b_{n_{xy}(R)} = (m-1)\frac{2}{m(m-1)} = \frac{2}{m}$, which means that the randomized Copeland rule is $\frac{2}{m}$-Condorcet consistent. For the second part, let f denote an SDS that satisfies anonymity, neutrality, strategyproofness, and $\frac{2}{m}$-Condorcet consistency. We show that f is the randomized Copeland rule. First, note that f is a supporting size SDS as otherwise, it can be represented as a proper mixture of a point voting SDS and a supporting size SDS. However, it follows then from Lemma 3 and Lemma 4 that f is only α-Condorcet-consistent for $\alpha < \frac{2}{m}$. Next, observe that an alternative x can only receive a probability of $\frac{2}{m}$ in a supporting size SDS if $b_{n_{xy}(R)} = \frac{2}{m(m-1)}$ for all $y \in A \setminus \{x\}$. Furthermore, Condorcet winners are indifferent about the exact supporting sizes and therefore, $b_i = \frac{2}{m(m-1)}$ for all $i > \frac{n}{2}$. The remaining entries of the scoring vector are given by the constraints $b_i + b_{n-i} = \frac{2}{m(m-1)}$, which imply that $b_i = 0$ for $i < \frac{n}{2}$ and $b_{\frac{n}{2}} = \frac{1}{m(m-1)}$. Hence, f is a supporting size SDS with the same scoring vector as the randomized Copeland rule which means that $f = f_C$.

Remark 4. All axioms in the proof of Theorem 5 are independent. The SDS that picks the Condorcet winner with probability $\frac{2}{m}$ if one exists and distributes the remaining probability uniformly at random between the other alternatives only violates strategyproofness. The randomized Borda rule satisfies all axioms of Theorem 5 but $\frac{2}{m}$-Condorcet-consistency. An SDS that satisfies strategyproofness, anonymity and $\frac{2}{m}$-Condorcet-consistency can be defined based on an arbitrary order of alternatives $x_0, \dots, x_{m-1}$. Then, we pick an index $i \in \{0, \dots, m-1\}$ uniformly at random and return the winner of the majority comparison between x_i and $x_{i+1 \bmod m}$ (if there is a majority tie, a fair coin toss decides the winner). This SDS is strategyproof as it is a mixture of duples, and $\frac{2}{m}$-Condorcet-consistent as each alternative has a chance of $\frac{2}{m}$ of being chosen for the pairwise majority comparison, which is always won by a Condorcet winner. Finally, we use the randomized Copeland rule f_C to construct an SDS that fails only anonymity for even n: we just ignore a voter in the calculation of f_C. If n is even and x is the Condorcet winner in R, then $n_{xy}(R) - n_{yx}(R) \geq 2$ for all $y \in N\setminus\{x\}$. Hence, the Condorcet winner remains a Condorcet winner after removing a single voter and the SDS satisfies all axioms but anonymity.

Remark 5. The randomized Copeland rule has multiple appealing interpretations. Firstly, it can be defined as a supporting size SDS as shown in Section 2.1. Alternatively, it can be defined as the SDS that picks two alternatives uniformly at random and then picks the majority winner between them; majority ties are broken by a fair coin toss. Next, Theorem 5 shows that the randomized Copeland rule is the SDS that maximizes the value of α for α-Condorcet-consistency among all anonymous, neutral, and strategyproof SDSs.

3.2 β-*ex post* Efficiency

Gibbard's random dictatorship theorem implies that random dictatorships are the only strategyproof SDSs that satisfy *ex post* efficiency. In this section, we show that this result is rather robust because every strategyproof SDS that assigns Pareto-dominated alternatives at most $\beta < 1/m$ probability will be a random dictatorship with positive probability. More formally, we show that no 0-randomly dictatorial SDS satisfies β-*ex post* efficiency for $\beta < \frac{1}{m}$. It follows from this result that for every $\epsilon \in [0, 1]$, all strategyproof and $\frac{1-\epsilon}{m}$-*ex post* efficient SDSs are γ-randomly dictatorial for $\gamma \geq \epsilon$.

Next, we start proving our impossibility result. Since Theorem 1 states that every strategyproof SDS can be represented as a convex combination of duples and unilaterals, we consider these two types of SDSs separately. First, we show that no SDS that can be represented as a mixture of duples satisfies β-*ex post* efficiency for $\beta < \frac{1}{m}$.

Lemma 5. *No SDS that can be represented as a convex combination of duples satisfies β-*ex post *efficiency for $\beta < \frac{1}{m}$ if $m \geq 3$.*

For the proof of this lemma, we first apply the averaging construction of Lemma 2 because this construction also preserves β-*ex post* efficiency. Hence, we can focus on SDSs that are anonymous and neutral, and that can be represented as mixtures of duples. This class of SDSs is equivalent to Barberà's supporting size SDSs, which cannot satisfy β-*ex post* efficiency for $\beta < \frac{1}{m}$. This follows by considering a preference profile in which each voter submits the same preference relation. In this profile, the probabilities of the unanimously best and the unanimously worst alternative sum up to $\frac{2}{m}$ because of the definition of supporting size SDSs. This means that $\frac{m-2}{m}$ probability is assigned to $m-2$ Pareto-dominated alternatives and hence, at least one of them has a probability of at least $\frac{1}{m}$. The contraposition of Lemma 2 entails therefore that no SDS satisfies β-*ex post* efficiency for $\beta < \frac{1}{m}$ if it can be represented as a mixture of duples.

Next, we focus on SDSs that can be represented as mixtures of unilaterals. The next lemma shows that such SDSs cannot be both 0-randomly dictatorial and β-*ex post* efficient for $\beta < \frac{1}{m}$.

Lemma 6. *No 0-randomly dictatorial SDS that can be represented as a convex combination of unilaterals satisfies β-*ex post *efficiency for $\beta < \frac{1}{m}$ if $m \geq 3$.*

The proof of this lemma is rather involved. First, we discuss a construction that transforms a 0-randomly dictatorial mixture of unilaterals f into a function similar to a point voting rule while ensuring that the resulting SDS is strategyproof, 0-randomly dictatorial, and β-*ex post* efficient for the same β as f. Note that Lemma 2 cannot be used here as it does not guarantee that the resulting SDS is 0-randomly dictatorial. For the resulting SDS, we then show that a profile exists where every alternative receives a probability of at most β. This results in a contradiction because the sum of the probabilities of all alternatives is less than 1 if $\beta < \frac{1}{m}$.

Finally, we use Lemma 5 and Lemma 6 to show that no strategyproof SDS is both 0-randomly dictatorial and β-*ex post* efficient for $\beta < \frac{1}{m}$. As a consequence of this result, no 0-randomly dictatorial SDS outperforms the uniform lottery with respect to β-*ex post* efficiency. If we are only interested in this axiom, we may therefore dismiss the voters' preferences completely instead of using a 0-randomly dictatorial SDS.

Theorem 6. *There is no strategyproof SDS that is both* 0*-randomly dictatorial and* β-ex post *efficient for* $\beta < \frac{1}{m}$ *if* $m \geq 3$.

The proof of this result is quite similar to the one of Theorem 4. We consider a 0-randomly dictatorial SDS f that satisfies β-*ex post* efficiency for some $\beta \in [0, 1]$ and show that $\beta \geq \frac{1}{m}$ must be true. Note that we can use Theorem 1 to represent f as the convex combination of a 0-randomly dictatorial mixture of unilaterals f_{uni} and a mixture of duples f_{duple}. We even know that these SDSs violate β-*ex post* efficiency for $\beta < \frac{1}{m}$ because of Lemma 5 and Lemma 6. However, there is no direct implication for f since f_{uni} and f_{duple} might violate β-*ex post* efficiency for different profiles or alternatives. We solve this problem by transforming f into a 0-randomly dictatorial SDS f^* that is β-*ex post* efficient for the same β as f and satisfies additional properties. In particular, f^* can be represented as a convex combination of a 0-randomly dictatorial mixture of unilaterals f^*_{uni} and a mixture of duples f^*_{duple} such that $f^*_{uni}(R, x) \geq \frac{1}{m}$ and $f^*_{duple}(R, x) \geq \frac{1}{m}$ for some profile R in which alternative x is Pareto-dominated. Consequently, f^* fails β-*ex post* efficiency for $\beta < \frac{1}{m}$, which implies that also f violates this axiom.

Theorem 6 identifies a trade-off between β-*ex post* efficiency and the similarity to a random dictatorship. In more detail, we now show that for every $\epsilon \in [0, 1]$, every strategyproof and $\frac{1-\epsilon}{m}$-*ex post* efficient SDS is γ-randomly dictatorial for $\gamma \geq \epsilon$.

Corollary 1. *For every* $\epsilon \in [0, 1]$*, every strategyproof and* $\frac{1-\epsilon}{m}$-ex post *efficient SDS is* γ*-randomly dictatorial for* $\gamma \geq \epsilon$.

Proof. Consider an SDS f that is strategyproof, γ-randomly dictatorial for some $\gamma \in [0, 1]$, and $\frac{1-\epsilon}{m}$-*ex post* efficient for some $\epsilon \in [0, 1]$. We show in the sequel that $\gamma \geq \epsilon$. First, we use the definition of γ-randomly dictatorial SDSs to derive that $f = \gamma d + (1-\gamma)g$, where d is a random dictatorship and g is another strategyproof SDS. In particular, the maximality of γ implies that g is 0-randomly dictatorial. Hence, Theorem 6 shows that g is at best $\frac{1}{m}$-*ex post* efficient, i.e, there is a profile R with a Pareto-dominated alternative x such that $g(R, x) \geq \frac{1}{m}$. This results in the following inequality for $f(R, x)$.

$$f(R, x) = \gamma d(R, x) + (1-\gamma)g(R, x) \geq \gamma 0 + (1-\gamma)\frac{1}{m} = \frac{1-\gamma}{m}.$$

On the other hand, we know that $f(R, x) \leq \frac{1-\epsilon}{m}$ because f is $\frac{1-\epsilon}{m}$-*ex post* efficient. Therefore, we derive that $\frac{1-\gamma}{m} \leq \frac{1-\epsilon}{m}$, which is equivalent to $\epsilon \leq \gamma$. Hence, all strategyproof and $\frac{1-\epsilon}{m}$-*ex post* efficient SDSs are γ-randomly dictatorial for $\gamma \geq \epsilon$. □

Corollary 1 is a continuous strengthening of Gibbard's random dictatorship theorem. If we set $\epsilon = 1$, then Corollary 1 is equivalent to the random dictatorship theorem as every strategyproof and 0-*ex post* efficient SDSs is 1-randomly dictatorial. Moreover, increasing the ϵ of $\frac{1-\epsilon}{m}$-*ex post* efficiency allows for strategyproof SDSs that are less similar to random dictatorships. On the other hand, Corollary 1 entails that a γ-randomly dictatorial SDS can only satisfy $\frac{1-\epsilon}{m}$-*ex post* efficiency for $\epsilon \leq \gamma$. Hence, the more *ex post* efficiency is required, the closer a strategyproof SDS gets to a random dictatorship.

