Positional Social Decision Schemes: Fair and Efficient Portioning

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Abstract

We introduce a new family of social decision schemes, which can be viewed as a probabilistic counterpart of positional scoring rules. A rule in this family is defined by a scoring vector associating a positive value with each rank in a vote, and an aggregation function. We focus on two types of aggregation functions: those corresponding to egalitarianism (min, and leximin) and the Nash product. We examine the computation of the rules and their normative properties. We argue that some of these rules are particularly useful for time-sharing in an efficient and fair manner, or more generally for portioning.

1 Introduction

Voting concerns making a collective decision based on the preferences of voters over a set of alternatives. Most common voting rules take as input a collection of linear orders over the alternatives. Resolute rules output a single alternative, irresolute rules output a nonempty subset of alternatives, and randomized rules, also known as *social decision schemes* (SDSs), output a probability distribution over the set of alternatives. There are at least two reasons why one would like to use randomized rules. The first one has to do with normative properties: randomized rules tend to offer more strategyproofness guarantees than resolute rules, while keeping anonymity and neutrality: notably, *random dictatorship* is anonymous, neutral, and strategyproof; and they are directly applicable, unlike irresolute rules that have to be paired with a tie-breaking mechanism to output a winning alternative (this pairing implying resoluteness and the consequent drawbacks). The second reason has to do with *portioning*, that is, deciding how each of the candidates should be represented in the output. For example, one may want to decide which fraction of the time to listen to each of a set of radio channels, which fraction of the time the lab meeting will be on each particular day of the week, or which fraction of the seats of a parliament will go to each party: in such cases, the probability p_a of alternative a being the winner corresponds to the fraction of the resource (time, number of seats etc.) given to each alternative; it is a probability distribution only in a technical sense. While our rules make sense for both contexts, *the portioning interpretation is the main interpretation we have in mind.*

To illustrate the use of randomized voting for time-sharing, consider the following example, adapted from [17]. There are five radio channels (a, b, c, d, e), and only one radio. 1000 voters express their preferences on the channel they want to listen: the first 501 have the preference order $a \succ b \succ c \succ d \succ e$ and the other 499 voters have the preference order $e \succ b \succ c \succ d \succ a$. How should they decide which fraction of the time they will listen to each channel?[1] Let us review the randomized rules that are most commonly studied and see how they work on this example.

[1]Note that although this example may look at first glance like a fair division problem, *it is not* one. In fair division, every voter is allocated a share of the available resources and draws utility from her share: an allocation is a partition of the resources between the voters. Here, there is no allocation: every voter has to listen to the radio channel that is collectively chosen, even if it is not her preferred one; just like every voter has no choice but live with the alternative chosen by a voting rule. Notions such as envy-freeness, for instance, would not make sense in our setting.

Random dictatorship will result in listening to a and e a fraction, respectively, .501 and .499 of the time. In some settings, this may be a good choice, especially if voters like their first choice much better than any other choice. In other settings, less so, for instance when voters' utilities are typically decreasing regularly. In this case, rather than having every voter listen to her worst channel about half of the time, it may be better to listen to channel b (which seems like a good compromise) a significant part of the time.

Egalitarian simultaneous reservation [2] outputs all candidates ranked first by at least one voter with uniform probability (voters listen to a and e half of the time) thus also rules out potential compromise alternatives.

The *maximal lottery* rule [15] outputs a lottery p such that for any lottery q, a weak majority of voters prefers p to q (such a lottery is guaranteed to exist). It is appealing for normative reasons [8]. It is Condorcet-consistent, i.e., it outputs the Condorcet winner with probability 1 when there is one. As a result, it may be overly unfair to the minority: on our example, a will be listened to all the time although it is the worst channel for almost half of the voters.