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References

A. Abdulkadiroğlu and T. Sönmez. Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3):689–701, 1998.

H. Aziz, F. Brandl, F. Brandt, and M. Brill. On the tradeoff between efficiency and strategyproofness. *Games and Economic Behavior*, 110:1–18, 2018.

S. Barberà. A note on group strategy-proof decision schemes. *Econometrica*, 47(3):637–640, 1979a.

S. Barberà. Majority and positional voting in a probabilistic framework. *Review of Economic Studies*, 46(2):379–389, 1979b.

J.-P. Benoît. Strategic manipulation in voting games when lotteries and ties are permitted. *Journal of Economic Theory*, 102(2):421–436, 2002.

A. Bogomolnaia and H. Moulin. A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328, 2001.

F. Brandl, F. Brandt, M. Eberl, and C. Geist. Proving the incompatibility of efficiency and strategyproofness via SMT solving. *Journal of the ACM*, 65(2):1–28, 2018.

S. Chatterji, A. Sen, and H. Zeng. Random dictatorship domains. *Games and Economic Behavior*, 86:212–236, 2014.

Y.-K. Che and F. Kojima. Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, 78(5):1625–1672, 2010.

V. Conitzer and T. Sandholm. Nonexistence of voting rules that are usually hard to manipulate. In *Proceedings of the 21st National Conference on Artificial Intelligence (AAAI)*, pages 627–634, 2006.

J. Duggan. A geometric proof of Gibbard's random dictatorship theorem. *Economic Theory*, 7(2):365–369, 1996.

B. Dutta, H. Peters, and A. Sen. Strategy-proof probabilistic mechanisms in economies with pure public goods. *Journal of Economic Theory*, 106(2):392–416, 2002.

B. Dutta, H. Peters, and A. Sen. Strategy-proof cardinal decision schemes. *Social Choice and Welfare*, 28(1):163–179, 2007.

L. Ehlers, H. Peters, and T. Storcken. Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences. *Journal of Economic Theory*, 105(2):408–434, 2002.

P. C. Fishburn. Condorcet social choice functions. *SIAM Journal on Applied Mathematics*, 33(3):469–489, 1977.

A. Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601, 1973.

A. Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45(3):665–681, 1977.

J. C. Heckelman. Probabilistic Borda rule voting. *Social Choice and Welfare*, 21:455–468, 2003.

J. C. Heckelman and F. H. Chen. Strategy proof scoring rule lotteries for multiple winners. *Journal of Public Economic Theory*, 15(1):103–123, 2013.

A. Hylland. Strategyproofness of voting procedures with lotteries as outcomes and infinite sets of strategies. Mimeo, 1980.

S. Nandeibam. An alternative proof of Gibbard's random dictatorship result. *Social Choice and Welfare*, 15(4):509–519, 1997.

S. Nandeibam. The structure of decision schemes with cardinal preferences. *Review of Economic Design*, 17(3):205–238, 2013.

A. D. Procaccia. Can approximation circumvent Gibbard-Satterthwaite? In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI)*, pages 836–841, 2010.

M. A. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

A. Sen. The Gibbard random dictatorship theorem: a generalization and a new proof. *SERIEs*, 2(4):515–527, 2011.

J. H. Smith. Aggregation of preferences with variable electorate. *Econometrica*, 41(6): 1027–1041, 1973.

Y. Tanaka. An alternative proof of Gibbard's random dictatorship theorem. *Review of Economic Design*, 8:319–328, 2003.

Appendix: Omitted Proofs

Here, we discuss the missing proofs of all lemmas and of Theorem 5 and Theorem 6. Proof sketches providing intuition for the lemmas can be found in the main body. First, we discuss the proof of Lemma 1. Recall for this proof that $R^{i:yx}$ is the profile derived from R by letting voter i reinforce y against x.

Lemma 1. *A strategyproof SDS f is γ-randomly dictatorial if and only if there are non-negative values $\gamma_1, \dots, \gamma_n$ that satisfy the following conditions:*

i) $\sum_{i\in N} \gamma_i = \gamma$.

ii) $f(R^{i:yx}, y) - f(R, y) \geq \gamma_i$ *for all alternatives $x, y \in A$, voters $i \in N$, and preference profiles R in which voter i prefers x the most and y the second most.*

iii) for every voter $i \in N$ there are alternatives $x, y \in A$ and a profile R such that voter i prefers x the most and y the second most in R, and $f(R^{i:yx}, y) - f(R, y) = \gamma_i$.

Proof. " $\Longleftarrow$ " Assume that f is a strategyproof SDS for which there are values $\gamma_1, \dots, \gamma_n$ such that $f(R^{i:yx}, y) - f(R, y) \geq \gamma_i \geq 0$ for all alternatives $x, y \in A$, voters $i \in N$, and profiles R such that voter i prefers x the most and y the second most in R. Furthermore, we assume that for every voter $i \in N$, this inequality is tight for at least one pair of alternatives $x, y \in A$ and one such profile R. We show next that f is γ-randomly dictatorial for $\gamma = \sum_{i\in N} \gamma_i$. Therefore, we define $g = \frac{1}{1-\gamma}\Big(f - \sum_{i\in N} \gamma_i d_i\Big)$, where d_i is the SDS that assigns the best alternative of voter i probability 1. Note that g is a well-defined SDS because for every voter $i \in N$, his best alternative receives probability of at least γ_i. The reason for this is that reinforcing an alternative from second to first place in voter i's preferences increases its probability by at least γ_i.

Next, we show that g is strategyproof, which implies that f is γ'-randomly dictatorial for $\gamma' \geq \gamma$ because $f = \sum_{i\in N} \gamma_i d_i + (1-\gamma)g$. Hence, we show that g is localized and non-perverse, which implies that it is strategyproof because of Theorem 2. In more detail, g is localized because the SDS f and all SDSs d_i are localized. Hence, swapping two alternatives in the preferences of a voter only affects these two alternatives. For seeing that g is non-perverse, consider a voter i, two alternatives $x, y \in A$ and a profile R such that x is voter i's k-th best alternative and y is his $k+1$-th best one. We show that $g(R^{i:yx}) \geq g(R, y)$, which entails that g is non-perverse. Note for this that $d_j(R^{i:yx}) = d_j(R)$ for all $j \in N \setminus \{i\}$ because the preferences of these voters did not change, and $f(R^{i:yx}, y) - f(R, y) \geq 0$ because f is strategyproof and therefore non-perverse. Furthermore, if x and y are not the two best alternatives of voter i, then $d_i(R^{i:yx}) = d_i(R)$. Hence, it immediately follows that $g(R^{i:yx}, y) - g(R, y) = \frac{1}{1-\gamma}\Big(f(R^{i:yx}, y) - f(R, y)\Big) \geq 0$. On the other hand, if x and y are voter i's two best alternative, we have that $d_i(R^{i:yx}, y) = 1$ and $d_i(R, y) = 0$. Moreover, it holds that $f(R^{i:yx}, y) - f(R, y) \geq \gamma_i$ because x and y voter i's two best alternatives and thus, our initial assumption applies. Hence, we calculate that $g(R^{i:yx}, y) - g(R, y) = \frac{1}{1-\gamma}\Big(f(R^{i:yx}, y) - f(R, y) + \gamma_i(d_i(R^{i:yx}, y) - d_i(R, y))\Big) \geq \frac{1}{1-\gamma}\Big(\gamma_i - \gamma_i \cdot 1\Big) = 0$, which shows that g is non-perverse.

Finally, we show that f cannot be γ'-randomly dictatorial for $\gamma' > \gamma$. If this was the case, we can represent f as $f = \sum_{i\in N} \gamma'_i d_i + (1-\gamma')g'$, where $\gamma'_i \geq 0$ are values such that $\sum_{i\in N} \gamma'_i = \gamma$ and g' is a strategyproof SDS. Since $\gamma' > \gamma$, there is a voter i with $\gamma'_i > \gamma_i$. Furthermore, we know from our assumptions that there is a profile R and alternatives x, y such that voter i prefers x the most and y the second most in R, and $f(R^{i:yx}, y) - f(R, y) = \gamma_i$. Moreover, we have that $d_i(R^{i:yx}, y) - d_i(R, y) = 1$ and $d_j(R^{i:yx}, y) - d_j(R, y) = 0$ for all $j \in N \setminus \{i\}$. Hence, it follows the contradiction assumption that $\gamma_i = f(R^{i:yx}, y) - f(R, y) = \gamma'_i + (1-\gamma)\Big(g'(R^{i:yx}, y) - g'(R, y)\Big)$. This is only possible if $g'(R^{i:yx}, y) - g'(R, y) < 0$, which means that g' violates non-perversity and therefore also strategyproofness. Hence, the assumption that f is γ'-randomly dictatorial for $\gamma' > \gamma$ is wrong and f is therefore γ-randomly dictatorial.

"$\implies$" Let f be a strategyproof γ-randomly dictatorial SDS. We show next that there are values γ_i that satisfy the requirements of the lemma. Since f is γ-randomly dictatorial, it can be represented as $f = \gamma d + (1-\gamma)g$, where d is a random dictatorship and g is another strategyproof SDS. Moreover, as d is a random dictatorship, there are values $\delta_1, \dots, \delta_n$ such that $\delta_i \geq 0$ for all $i \in N$, $\sum_{i\in N} \delta_i = 1$, and $d = \sum_{i\in N} \delta_i d_i$, where d_i denotes the SDS that assigns probability 1 to voter i's best alternative. Combining these two equations, we get that $f = \gamma \sum_{i\in N} \delta_i d_i + (1-\gamma)g$. We show in the sequel that the values $\gamma_i = \gamma\delta_i$ satisfy all requirements of our lemma. First, note that the conditions $\gamma_i \geq 0$ for all $i \in N$ and $\sum_{i\in N} \gamma_i = \gamma$ are obviously true.

Next, consider two alternatives $x, y \in A$, an arbitrary voter $i \in N$, and a profile R in which voter i reports x as his best alternative and y as his second best one. It holds that $g(R^{i:yx}, y) - g(R, y) \geq 0$ because g is strategyproof and therefore non-perverse, $d_j(R^{i:yx}, y) - d_j(R, y) = 0$ for all $j \in N \setminus \{i\}$ because $R_j^{i:yx} = R_j$, and $d_i(R^{i:yx}, y) - d_i(R, y) = 1$ as y is voter i's best alternative in $R^{i:yx}$, but not in R. Hence, it follows that $f(R^{i:yx}, y) - f(R, y) \geq \gamma\delta_i = \gamma_i$ for all voters $i \in N$, alternatives $x, y \in A$, and preference profiles R in which voter i reports x as his best and y as his second best alternative.