Given an irresolute voting rule F, the *uniform* F rule outputs the winners of F with uniform probability: for instance, *uniform Borda* here outputs the unique Borda winner b with probability 1. While this may seem acceptable, it would no longer be if we add 200 voters with preference $c \succ a \succ e \succ d \succ b$. Alternative b is still the Borda winner, and uniform Borda leads to one sixth of the population listening to their worst channel all the time.

Given an irresolute voting rule F defined by the maximization of a numerical score, *proportional* F outputs each alternative with a probability proportional to its score.[2] Proportional Borda, on our orginal example without the extra 200 voters, outputs a, b, c, d, e with probabilities close to 0.2, 0.3, 0.2, 0.1, 0.2. Now, c and d are Pareto-dominated[3] by b: our voters will listen to a Pareto-dominated station 30 % of the time! Starting from another score-maximizing rule would not do better in general.[4]

Having mostly portioning in mind, our goal is to define and study a family of probabilistic rules whose members are desired to satisfy some or all of the following desiderata: (1) be reasonably fair to the voters, (2) be ex-post Pareto-efficient (no Pareto-dominated alternative should get a nonzero probability) and (3) potentially allow for compromise alternatives to be selected. Of the rules mentioned above, random dictatorship seems to best fulfill (1) and (2), but leaves no room for selecting compromise alternatives (3). Simultaneous reservation is Pareto-efficient and seems at least to partially satisfy (1). Uniform Borda (and more generally uniform F) and the maximal lottery rule also satisfy (2), but do not seem very fair. Proportional Borda (and more generally proportional F) arguably satisfies (1) but clearly violate (2). A time-sharing view of voting also applies to scenarios where repeated decisions are made with fairness and efficiency in mind [9, 11].

We will now define a *randomized version* of the well-known family of *positional scoring rules*. This family contains some known rules, namely random dictatorship and uniform rules, but that also includes other rules, which can be much more satisfactory than these in specific contexts. The intuition is the following: standard positional scoring rules can be seen as rules maximizing social welfare, where a voter's utility function is induced from his ranking by a fixed scoring vector, the winners being those alternatives maximizing utilitarian social welfare. For SDSs, this principle can be made more powerful: we will still assume that ranks induce utilities; then we will assume that social welfare is determined by some symmetric, non-decreasing aggregation function W. Finally, the SDS is defined by (ex ante)

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[2] As noted by an anonymous reviewer, two numerical scores may define irresolute rules F and F' that coincide, whilst at the same time proportional F and proportional F' are distinct.

[3] A first alternative Pareto dominates a second when all voters weakly prefer and some strictly prefer the first to the second.

[4] Note that, on the other hand, proportional Borda and proportional Copeland are strategyproof [3].

social welfare maximization.[5]

Although positional SDSs do not seem to have been studied before, Bogomolnaia et al. [5] consider a setting similar to ours, where the output lottery maximizes the aggregation of ex ante utilities, but in the context of *dichotomous preferences*, and having strategyproofness in mind. Like us, they consider the Nash and utilitarian social welfare. In a sense, we generalise their work in that we consider rankings instead of approval ballots, but our motivations are different.

Barberà [3] also relates positional scoring rules to social decision schemes, but in a completely different way. His aim is to define rules (such as proportional Borda, cf. above) that are strategyproof, and whose definition uses the candidate's positions in the rankings. These rules are not based on social welfare maximization.

The rest of the paper is structured as follows. In Section 2, we formally present positional social decision schemes. In Section 3 we focus on egalitarian rules, and show that they can be computed efficiently. In Section 4 we focus on rules obtained by aggregating expected scores by the Nash product, and also discuss their computation in the general case as well as in specific cases. In Section 5 we study a few important properties that should ideally be satisfied by a social decision scheme (proportional share, Pareto-efficiency, reinforcement, participation) and identify which of our rules is known to satisfy them. In Section 6 we give a final summary.