Finally, it remains to show that there is for every voter i a pair of alternatives $x, y \in A$ and a profile R such that voter i prefers x the most in R and y the second most and $f(R^{i:yx}, y) - f(R, y) = \gamma_i$. Assume this is not the case for some voter i, i.e, that $f(R^{i:yx}, y) - f(R, y) > \gamma_i$

for all alternatives $x, y \in A$ and profiles R in which x is voter i's best alternative and y his second best one. Hence, let $\gamma_i' > \gamma_i$ denote the minimal value of $f(R^{i:yx}, y) - f(R, y)$ among all alternatives $x, y \in A$ and preference profiles R in which voter i reports x as his best alternative and y as his second best one. Moreover, define $\gamma' = \gamma_i + \sum_{j \in N\setminus\{i\}} \gamma_j$ and $g = \frac{1}{1-\gamma'}\left(f - \sum_{j\in N\setminus\{i\}} \gamma_j d_j - \gamma_i' d_i\right)$. It follows from the same arguments as in the inverse direction of the proof that g is strategyproof and thus, $f = \gamma_i' d_i + \sum_{j \in N\setminus\{i\}} +(1-\gamma')g$ is γ'-randomly dictatorial for $\gamma' > \gamma$. This contradicts our assumption that f is γ-randomly dictatorial as γ must be the maximal value such that f can be represented as $f = \gamma d + (1-\gamma)g$, where d is a random dictatorship and g is another strategyproof SDS. Hence, the assumption that $f(R^{i:yx}, y) - f(R, y) > \gamma_i$ for all alternatives $x, y \in A$ and profiles R in which x is voter i's best alternative and y his second best one is wrong, which shows that our choice of γ_i satisfies all requirements of the lemma. □

Next, we discuss the lemmas required in the proof of Theorem 4 and Theorem 5. First, we discuss the averaging construction of Lemma 2 in detail.

Lemma 2. *If a strategyproof SDS satisfies α-Condorcet-consistency for some $\alpha \in [0, 1]$, there is also a strategyproof SDS that satisfies anonymity, neutrality, and α-Condorcet-consistency for the same α.*

Proof. Let f denote an arbitrary strategyproof SDS that is α-Condorcet-consistent for some $\alpha \in [0, 1]$. We construct in the sequel an anonymous and neutral SDS f^* that satisfies strategyproofness and α-Condorcet-consistency for the same α as f. As first step, we define the SDS $f^{\pi\tau}$ for arbitrary permutations $\pi : N \to N$ and $\tau : A \to A$ as follows. First, $f^{\pi\tau}$ permutes the voters in the input profile R according to π and the alternatives according to τ. Next, we compute f on the resulting profile $\tau(\pi(R))$ and finally, we define $f^{\pi\tau}(R, x)$ as the probability assigned to $\tau(x)$ by f in $\tau(\pi(R))$. More formally, $f^{\pi\tau}$ is defined as $f^{\pi\tau}(R, x) = f(\tau(\pi(R)), \tau(x))$, where the profile $\tau(\pi(R))$ satisfies for all $i \in N$ and $x, y \in A$ that $\tau(x) \succ_{\pi(i)} \tau(y)$ in $\tau(\pi(R))$ if and only if $x \succ_i y$ in R. Note that $f^{\pi\tau}$ is strategyproof for all permutations π and τ because every manipulation of $f^{\pi\tau}$ implies a manipulation of f. Furthermore, $f^{\pi\tau}$ is α-Condorcet-consistent because for every preference profile R with Condorcet winner x, $\tau(x)$ is the Condorcet winner in $\tau(\pi(R))$. Hence, if $f^{\pi\tau}$ violates α-Condorcet-consistency in some profile R, then f violates this axiom in the profile $\tau(\pi(R))$.

Finally, we define the SDS f^* by averaging over $f^{\pi\tau}$ for all permutations π and τ. Hence, let Π denote the set of all permutations on N and let T denote the set of all permutations on A. Then, f^* is defined as follows.

$$f^*(R, x) := \sum_{\pi\in\Pi} \frac{1}{|\Pi|} \sum_{\tau\in\mathrm{T}} \frac{1}{|\mathrm{T}|} f^{\pi\tau}(R, x) = \sum_{\pi\in\Pi}\sum_{\tau\in\mathrm{T}} \frac{1}{n!m!} f(\tau(\pi(R)), \tau(x))$$

Next, we show that f^* satisfies all axioms required by the lemma. First, f^* is strategyproof since all SDSs $f^{\pi\tau}$ are strategyproof. The α-Condorcet-consistency of f^* is shown by the following inequality, where R denotes a profile in which x is the Condorcet winner.

$$f^*(R, x) = \sum_{\pi\in\Pi}\sum_{\tau\in\mathrm{T}} \frac{1}{n!m!} f(\tau(\pi(R)), \tau(x)) \geq \sum_{\pi\in\Pi}\sum_{\tau\in\mathrm{T}} \frac{1}{n!m!}\alpha = \alpha$$

Furthermore, observe that f^* is anonymous because it averages over all possible permutations of the voters, i.e., for all permutations of the voters $\pi \in \Pi : f^*(R) = f^*(\pi(R))$. It follows from a similar argument that f^* is neutral: since f^* averages over all permutations of the alternatives, it holds that $f^*(R, x) = f^*(\tau(R), \tau(x))$ for every $\tau \in \mathrm{T}$. Hence, f^* is strategyproof, α-Condorcet-consistent, anonymous, and neutral. □

2	1
x_1	x_2
x_2	x_3
x_3	x_1

R_1

2	1
x_1	x_4
x_2	x_2
x_3	x_1
x_4	x_3

R_2

2	2
x_1	x_2
x_2	x_1
x_3	x_3

R_3

2	1	1
x_1	x_2	x_4
x_2	x_1	x_2
x_3	x_3	x_1
x_4	x_4	x_3

R_4

Figure 2: Profiles used in the base cases of the proof of Lemma 3 if $m \in 3, 4$. The profile R^i shows the profile corresponding to case i.

Next, we present the proof of Lemma 3 which states that point voting SDSs cannot satisfy α-Condorcet-consistency for $\alpha \geq \frac{2}{m}$. Note that we use additional notation for this proof. The *rank* $r(x, R_i)$ of an alternative x in the preferences of a voter i is the number of alternatives that are weakly preferred to x by voter i, i.e., $r(x, R_i) = |\{y \in A \colon y \succsim_i x\}|$. Moreover, the *rank vector* $r^*(x, R)$ of an alternative x in a preference profile R is the vector that contains the rank of x with respect to every voter in increasing order. An important observation for point voting SDSs f is that $f(R, x) = f(R', x)$ if $r^*(x, R) = r^*(x, R')$. The reason for this is that a point voting SDSs assign an alternative every time probability a_i when it is ranked i-th. Finally, the proof focuses mainly on *Condorcet winner candidates*, which are alternatives that can be made into the Condorcet winner without changing their rank vectors.

Lemma 3. *No point voting SDS is α-Condorcet-consistent for $\alpha \geq \frac{2}{m}$ if $n \geq 3$ and $m \geq 3$.*

Proof. Let f be a point voting SDS for $m \geq 3$ alternatives and let $a = (a_1, \dots, a_m)$ be the scoring vector that defines f. Furthermore, assume for contradiction that f is α-Condorcet-consistent for $\alpha \geq \frac{2}{m}$. In the sequel, we show that there can be many Condorcet winner candidates in a profile R. Since we can turn Condorcet winner candidates into Condorcet winners without changing their rank vector and since $f(R, x) = f(R', x)$ for all profiles R and R' with $r^*(x, R) = r^*(x, R')$, it follows that each Condorcet winner candidate has at least probability α in R. This observation is in conflict with $\sum_{x \in A} f(R, x) = 1$ if $\alpha > \frac{2}{m}$ because there can be $\lceil \frac{m}{2} \rceil$ Condorcet winner candidates. By investigating our profiles in more detail, we also deduce that $\alpha = \frac{2}{m}$ is not possible.

We use a case distinction with respect to the parity of n and m to construct profiles with $\lceil \frac{m}{2} \rceil$ Condorcet winner candidates. Moreover, we first focus on cases with fixed n, and provide in the end an argument for generalizing the impossibility to all $n \geq 3$. Figure 2 illustrates our construction for all four base cases with $m \in \{3, 4\}$.

Case 1: $n = 3$ and m is odd

In this case, we choose $k = \frac{m+1}{2}$ alternatives which are denoted by $x_1, \dots, x_k$. We construct the profile R^1 with k Condorcet winner candidates as follows. For every $i \in \{1, \dots, k\}$, voters 1 and 2 rank alternative x_i at position i, and voter 3 ranks it at position $m+2-2i$. The sum of ranks of x_i is then equal to $2i+m+2-2i = m+2$, which means that only $m-1$ alternatives can be ranked above x_i. Note for this that the sum of ranks of an alternative x is the number of voters n plus the number of alternatives that are ranked above x. Hence, for every $i \in \{1, \dots, k\}$, we can reorder the alternatives in $A \setminus \{x_i\}$ such that each alternative $y \in A \setminus \{x_i\}$ is preferred to x_i by a single voter. Consequently, x_i is a Condorcet winner candidate in R^1, and thus $f(R^1, x_i) \geq \alpha$ for all $i \in \{1, \dots, k\}$. Since there are $k = \frac{m+1}{2}$ Condorcet winner candidates and $\sum_{i=1}^{k} f(R^1, x_i) \leq 1$, we derive that $\alpha \frac{m+1}{2} \leq 1$. This is equivalent to $\alpha \leq \frac{2}{m+1} < \frac{2}{m}$, and hence, f cannot satisfy α-Condorcet-consistency for $\alpha \geq \frac{2}{m}$ in this case.

Case 2: $n = 3$ and m is even

If $n = 3$ and m is even, we construct a preference profile R^2 with $\frac{m}{2}$ Condorcet winner candidates similar to the last case. More precisely, we first choose an alternative z, and apply the construction of the last case to the alternatives $A \setminus \{z\}$. Then, we add z as the last-ranked alternative of voters 1 and 2 and as first-ranked alternative of voter 3. Note that adding z does not affect whether an alternative is a Condorcet winner candidate because it is last-ranked by two out of three voters. Thus, there are $\frac{m}{2}$ Condorcet winner candidates in R^2 and it follows analogously to the last case that $\alpha\frac{m}{2} \leq 1$, which is equivalent $\alpha \leq \frac{2}{m}$. Finally, note that $\alpha = \frac{2}{m}$ is also not possible. Otherwise, each of the $\frac{m}{2}$ Condorcet winner candidates has a probability of $\frac{2}{m}$. As a consequence, all other alternatives have a probability of 0. In particular, this means that $f(R^2, z) = 0$ for voter 3's best alternative z. Hence, we derive for f's scoring vector $a = (a_1, \dots, a_m)$ that $a_1 = 0$. This is not possible because the scoring vector of a point voting SDS is monotone, i.e, $a_i \geq a_j$ if $i < j$, and $\sum_{i=1}^{m} a_i = \frac{1}{n}$. Hence, we deduce also for this case that $\alpha < \frac{2}{m}$ holds.

Case 3: $n = 4$ and m is odd

Just as in the first case, we choose $k = \frac{m+1}{2}$ alternatives which are denoted by $x_1, \dots, x_k$. Next, we construct a profile R^3 with k Condorcet winner candidates as follows. For every $i \in \{1, \dots, k\}$, voters 1 and 2 rank alternative x_i at position i, and voters 3 and 4 rank it at position $\frac{m+1}{2} + 1 - i$. The sum of ranks of x_i is then equal to $2i + 2\left(\frac{m+1}{2} + 1 - i\right) = m + 3$. Since the sum of ranks of an alternative x is the number of voters plus the number of alternatives ranked above x, we derive that only $m - 1$ alternatives can be ranked above x_i. Hence, for every $i \in \{1, \dots, k\}$, we can reorder the alternatives such that each alternative $y \in A \setminus \{x_i\}$ is ranked above x_i once without changing the rank vector of x_i. This entails that each alternative x_i is a Condorcet winner candidate and thus, we derive that $\alpha \leq \frac{2}{m+1} < \frac{2}{m}$ analogously to Case 1.

Case 4: $n = 4$ and m is even

Finally, consider the case that $n = 4$ and m is even. In this situation, we construct the profile R^4 with $\frac{m}{2}$ Condorcet winner candidates as follows: we choose an alternative z, and apply the construction of Case 3 to the alternatives in $A \setminus \{z\}$. Then, voters 1 to 3 add z as their least preferred alternative and voter 4 adds it as his best alternative. Just as in Case 2, every alternative that is a Condorcet winner candidate before adding z is also a Condorcet winner candidate after adding this alternative because for every other alternative y there is only a single voter who prefers z to y. Hence, there are $\frac{m}{2}$ Condorcet winner candidates in R^4, which implies that $\alpha \leq \frac{m}{2}$. Finally, we show that $\alpha = \frac{2}{m}$ is not possible either. If this was the case, it follows that $f(R^4, x_i) \geq \frac{2}{m}$ for all $i \in \{1, \dots, k\}$, and consequently that the remaining alternatives have probability 0. In particular, $f(R^4, z) = 0$. However, this entails for the scoring vector a of f that $a_1 = 0$ as voter 4 ranks z first. This is impossible because $a_1 \geq a_j$ for all $j \in \{1, \dots, m\}$ and $\sum_{i=1}^{m} a_i = \frac{1}{n}$ must be true for the scoring vector of a point voting SDS. Therefore, it follows that f can only satisfy α-Condorcet-consistency for $\alpha < \frac{2}{m}$.

Case 5: Generalizing the impossibility to larger n

Finally, we explain how to generalize the last four cases to an arbitrary number of voters $n \geq 3$. In this case, we also construct a profile with $\lceil \frac{m}{2} \rceil$ Condorcet winner candidates. In more detail, we choose the suitable base case and add repeatedly pairs of voters with inverse preferences until there are n voters. Note that voters with inverse preferences do not change the majority margins, and therefore they do not change whether an alternative is a Condorcet winner candidate. Hence, every alternative that is a Condorcet winner candidate in the base case is also a Condorcet winner candidate in the extended profile, which means that the arguments in the base cases also apply for larger numbers of voters. Therefore, no point voting SDS can satisfy α-Condorcet-consistency for $\alpha \geq \frac{2}{m}$ □

Next, we prove Lemma 4, which bounds the probability that can be guaranteed to Condorcet winners by supporting size SDSs.

Lemma 4. *No supporting size SDS can assign more than $\frac{2}{m}$ probability to an alternative.*

Proof. Let f be a supporting size SDS and let $b = (b_n, \dots, b_0)$ be the scoring vector that defines f. Furthermore, let R denote a profile and x denote an alternative. The probability that the SDS f assigns to alternative x in profile R is $f(R, x) = \sum_{y \in A\setminus\{x\}} b_{n_{xy}(R)}$. The definition of supporting size SDSs requires that the scoring vector b satisfies $b_n \geq \dots \geq b_0 \geq 0$ and $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{1, \dots \lfloor \frac{n}{2} \rfloor\}$. Therefore, it follows that $b_i \leq \frac{2}{m(m-1)}$ for all $i \in \{0, \dots, n\}$ and we derive that $f(R, x) = \sum_{y \in A\setminus\{x\}} b_{n_{xy}} \leq (m-1)\frac{2}{m(m-1)} = \frac{2}{m}$ for all preference profiles R and alternatives x. □

Lemma 4 is the last lemma required for the proofs of Theorem 4 and Theorem 5. We now show Theorem 5 before we continue with the lemmas required for Theorem 6.

Theorem 5. *The randomized Copeland rule is the only strategyproof SDS that satisfies anonymity, neutrality, and $\frac{2}{m}$-Condorcet-consistency if $m \geq 3$ and $n \geq 3$.*

Proof. The randomized Copeland rule f_C is a supporting size SDS and satisfies therefore anonymity, neutrality and strategyproofness. Furthermore, it satisfies also $\frac{2}{m}$-Condorcet-consistency because a Condorcet winner x wins every pairwise majority comparison in R. Hence, $n_{xy}(R) > \frac{n}{2}$ for all $y \in A \setminus \{x\}$, which implies that $f_C(R, x) = \sum_{y \in A\setminus\{x\}} b_{n_{xy}(R)} = (m-1)\frac{2}{m(m-1)} = \frac{2}{m}$ for all alternatives x and profiles R in which x is the Condorcet winner.

Next we show that the randomized Copeland rule is the only SDS that satisfies all of these axioms. Let f be an SDS satisfying anonymity, neutrality, strategyproofness, and $\frac{2}{m}$-Condorcet-consistency. We show that f is a supporting size SDS with the same scoring vector as f_C and thus, f is the randomized Copeland rule. Since f is anonymous, neutral, and strategyproof, we can apply Theorem 3 to represent f as $f = \lambda f_{point} + (1-\lambda) f_{sup}$, where $\lambda \in [0, 1]$, f_{point} is a point voting SDS, and f_{sup} is a supporting size SDS. Lemma 3 states that there is a profile R with Condorcet winner x such that $f_{point}(R, x) < \frac{2}{m}$, and it follows from Lemma 4 that $f_{sup}(R, x) \leq \frac{2}{m}$. Hence, $f(R, x) = \lambda f_{point}(R, x) + f_{sup}(R, x) < \frac{2}{m}$ if $\lambda > 0$. Therefore, f is a supporting size SDS as it satisfies $\frac{2}{m}$-Condorcet-consistency.

Next, we show that f has the same scoring vector as the randomized Copeland rule. Since f is a supporting size SDS, there is a scoring vector $b = (b_n, \dots, b_0)$ with $b_n \geq b_{n-1} \geq \dots \geq b_0 \geq 0$ and $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{1, \dots, n\}$ such that $f(R, x) = \sum_{y \in A\setminus\{x\}} b_{n_{xy}(R)}$. Moreover, $f(R, x) = \frac{2}{m}$ if x is the Condorcet winner in R because of $\frac{2}{m}$-Condorcet-consistency and Lemma 4. We derive from the definition of supporting size SDSs that the Condorcet winner x can only achieve this probability if $b_{n_{xy(R)}} = \frac{2}{m(m-1)}$ for every other alternatives $y \in A \setminus \{x\}$. Moreover, observe that the Condorcet winner needs to win every majority comparison but is indifferent about the exact supporting sizes. Hence, it follows that $b_i = \frac{2}{m(m-1)}$ for all $i > \frac{n}{2}$ as otherwise, there is a profile in which the Condorcet winner does not receive a probability of $\frac{2}{m}$. We also know that $b_i + b_{n-i} = \frac{2}{m(m-1)}$, so $b_i = 0$ for all $i < \frac{n}{2}$. If n is even, then $b_{\frac{n}{2}} = \frac{1}{m(m-1)}$ is required by the definition of supporting size SDSs as $\frac{n}{2} = n - \frac{n}{2}$. Hence, the scoring vector of f is equivalent to the scoring vector of the randomized Copeland rule f_C, which entails that f is f_C. □

We focus next on the proofs of the lemmas that are required for Theorem 6. Hence, our goal is to derive a lower bound for the β-*ex post* efficiency of strategyproof 0-randomly dictatorial SDSs. Since Theorem 1 allows us to represent every strategyproof SDS as a mixture of duples and unilaterals, we focus next on these two classes.

Lemma 5. *No SDS that can be represented as a convex combination of duples satisfies* β-ex post *efficiency for* $\beta < \frac{1}{m}$ *if* $m \geq 3$.

Proof. Let the SDS f denote a mixture of duples. First, we apply the construction in the proof of Lemma 2 to turn f into an anonymous, neutral, and strategyproof SDS f^* defined as $f^*(R,x) = \sum \pi \in \Pi \sum_{\tau\in \mathrm{T}} \frac{1}{m!n!} f(\tau(\pi(R)), \tau(x))$, where Π denotes the set of all permutation on N and T the set of all permutations on A. Next, we show that f^* satisfies β-*ex post* efficiency for the same β as f. This follows immediately from the observation that, if x is Pareto-dominated in R, then $\tau(x)$ is Pareto-dominated in $\tau(\pi(R))$. Since f is β-*ex post* efficient, it follows that $f(\tau(\pi(R)), \tau(x)) \leq \beta$ for all permutations $\pi : N \to N$ and $\tau : A \to A$. Hence, it holds for all profiles R and alternatives x such that x is Pareto-dominated in R that $f^*(R,x) = \sum_{\pi\in\Pi} \sum_{\tau\in\mathrm{T}} \frac{1}{m!n!} f(\tau(\pi(R)), \tau(x)) \leq \sum_{\pi\in\Pi} \sum_{\tau\in\mathrm{T}} \frac{1}{m!n!}\beta = \beta$, which proves that f^* is β-*ex post* efficient.

Since f is a mixture of duples, f^* is an anonymous and neutral mixtures of duples. We now show that f^* can be represented as a supporting size SDS. If one pair of alternatives has a duple component in f^*, then neutrality implies that every pair has an identical duple component with the same weight. In particular, this means that every pair of alternatives has a duple SDS with weight $\frac{2}{m(m-1)}$. We can then construct the scoring vector $(b_n, \dots, b_0)$ for formalizing f^* as a supporting size SDS by choosing a pair of alternatives x, y, the corresponding duple f_{xy}, and a profile R^j for every $j \in \{1, \dots, n\}$ such that $n_{xy}(R^j) = j$. Then, $b_j = \frac{2}{m(m-1)} f_{xy}(R^j, x)$ since anonymity implies that the probability assigned to x by f_{xy} only depends on the number of voters who prefer x to y. In particular, observe that $f(R^j, x)$ is independent of the preferences on the alternatives $A \setminus \{x, y\}$ because of strategyproofness and the definition of duples. Finally, we check that the scoring vector $b = (b_n, \dots, b_0)$ satisfies the conditions of a supporting size SDS: $b_n \geq \dots \geq b_0 \geq 0$ and $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Start with the profile R_0. Clearly, $b_0 = \frac{2}{m(m-1)} f_{xy}(R_0, x) \geq 0$ by the definition of an SDS. We now repeatedly let one voter push x above y. The duples are strategyproof and therefore non-perverse, so the probability of x cannot decrease during these steps. Thus, $b_n \geq \dots \geq b_1 \geq b_0$. For the second condition $b_i + b_{n-i} = \frac{2}{m(m-1)}$ for all $i \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$, observe that f_{xy} is a duple and thus, $f_{xy}(R,x) + f_{xy}(R,y) = 1$. Moreover, if $n_{xy}(R_j) = j$, then $n_{yx}(R_j) = n - j$, and therefore, $b_j + b_{n-j} = \frac{2}{m(m-1)} (f_{xy}(R_j, x) + f_{xy}(R_j, y)) = \frac{2}{m(m-1)}$. Hence, f^* is indeed a supporting size SDS.

Next, we show that all supporting size SDSs have $\beta \geq \frac{1}{m}$. Let R denote the profile in which all voters report $x_1 > x_2 > \dots > x_m$. Then, $f^*(R, x_1) = (m-1)b_n$ and $f^*(R, x_m) = (m-1)b_0$. Furthermore, the scoring vector b satisfies that $b_n + b_0 = \frac{2}{m(m-1)}$. Therefore, the probabilities of x_1 and x_m sum up to $\frac{2}{m}$. This means that f^* distributes a probability of $\frac{m-2}{m}$ among the alternatives $x_2, \dots, x_{m-1}$, so at least one Pareto-dominated alternative receives a probability of $\frac{1}{m}$ or more. Hence, f^* cannot satisfy β-*ex post* efficiency for $\beta < \frac{1}{m}$, and it follows from the construction of f^* that f also violates β-*ex post* efficiency for $\beta < \frac{1}{m}$. □

Next, we aim to show that no 0-randomly dictatorial SDS that can be represented as a mixture of unilaterals satisfies β-*ex post* efficiency for $\beta < \frac{1}{m}$. Similar to the proof of Lemma 5, we would like to use such an SDS f to construct a 0-randomly dictatorial SDS f^* that satisfies β-*ex post* efficiency for the same β as f, and that is additionally neutral and anonymous. Unfortunately, we cannot use Lemma 2 here as this lemma does not preserve that f^* is 0-randomly dictatorial. This follows by considering the unilaterals f^i_{xy}, which are defined as follows: if voter i's most preferred alternative a is x or y, then $f^i_{xy}(R, a) = \frac{1}{2}$ and $f^i_{xy}(R, b) = \frac{1}{2}$, where b denotes voter i's second best alternative; otherwise, $f^i_{xy}(R, a) = 1$. It follows from Lemma 1 that f^i_{xy} is 0-randomly dictatorial as only voter i can affect the outcome, and the probability of y does not increase if it swapped with x when x is voter i's best alternative and y is his second best one. As a consequence, the following SDS f^+ for $\binom{m}{2}$ voters and $m \geq 3$ alternatives is 0-randomly dictatorial. First, every voter i is assigned a different pair of alternatives $x, y \in A$. Next, f^+ chooses a voter uniformly at random and returns f^i_{xy}. If we apply the construction of Lemma 2 to f^+, we derive the point voting SDS that is defined by the scoring vector $(\frac{1}{n} - \frac{m-1}{2n\binom{m}{2}}, \frac{m-1}{2n\binom{m}{2}}, 0, \dots, 0)$. It follows immediately from Lemma 1 that this SDS is not 0-randomly dictatorial as pushing an alternative from second place to first place increases its probability always by $1 - \frac{m-1}{n\binom{m}{2}} \geq \frac{1}{3n}$.

Therefore, we propose another construction in the next lemma that, given an arbitrary strategyproof 0-randomly dictatorial SDS that can be represented as a mixture of unilaterals, constructs a strategyproof 0-randomly dictatorial SDS that is β-*ex post* efficient for the same β as the original SDS and that has a lot of symmetries. Unfortunately, this construction does not result in a neutral or anonymous SDS. Nevertheless, the resulting SDS is significantly easier to work with and its properties are crucial for the proof of Lemma 6,

Note that we require some additional terminology for the next lemma. In the sequel, we say that voter i or his unilateral SDS f_i is 0-randomly dictatorial for alternatives x, y if $f(R) = f(R^{i:yx})$ for *all* preference profiles R in which x is voter i's best alternative and y is his second best alternative.

Lemma 7. *Let f be a strategyproof 0-randomly dictatorial SDS that satisfies β-*ex post *efficiency for some $\beta \in [0, 1]$ and that can be represented as a mixture of unilaterals. Then, there is a strategyproof 0-randomly dictatorial SDS f^* for $\binom{m}{2}$ voters that can be represented as a mixture of unilaterals and that is β-*ex post *efficient for the same β as f. Moreover, f^* satisfies the following conditions:*

(i) For every voter $i \in N$, there is a set $\{x_i, y_i\}$ such that i is 0-randomly dictatorial for x_i, y_i and $\{x_i, y_i\} \neq \{x_j, y_j\}$ if $i \neq j$.

(ii) There is a constant δ such that $f^*(R^{i:cb}, c) - f^*(R, c) = \delta$ for all voters $i \in N$, alternatives $\{a, b\} = \{x_i, y_i\}$, $c \in A \setminus \{x_i, y_i\}$, and preference profiles R such that voter i reports a as his best alternative, b as his second best one, and c as his third best one.

(iii) If every voter $i \in N$ reports x_i and y_i as their two best alternatives, then there exists a scoring vector $a = (a_1, \dots, a_m)$ such that $a_1 = a_2 \geq 0$, $a_3 \geq \dots \geq a_m \geq 0$, and $f^*(R, x) = \sum_{i \in N} a_{|\{y \in A : y \succsim_i x\}|}$.

Proof. Let $\beta \in [0, 1]$ and let f denote a strategyproof 0-randomly dictatorial SDS that is β-*ex post* efficient and that can be represented as a mixture of unilaterals. In the sequel, we use f to construct the SDS f^* that satisfies all requirements of the lemma. Note that this proof is quite involved and therefore, we use some auxiliary observations that are proven in the end.

We start by representing f as $f(R) = \sum_{i \in N} \lambda_i f_i(R_i)$, where f_i denotes the unilateral SDS of voter i and $\lambda_i \geq 0$ is its weight. Note that we interpret unilaterals in this proof as SDSs that take the preference of a single voter as input. This is possible as unilaterals only rely on the preferences of a single voter. Observation 1 states that for every voter $i \in N$ there are alternatives x, y such that voter i is 0-randomly dictatorial for x and y. Note that there can be multiple such pairs of alternatives. Nevertheless, we associate from now on every voter i with exactly one such pair x_i, y_i because we can choose an arbitrary pair if there are multiple.

Next, we define the SDSs f_i^τ as $f_i^\tau(R, x) = f_i(\tau(R), \tau(x))$ for all voters $i \in N$ and permutations τ on the alternatives. Note that f_i^τ is again a unilateral SDS and thus, it only takes a single preference relation as input. Moreover, observation 2 states that these SDSs are all strategyproof and 0-randomly dictatorial for $\tau^{-1}(x_i)$, $\tau^{-1}(y_i)$, where τ^{-1} is the inverse permutation of τ and x_i and y_i are the alternatives for which f_i is 0-randomly dictatorial. Note again that f_i^τ can be 0-randomly dictatorial for many pairs of alternatives, but we only need the pair $\tau^{-1}(x_i)$, $\tau^{-1}(y_i)$. Hence, we assume for the sake of simplicity that f_i^τ is only for this pair 0-randomly dictatorial. This assumption does not restrict the generality of the proof as we only use it to partition the functions f_i^τ in the next step and we could derive this partition also by another relation that associates f_i^τ always with the correct pair of alternatives.

We group the SDSs f_i^τ with respect to the alternatives for which they are 0-randomly dictatorial. In more detail, let $F_{xy} = \{f_i^\tau : i \in N, \tau \in \mathrm{T}, f_i^\tau \text{ is 0-randomly dictatorial for } x, y\}$ denote the set of SDSs f_i^τ that are 0-randomly dictatorial for x and y. Note that these sets form a partition of all SDSs f_i^τ as every such SDS is by assumption only for a single pair of alternatives 0-randomly dictatorial. In more detail, f_i^τ is in F_{xy} if and only if $\{\tau(x), \tau(y)\} = \{x_i, y_i\}$ (where x_i, y_i are the alternatives for which f_i is 0-randomly dictatorial). There are for every f_i exactly $2(m-2)!$ permutations τ such that $\{\tau(x), \tau(y)\} = \{x_i, y_i\}$ because τ can behave arbitrarily on $A \setminus \{x, y\}$ and there are two possibilities to achieve $\{\tau(x), \tau(y)\} = \{x_i, y_i\}$. Hence, we derive that each set F_{xy} contains $2n(m-2)!$ SDSs.

Let N' be a set of $\binom{m}{2}$ voters and let $j \in N'$ be a voter of the new SDS f^*. We use the sets F_{xy} to define the unilateral SDS f_{xy} for a single voter j as $f_{xy} = \sum_{f_i^\tau \in F_{xy}} \frac{\lambda_i}{2(m-2)!} f_i^\tau$. First, note that f_{xy} is well-defined because the original SDS f is defined for n voters and for each unilateral SDS f_i of these voters, there are exactly $2(m-2)!$ functions in F_{xy}. Hence, $\sum_{f_i^\tau \in F_{xy}} \frac{\lambda_i}{2(m-2)!} = 2(m-2)! \sum_{i \in N} \frac{\lambda_i}{2(m-2)!} = 1$. Moreover, f_{xy} is strategyproof because it is a mixture of strategyproof SDSs, and it is 0-randomly dictatorial for x, y because all SDSs in F_{xy} are 0-randomly dictatorial for these alternatives.

Finally, we use these SDSs f_{xy} to define the SDS f^* for the $\binom{m}{2}$ voters in N'. Therefore, associate every voter $j \in N'$ with a different pair of alternatives x_j, y_j and define $f_j^* = f_{x_j y_j}$. Then, $f^*(R) = \frac{1}{\binom{m}{2}} \sum_{j=1}^{\binom{m}{2}} f_j^*(R_j)$. Clearly, f^* is strategyproof because it is a mixture of strategyproof SDSs. Moreover, it is 0-randomly dictatorial because for every voter j, there is a pair of alternatives for which f_j^* is 0-randomly dictatorial. Furthermore, Observation 3 shows that f^* is β-*ex post* efficient for the same β as f.

Next, we show that the SDS f^* satisfies the properties *(i)*, *(ii)*, and *(iii)*. First, note that it satisfies *(i)* by construction as every voter is 0-randomly dictatorial for a different pair of alternatives. For *(ii)* and *(iii)*, we show first the auxiliary claim that the unilateral of every voter is equal up to a permutation that maps the 0-randomly dictatorial pairs of the unilaterals onto each other. Consider for this claim two sets F_{xy} and $F_{x'y'}$, and choose a permutation τ' such that $\{\tau'(x), \tau'(y)\} = \{x', y'\}$. Then, we can map the set $F_{x'y'}$ to the set F_{xy} with the permutation τ'. In more detail, $F_{\tau'(x)\tau'(y)} = \{f_i^{\tau' \circ \tau} : i \in N, \tau' \circ \tau \in \mathrm{T}, f_i^{\tau' \circ \tau} \text{ is 0-randomly dictatorial for } \tau'(x), \tau'(y)\} = \{f_i^\tau : i \in N, \tau \in \mathrm{T}, f_i^\tau \text{ is 0-randomly dictatorial for } x, y\} = F_{xy}$. Therefore, the sets F_{xy} and $F_{x'y'}$ can be mapped to each other by every permutation that maps the pair x, y to the pair x', y'. The functions f_{xy} and $f_{x'y'}$ are defined by F_{xy} and $F_{x'y'}$ and thus, we derive that $f_i^*(R_i, a) = f_{x_i y_i}(R, a) = f_{\tau'(x_j)\tau'(y_j)}(\tau'(R_i), \tau'(a)) = f_j^*(\tau'(R_i), \tau'(a))$ for every permutation τ' that maps x_i and y_i to x_j and y_j. In particular, note $f_i^*(R_i, a) = f_i^*(\tau'(R_i), \tau'(a))$ if τ' maps x_i and y_i to x_i and y_i.

We now use the auxiliary claim to show *(ii)*. Hence, consider an arbitrary voter i and let x_i, y_i denote the alternatives for which he is 0-randomly dictatorial. Furthermore, consider a profile R in which voter i prefers x_i the most, y_i the second most, and some arbitrary alternative $z \in A \setminus \{x_i, y_i\}$ the third most. We define $\delta = f^*(R^{i:zy_i}, z) - f^*(R, z)$. First, note that $R^{i:zy_i}$ and R only differ in the preferences of voter i and thus, $f^*(R^{i:zy_i}, z) - f^*(R, z) = f_i^*(R_i^{i:zy_i}, z) - f_i^*(R_i, z)$. Next, observe that the order of the alternatives $z' \in A \setminus \{x_i, y_i, z\}$ in

R_i does not matter as f_i^* is strategyproof and therefore localized. Furthermore, we can use the auxiliary claim to show that $\delta = f_i^*(R_i^{i:z'y_i}, z') - f_i^*(R_i, z')$ for every $z' \in A \setminus \{x_i, y_i\}$. Therefore, consider another alternative $z' \in A \setminus \{x_i, y_i, z\}$ and the permutation τ such that $\tau(z) = z'$, $\tau(z') = z$, and $\tau(x) = x$ for all other alternatives. Our auxiliary claim states that $f_i^*(R', a) = f_{x_iy_i}(R', a) = f^*_{\tau(x_i)\tau(y_i)}(\tau(R'), \tau(a)) = f^*_{x_iy_i}(\tau(R'), \tau(a))$ for all alternatives $a \in A$ and preference profiles R'. Hence, if we exchange z with z' in R_i and $R_i^{i:zy_i}$, the probabilities of z and z' are exchanged, too. Consequently, it holds that $\delta = f_i^*(R_i^{i:z'y_i}, z') - f_i^*(R_i, z')$ for every $z' \in A \setminus \{x_i, y_i\}$. Finally, consider a voter $j \neq i$, let x_j, y_j denote the alternatives for which he is 0-randomly dictatorial, and let R denote a profile in which he reports x_j as his best alternative, y_j as his second best one, and an alternative $z \in A \setminus \{x_j, y_j\}$ as his third best one. Furthermore, let τ denote a permutation such that $\tau(x_j) = x_i$, $\tau(x_i) = x_j$, $\tau(x_j) = x_i$, $\tau(y_i) = y_j$, and $\tau(z) = z$ for all remaining alternatives, i.e., τ exchanges x_i with x_j, and y_i with y_j. The following equation shows that $f_j^*(R_j^{j:zy_j}, z) - f_j^*(R_j, z) = \delta$ for all voters $j \in N \setminus \{i\}$ and alternatives $z \in A \setminus \{x_j, y_j\}$. The last equality is true because x_i is the best alternative and y_i the second best one in $\tau(R_j)$. Hence, property *(ii)* is proven.

$$\begin{aligned} f_j^*(R_j^{j:zy_j}, z) - f_j^*(R_j, z) &= f^*_{x_jy_j}(R_j^{j:zy_j}, z) - f^*_{x_jy_j}(R_j, z) \\ &= f^*_{x_iy_i}(\tau(R_j^{j:zy_j}), z) - f^*_{x_iy_i}(\tau(R_j), z) \\ &= f_i^*(\tau(R_j^{j:zy_j}), z) - f_i^*(\tau(R_j), z) \\ &= \delta \end{aligned}$$

Finally, we discuss why property *(iii)* is true. We show first that if a fixed voter i top-ranks x_i, y_i, then f_i^* can be described by a scoring vector. This is equivalent to showing that every alternative that is ranked at position k by voter i receives probability a_k. Note that we always assume in the sequel that x_i and y_i are voter i's two best alternatives. First, consider an alternative $z \in A \setminus \{x_i, y_i\}$ and an arbitrary permutation τ that maps each alternative in $\{x_i, y_i\}$ to an alternative in $\{x_i, y_i\}$. It follows from our auxiliary claim that $f_i^*(R_i, z) = f_{x_iy_i}(R_i, z) = f_{\tau(x_i)\tau y_i}(\tau(R_i), \tau(z)) = f_{x_iy_i}(\tau(R_i), \tau(z))$. This means that permuting the alternatives $A \setminus \{x_i, y_i\}$ in R_i, also permutes the probabilities of these alternative. Hence, if an alternative $z \in A \setminus \{x_i, y_i\}$ is ranked k-th, it always receives a fixed probability a_k if voter i ranks x_i and y_i top. Moreover, it follows from strategyproofness that $a_k \geq a_{k+1}$ for all $k \in \{3, \dots, m-1\}$ and from the definition of an SDS that $a_m \geq 0$. Next, consider the alternatives x_i and y_i. Since we assume that these alternatives are top-ranked by voter i, it follows from strategyproofness that reordering the alternatives $A \setminus \{x_i, y_i\}$ does not affect their probabilities. Furthermore, we derive that each of these alternatives receives probability a_k when ranked k-th for $k \in \{1, 2\}$ by considering the permutation τ that swaps x_i and y_i. Then, $f_i^*(R_i, x_i) = f^*_{x_iy_i}(R_i, x_i) = f^*_{x_iy_i}(\tau(R_i), \tau(x_i))$. Hence, the probabilities assigned to these alternatives can be described by values a_1 and a_2. Moreover, note that f_i^* is 0-randomly dictatorial for x_i, y_i, which means that the probability of y_i does not change if it is reinforced against x_i. This entails that $a_1 = a_2$. Hence, the scoring vector of f_i^* satisfies all requirements of the lemma. Finally, our auxiliary claim implies that all f_j^* are described by the same scoring vector if x_j and y_j are the best alternatives of voter j. The reason for this is that we can just consider a permutation τ that swaps x_i and x_j, and y_i and y_j. Then, $f_i^*(R_i, x) = f_j^*(\tau(R_i), \tau(x))$ for all voters $i, j \in N$ and alternatives $x \in A$. Since $\tau(x)$ is ranked k-th in $\tau(R_i)$ if x is ranked k-th in R_i, this shows that the scoring vector a describes all f_i^* and therefore also f^*.

Observation 1: For every voter i, there exists a pair of alternatives x_i, y_i such

that $f(R) = f(R^{i:y_i x_i})$ for all preference profiles R in which voter i reports x as best alternative and y as second best one.

Since f_i is a strategyproof 0-randomly dictatorial SDS, it follows from Lemma 1 that for every voter i there exists a pair of alternatives x_i, y_i and preference profiles R such that $f_i(R) = f_i(R^{i:y_i x_i})$ if x_i is top-ranked and y_i is second-ranked by voter i in R. We show in the sequel that $f(R) = f(R^{i:y_i x_i})$ for all preference profiles R in which voter i reports x_i and y_i as his best and second best alternative. Since f is a mixture of strategyproof unilateral SDSs, it follows that $f(R) = f(R^{i:y_i x_i})$ if $f_i(R_i) = f(R_i^{i:y_i x_i})$ because the remaining voters $j \in N \setminus \{i\}$ do not change their preferences. Moreover, it follows from strategyproofness, which entails localizedness, that $f_i(\bar{R}_i, z) = f_i(R_i, z) = f_i(R_i^{i:y_i x_i}, z) = f_i(\bar{R}_i^{i:y_i x_i}, z)$ for $z \in \{x_i, y_i\}$ and all preferences profiles $\bar{R}$ that only differ from R in the order of the alternatives $A \setminus \{x_i, y_i\}$ in voters i's preference. Because $\bar{R}$ and $\bar{R}^{i:y_i x_i}$ differ by definition only in voter i's preferences on x_i and y_i, another application of localizedness implies that $f_i(\bar{R}) = f_i(\bar{R}^{i:y_i x_i})$. Hence, it holds indeed that $f(R) = f(R^{i:y_i x_i})$ for all preference profiles in which voter i reports x_i and y_i as his two best alternatives.

Observation 2: The SDS f_i^τ defined by $f_i^\tau(R, x) = f_i(\tau(R), \tau(x))$ is strategyproof and 0-randomly dictatorial for $\tau^{-1}(x_i)$, $\tau^{-1}(y_i)$.

First, note that f_i^τ is strategyproof as every manipulation of this SDS could be mapped to a manipulation of f_i. In more detail, if voter i can manipulate f_i^τ by switching from R to R', he can also manipulate f_i by switching from $\tau(R)$ to $\tau(R')$. This is true because a manipulation requires an alternative x such that $\sum_{y \succ_i x} f_i^\tau(R', y) > \sum_{y \succ_i x} f_i^\tau(R, y)$, which entails by definition of f_i^τ that $\sum_{y \succ_i x} f_i(\tau(R'), \tau(y)) > \sum_{y \succ_i x} f_i(\tau(R), \tau(y))$. Finally, since $y \succ_i x$ in R if and only if $\tau(y) \succ_i \tau(x)$ in $\tau(R)$, we derive that switching from $\tau(R)$ to $\tau(R')$ is a manipulation for voter i in f_i.

Furthermore, f_i^τ is a 0-randomly dictatorial SDS because f_i is one: Observation 1 shows that for every voter i, there exists a pair of alternatives x_i, y_i such that $f(R) = f(R^{i:y_i x_i})$ for all preference profiles R in which voter i prefers x_i the most and y_i the second most. It follows from this observation that $f_i^\tau(\tau^{-1}(R), \tau^{-1}(x)) = f_i(R, x) = f_i(R^{i:y_i x_i}, x) = f_i^\tau(\tau^{-1}(R^{i:y_i x_i}), \tau^{-1}(x))$ for all $x \in A$, where τ^{-1} is the inverse permutation of τ, i.e., $\tau^{-1}(\tau(x)) = x$ for all $x \in A$. Therefore, $f_i^\tau(\tau^{-1}(R), \tau^{-1}(x_i)) = f_i^\tau(\tau^{-1}(R^{i:y_i x_i}), \tau^{-1}(x_i))$ and $f_i^\tau(\tau^{-1}(R), \tau^{-1}(y_i)) = f_i^\tau(\tau^{-1}(R^{i:y_i x_i}), \tau^{-1}(y_i))$. Moreover, the preference profiles $\tau^{-1}(R)$ and $\tau^{-1}(R^{i:y_i x_i})$ only differ in the order of the two best alternatives $\tau^{-1}(x)$ and $\tau^{-1}(y)$ of voter 1 and the proof of Observation 1 entails thus that f_i^τ is 0-randomly dictatorial for these two alternatives.

Observation 3: The SDS f^* is β-*ex post* efficient for the same β as f.

For proving this observation, we construct another SDS f^+ and show that this SDS is β-*ex post* efficient for the same β as f. Then, we show that f^* is β-*ex post* efficient for the same β as f^+, which proves this auxiliary claim. Before defining f^+, we first define f^τ: just as the SDSs f_i^τ, it is defined as $f^\tau(R, x) = f(\tau(R), \tau(x))$. In particular, f^τ is β-*ex post* efficient for the same β as f. This follows by considering an arbitrary profile R in which an alternative x is Pareto-dominated. It is easy to see that $\tau(x)$ is then Pareto-dominated in $\tau(R)$, and we derive therefore that $f^\tau(R, x) = f(\tau(R), \tau(x)) \leq \beta$ because f is β-*ex post* efficient. Next, we define the SDS f^+ for $nm!$ voters as follows: we partition the voters $\{1, \dots, nm!\}$ into $m!$ sets $N_1, \dots, N_{m!}$ with $|N_i| = n$ and associate with every set a different permutation $\tau_i : A \to A$. Then, $f^+(R) = \frac{1}{m!} \sum_{i=1}^{m!} f^{\tau_i}(R_{N_i})$, where R_{N_i} denotes the restriction of R to the voters in N_i. Observe that f^+ is β-*ex post* efficient for the same β as f because an alternative x that is Pareto-dominated in R is also Pareto-dominated in all R_{N_i} and all f^{τ_i} are β-*ex post* efficient. Hence, it follows that $f^+(R, x) = \frac{1}{m!} \sum_{i=1}^{m!} f^{\tau_i}(R_{N_i}, x) \leq \frac{1}{m!} \sum_{i+1}^{m!} \beta = \beta$.

Next, we show that f^+ and f^* satisfy β-*ex post* efficiency for the same β. Therefore, we change the representation of f^+. The central observation here is that $f^\tau = \sum_{i \in N} \lambda_i f_i^\tau$.

Hence, we can also associate every voter $j \in \{1, \dots, nm!\}$ with an index $i \in N$ and a permutation τ such that each index-permutation pair is assigned exactly once and define $f_j = f_i^\tau$ and $\lambda_j = \lambda_i$ (i.e., the weight of f_i^τ is the same as the weight of f_i in the original SDS f). Then, we can write f^+ as $f^+(R) = \frac{1}{m!}\sum_{j=1}^{nm!} \lambda_j f_j(R_j)$. Next, note that every f_i^τ appears once in $f^+(R, x)$ and once in the union of all F_{xy}. Therefore, we can write $f^+(R) = \frac{2}{m(m-1)} \sum_{\{x,y\}\subset \binom{A}{2}} \sum_{f_i^\tau \in F_{xy}} \frac{\lambda_i}{2(m-2)!} f_i^\tau(R_i)$. Next, we restrict our attention to profiles R such that for all $\{x, y\} \subset \binom{A}{2}$, all voters j with $f_j \in F_{xy}$ submit the same preference. In this case, we may replace the preferences of all voters j with $f_j \in F_{xy}$ with a single preference. Then, there are exactly $\binom{m}{2}$ voters left, each of which is associated with a different pair of alternatives. In particular, we can use the definition of $f_{xy}(R_i) = \sum_{f_i^\tau \in F_{xy}} \frac{\lambda_i}{2(m-2)!} f_i^\tau(R_i)$ now as we apply all unilateral SDSs in F_{xy} to the same preference relation R_i. Hence, we derive that f^+ returns the same outcomes as f^* if all voters j with $f_j \in F_{xy}$ report for each $\{x, y\} \subset \binom{A}{2}$ the same preferences as the voter corresponding to f_{xy} in f^*. Since f^+ is β-*ex post* efficient, it follows therefore also that f^* is β-*ex post* efficient. □

Finally, we use Lemma 7 to prove that no 0-randomly dictatorial SDS that can be represented as a mixture of unilaterals is β-*ex post* efficient for $\beta < \frac{1}{m}$.

Lemma 6. *No 0-randomly dictatorial SDS that can be represented as a convex combination of unilaterals satisfies β-*ex post *efficiency for $\beta < \frac{1}{m}$ if $m \geq 3$.*

Proof. Let the SDS f denote a mixture of unilaterals. First, we apply Lemma 7 to construct the SDS f^* as specified by this lemma. In the sequel, we show that f^* is β-*ex post* efficient for $\beta \geq \frac{1}{m}$ and therefore f is also β-*ex post* efficient for $\beta \geq \frac{1}{m}$. In our proof, we construct a profile R^* in which every alternative must receive a probability of at most β which leads to a contradiction if $\beta < \frac{1}{m}$. Let N with $|N| = \binom{m}{2}$ be the set of voters of f^*. Furthermore, recall that every voter $j \in N$ is associated with a different pair of alternatives $\{x_j, y_j\}$ for which he is 0-randomly dictatorial because of Lemma 7 *(i)*.

First, we explain the construction of an auxiliary profile R. For this profile, we choose an arbitrary pair of alternatives a, b and assume without loss of generality that voter 1 is 0-randomly dictatorial for a, b, i.e, $\{a, b\} = \{x_1, y_1\}$. Voter 1 submits the preference relation $R_1 = b \succ_1 a \succ_1 \dots$ in R. Furthermore, there are $m-2$ other voters $j \in N$ with $a \in \{x_j, y_j\}$ and $b \notin \{x_j, y_j\}$. We assume without loss of generality that the voters in $\{2, \dots, m-1\}$ are these $m-2$ voters and that $a = x_j$. The preferences of voter $j \in \{2, \dots, m-2\}$ in R is $R_j = y_j \succ_j a \succ_j b \succ_j \dots$. Also, there are $m-2$ voters j with $a \notin \{x_j, y_j\}$ and $b \in \{x_j, y_j\}$. We assume that these voters are the ones in $\{m, \dots, 2m-3\}$ and that $b = y_j$. The preferences of these voters is $R_j : b \succ_j x_j \succ_j a \succ_j \dots$. Finally, the remaining voters $j \in \{2m-2, \dots, \binom{m}{2}\}$ have $a, b \notin \{x_j, y_j\}$. These voters report $R_j = x_j \succ_j y_j \succ_j b \succ_j a$ in R. Note that if $m = 3$, there are no voters of the fourth type. Furthermore, every voter $j \in N$ ranks the alternatives x_j, y_j for which he is 0-randomly dictatorial at the top. The full profile for $m = 4$ is shown in Figure 3.

We show next that $f^*(R, a) \leq \beta$ by constructing a new preference profile R' such that $f^*(R, a) = f^*(R', a) \leq \beta$. For the construction of R', let all voters in the second group $j \in \{2, \dots, m-1\}$ swap a and b, and all voters in the third group $j \in \{m, \dots, 2m-3\}$ swap a and x_j. The resulting preference profile is shown in Figure 4 for the case that $m = 4$. It is easy to see that b Pareto-dominates a in R' and, as f^* is β-*ex post* efficient, $f^*(R', a) \leq \beta$. Alternative a was moved from third to second and from second to third place by $m-2$ voters. It follows therefore from Lemma 7 *(ii)* and localizedness that the probability that alternative a gains when $m-2$ voters swap it from third to second place is the same as the probability that a looses when $m-2$ voters swap it from second to third place. Thus, we derive that $f^*(R, a) = f^*(R', a) \leq \beta$.

1	1	1	1	1	1
b	c	d	b	b	c
a	a	a	c	d	d
c	b	b	a	a	b
d	d	c	d	c	a

Figure 3: The preference profile R that in the proof of Lemma 6 for $m = 4$. There are four groups of voters. The first group contains only the first voter who is 0-randomly dictatorial for a and b. The next two groups have both $m-2$ voters and are 0-randomly dictatorial for one of a and b. The last group contains the remaining $\binom{m-2}{2}$ voters that are not 0-randomly dictatorial a or b. All voters have the pair for which they are 0-randomly dictatorial ranked at the top.

1	1	1	1	1	1
b	c	d	b	b	c
a	b	b	a	a	d
c	a	a	c	d	b
d	d	c	d	c	a

Figure 4: The preference profile R' for $m = 4$ alternatives that results from R by swapping the second and third alternative of voters $j \in \{2, \dots, 2m-3\}$. Alternative a is Pareto-dominated by alternative b.

Finally, note that in R, all voters $j \in N$ report the pair x_j, y_j for which they are 0-randomly dictatorial as their two best alternatives. Hence, Lemma 7 *(iii)* entails the existence of a scoring vector $(a_1, \dots, a_m)$ such that $a_1 = a_2 \geq 0$, $a_3 \geq \dots \geq a_m \geq 0$ and $f^*(R, x) = \sum_{j \in N} a_{|\{y \in A : y \succsim_j x\}|}$ for all $x \in A$. In particular, observe that the probability of an alternative only depends on its rank vector $r^*(x, R)$. Recall that the rank vector $r^*(x, R)$ of an alternative x in a preference profile R is the vector that contains the rank of x with respect to every voter in increasing order. The rank vector of alternative a in R is

$$r^*(a, R) = (\overbrace{2, \dots, 2}^{m-1}, \overbrace{3, \dots, 3}^{m-2}, \overbrace{4, \dots, 4}^{\binom{m-2}{2}}).$$

Furthermore, observe that $f^*(\bar{R}, x) \leq f^*(R, a)$ in every profile $\bar{R}$ in which (i) each voter $j \in N$ reports the alternatives x_j, y_j as his two best alternatives and (ii) $r^*(x, \bar{R})_k \geq r^*(a, R)_k$ for all $k \in \{m, \dots, \binom{m}{2}\}$. Condition (i) implies that f^* can be computed based on the scoring vector $(a_1, \dots, a_m)$. Furthermore, it implies that every alternative $x \in A$ is among the two best alternatives of exactly $m-1$ voters, and since $a_1 = a_2$, it follows that we can ignore these entries when comparing the probability of a in R with the probability of x in $\bar{R}$. Finally, the claim follows as $a_3 \geq \dots \geq a_m$ and $r^*(x, \bar{R})_k \geq r^*(a, R)_k$ for all $k \in \{m, \dots, \binom{m}{2}\}$ entails thus that $f^*(R, a) \geq f^*(\bar{R}, x)$. We use this fact to construct a new profile R^* where $f^*(R^*, x) \leq f^*(R, a) \leq \beta$ for every $x \in A$. Let every voter $j \in N$ report the alternatives x_j, y_j for which he is 0-randomly dictatorial as his two best alternatives. Furthermore, distribute all other alternatives such that no alternative is ranked third by more than $m-2$ voters. This is always possible as there are $m \geq 3$ alternatives and $\frac{m(m-1)}{2}$ voters. It follows from the construction that $r^*(x, R^*)_k \geq r^*(a, R)_k$ for every $k \in \{m, \dots, \binom{m}{2}\}$ and every $x \in A$. Hence, we derive that $f^*(R^*, x) \leq f^*(R, a) \leq \beta$ for every $x \in A$. If $\beta < \frac{1}{m}$, this entails that $\sum_{x \in A} f^*(R^*, x) < 1$, a contradiction. Thus, f^* cannot satisfy β-*ex post* efficiency for $\beta < \frac{1}{m}$, and thus, f violates this axiom, too. This show that there exists no 0-randomly dictatorial SDS that can be represented as a mixture of unilaterals and that satisfies β-*ex post* efficiency for $\beta < \frac{1}{m}$ when $m \geq 3$. □

Finally, we use Lemma 5 and Lemma 6 to prove the impossibility of 0-randomly dictatorial SDSs that satisfy β-*ex post* efficiency for $\beta < \frac{1}{m}$.

Theorem 6. *There is no strategyproof SDS that is both* 0*-randomly dictatorial and* β-ex post *efficient for* $\beta < \frac{1}{m}$ *if* $m \geq 3$.

Proof. Let f denote a strategyproof SDS for n voters and $m \geq 3$ alternatives that is 0-randomly dictatorial. Our argument focuses mainly on the profiles $R^{x,y}$, in which all voters report x as their best choice and y as their second best choice. The reason for this is that if $f(R, y) > \beta$ for some profile R in which y is Pareto-dominated by x, then $f(R^{x,y}, y) > \beta$. This is a direct consequence of strategyproofness as we can transform R into $R^{x,y}$ by reinforcing x and y. Hence, non-perversity implies that $f(R^{x,y}, y) \geq f(R, y) > \beta$. Moreover, localizedness entails that the order of the alternatives $z \in A \setminus \{x, y\}$ in $R^{x,y}$ is not important as it does not affect the probabilities of x and y.

Next, we use Theorem 1 to represent f as mixture of duples and unilaterals, i.e, $f = \lambda f_{uni} + (1 - \lambda) f_{duple}$, where $\lambda \in [0, 1]$, f_{uni} is a mixture of unilaterals, and f_{duple} is a mixture of duples. While Lemma 5 and Lemma 6 imply that f_{uni} and f_{duple} are not β-*ex post* efficient for $\beta < \frac{1}{m}$, this does not imply that f violates β-efficiency for $\beta < \frac{1}{m}$, too. The reason for this is that f_{uni} and f_{duple} may violate β-*ex post* efficiency for different profiles or alternatives. We solve this problem by constructing a strategyproof SDS $f^* = \lambda f^*_{uni} + (1 - \lambda) f^*_{duple}$ that is 0-randomly dictatorial and β-*ex post* efficient for the same β as f, and for which f^*_{uni} and f^*_{duple} denote mixtures of unilaterals and duples such that $f^*_{uni}(R^{x,y}, y) = f^*_{uni}(R^{\tau(x),\tau(y)}, \tau(y))$ and $f^*_{duple}(R^{x,y}, y) = f^*_{duple}(R^{\tau(x),\tau(y)}, \tau(y))$ for all permutations $\tau : A \to A$.

For this construction, we define f^τ as $f^\tau(R, x) = f(\tau(R), \tau(x))$ for every permutation $\tau : A \to A$. We construct the SDS f^* for $m!n$ voters as follows: we partition the electorate in $m!$ sets N_k with $|N_k| = n$ and associate each of these $m!$ sets with a different permutation $\tau_k : A \to A$. Then, we choose one of these sets N_k uniformly at random and consider from now on only the preference profile R_{N_k} defined by the voters in N_k. Finally, return $f^{\tau_k}(R_{N_k})$, where τ_k denotes the permutation associated with N_k. More formally, $f^*(R) = \frac{1}{m!} \sum_{k=1}^{m!} f^{\tau_k}(R_{N_k})$.

First, note that f^* is 0-randomly dictatorial because of Lemma 1. Since f is a 0-randomly dictatorial, there is for every voter i a profile R and alternatives x, y such that voter i prefers x the most in R and y the second most, and $f(R, y) = f(R^{i:yx}, y)$. Consequently, there are such profiles and alternatives for every voter in each SDS f^τ. Finally, we derive that such profiles and alternatives exist also for f^*. For a voter $i \in N_k$, the corresponding alternatives x, y are the same as for f^{τ_k}, the preferences of the voters in N_k are the same as for f^{τ_k}, and the preferences of the remaining voters do not matter. If f^* does not choose N_k in the first step, the preferences of voter i do not matter, and if f^* chooses N_k, it only computes f^{τ_k}. Hence, if voter i now swaps x and y, the outcome of f^* does not change as the outcome of f^{τ_k} does not change. Consequently, Lemma 1 implies that f^* is 0-randomly dictatorial.

Next, observe that $f^*(R) = \frac{1}{m!} \sum_{k=1}^{m!} f^{\tau_k}(R_{N_k})$ is strategyproof as it is a mixture of strategyproof SDSs. In particular, we can interpret each term $f^{\tau_k}(R_{N_k})$ as SDS defined for $m!n$ voters that ignores the preferences of the voters in $N \setminus N_k$. It follows immediately from its definition that f^{τ_k} is strategyproof for the voters in N_k, and thus we derive that f^* is strategyproof. Hence, we can use Theorem 1 to represent f^* as $f^* = \lambda f^*_{uni} + (1 - \lambda) f^*_{duple}$, where f^*_{uni} is a mixture of unilaterals and f^*_{duple} is a mixture of duples. In more detail, the following equation shows that $f^*_{uni}(R) = \frac{1}{m!} \sum_{k=1}^{m!} f^{\tau_k}_{uni}(R_{N_k})$ and $f^*_{duple}(R) = \frac{1}{m!} \sum_{k=1}^{m!} f^{\tau_k}_{duple}(R_{N_k})$, where f^τ_{uni} and f^τ_{duple} are defined analogously to f^τ.

$$
\begin{aligned}
f^*(R) &= \frac{1}{m!}\sum_{k=1}^{m!} f^{\tau_k}(R_{N_k})\\
&= \frac{1}{m!}\sum_{k=1}^{m!} \lambda f^{\tau_k}_{uni}(R_{N_k}) + (1-\lambda) f^{\tau_k}_{duple}(R_{N_k})\\
&= \lambda \frac{1}{m!}\sum_{k=1}^{m!} f^{\tau_k}_{uni}(R_{N_k}) + (1-\lambda)\frac{1}{m!}\sum_{k=1}^{m!} f^{\tau_k}_{duple}(R_{N_k})\\
&= \lambda f^*_{uni}(R_{N_k}) + (1-\lambda) f^*_{duple}(R_{N_k})
\end{aligned}
$$

Note that the definition from f^*_{uni} and f^*_{duple} entails that $f^*_{uni}(R^{x,y}, y) = f^*_{uni}(R^{\tau(x),\tau(y)}, \tau(y))$ and $f^*_{duple}(R^{x,y}, y) = f^*_{duple}(R^{\tau(x),\tau(y)}, \tau(y))$ for every permutation $\tau : A \to A$. For f^*_{uni}, this follows from the following equations and a symmetric argument holds for f^*_{duple}.

$$
\begin{aligned}
f^*_{uni}(R^{x,y}, y) &= \frac{1}{m!}\sum_{k=1}^{m!} f^{\tau_k}_{uni}(R^{x,y}_{N_k}, y) = \frac{1}{m!}\sum_{k=1}^{m!} f_{uni}(\tau_k(R^{x,y}_{N_k}), \tau_k(y))\\
&= \frac{1}{m!}\sum_{k=1}^{m!} f_{uni}(\tau_k(\rho(R^{x,y}_{N_k})), \tau_k(\rho(y))) = f^*_{uni}(R^{\rho(x),\rho(y)}, \rho(y)))
\end{aligned}
$$

The first two equations rely only on our definitions. The third equation follows because $\{\tau \circ \rho \colon \tau \in \mathrm{T}\} = \mathrm{T} = \{\tau_k \colon k \in \{1, \dots, m!\}\}$ for every permutation $\rho : A \to A$, where T is the set of all permutations on A.

Finally, we show that f^* violates β-*ex post* efficiency for every $\beta < \frac{1}{m}$, which entails that f also violates this axiom. We use Lemma 5 and Lemma 6 for this as these lemmas imply that f^*_{duple} and f^*_{uni} violate β-*ex post* efficiency. Note for this that f^*_{uni} is 0-randomly dictatorial as otherwise, f^* cannot be 0-randomly dictatorial. Hence, there are profiles R^1 and R^2, and alternatives x_1, y_1, x_2, and y_2 such that x_i Pareto-dominates y_i in R^i for $i \in \{1, 2\}$, $f^*_{uni}(R^1, y_1) \geq \frac{1}{m}$, and $f^*_{duple}(R^2, y_2) \geq \frac{1}{m}$. Hence, we derive from strategyproofness that $f^*_{uni}(R^{x_1,y_1}, y_1) \geq \frac{1}{m}$ and $f^*_{duple}(R^{x_2,y_2}, y_2) \geq \frac{1}{m}$. Finally, it follows from the symmetry of f^*_{uni} and f^*_{duple} with respect to the profile $R^{x,y}$ that $f^*_{uni}(R^{x,y}, y) \geq \frac{1}{m}$ and $f^*_{duple}(R^{x,y}, y) \geq \frac{1}{m}$ for all alternatives $x, y \in A$. Consequently, we conclude that $f^*(R^{x,y}, y) = \lambda f^*_{uni}(R^{x,y}, y) + (1-\lambda) f^*_{duple}(R^{x,y}, y) \geq \frac{1}{m}$ for all $x, y \in A$. This means that f^* and therefore also f violate β-*ex post* efficiency for every $\beta < \frac{1}{m}$.

□

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