2 Positional social decision schemes

Let $X = \{x_1, \ldots, x_m\}$ be a set of alternatives and $N = \{1, \ldots, n\}$ be a set of voters. We use $\mathcal{L}(X)$ for the set of all linear orders over X. For $\succ \in \mathcal{L}(X)$, the *rank* of alternative x_j is the number of alternatives that are ranked higher than x_j in $\succ$ plus one, we write this as $r(\succ, x_j)$. We will write *abc* as shorthand for $a \succ b \succ c$. A *profile* $\boldsymbol{\succ} = (\succ_1, \ldots, \succ_n)$ is an element of $\mathcal{L}(X)^n$. We write $\Delta(X)$ for the set of probability distributions over X. An irresolute *social decision scheme* (SDS) is a function F from $\mathcal{L}(X)^n$ to a nonempty subset of $\Delta(X)$. We will typically denote a probability distribution, or lottery, by $p \in \Delta(X)$. For such a lottery we use p_a or $p(a)$ to denote the probability of selecting alternative a. When alternatives are indexed, for example $x_j \in X$, we write p_j instead of p_{x_j}. We also use the notation $(x_1 : p_1,\ x_2 : p_2, \ldots,\ x_m : p_m)$.

A *scoring vector* for m alternatives is a vector $\boldsymbol{s} = (s_1, \ldots, s_m)$ such that $s_1 \geq s_2 \ldots \geq s_m$ and $s_1 > s_m$. A scoring vector $\boldsymbol{s}$ is *strictly decreasing* if $s_j > s_{j+1}$ for all $j < m$. The Borda vector is $\mathbf{bor} = (m-1, m-2, \ldots, 0)$; the plurality vector is $\mathbf{plu} = (1, 0, \ldots, 0)$; the veto vector is $\mathbf{vet} = (1, \ldots, 1, 0)$.

Given a profile $\boldsymbol{\succ}$ and scoring vector $\boldsymbol{s}$, we use u_i^j to refer to the score that voter $i \in N$ assigns to alternative $x_j \in X$, that is, $u_i^j = s_{r(\succ_i, x_j)}$. We sometimes associate this score with the utility for voter i, as such, given a probability distribution $p \in \Delta(X)$, the expected utility for i is

$$u_i(p) = \sum_{j=1}^{m} p_j u_i^j \ .$$

Note that this definition supposes that there is a profile $\boldsymbol{\succ}$ and scoring vector $\boldsymbol{s}$ in the larger context. The vector of expected utilities for all voters is $\boldsymbol{u}(p)$. If p is clear from the context, we will simply write $\boldsymbol{u}$.

[5] A similar path—scoring followed by aggregation—has been followed for fair division in [6, 10, 4]. Also, Lesca and Perny [16] consider various aggregation functions like ours, in a fair division setting, starting from cardinal utilities. Goldsmith et al. [13] also generalize positional scoring rules with various aggregation functions (OWAs), but in a different context. None of these works makes use of randomization.

An *aggregation function* is a symmetric, non-decreasing function W mapping a collection of non-negative real numbers $(\alpha_1, \ldots, \alpha_n)$ to a non-negative real number. Standard choices are $W = \Sigma$, $W = \min$, and $W = \Pi$; these correspond respectively to utilitarianism, egalitarianism and the Nash product. Given an aggregation function W, the relation $>_W$ is the complete preorder on vectors that first applies the aggregation function then compares the results using the natural ordering on the real numbers. Leximin is a refinement of $>_{\min}$: given $\boldsymbol{u} \in \mathbb{R}^n$, let σ be a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(1)} \leq \ldots \leq u_{\sigma(n)}$, and let $\boldsymbol{u}^\uparrow = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. Given two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ of $\mathbb{R}^n$, $\boldsymbol{u} >_{\text{leximin}} \boldsymbol{v}$ if there is a $k \leq n$ such that $u_k^\uparrow > v_k^\uparrow$ and for all $i \leq k$, $u_i^\uparrow = v_i^\uparrow$.

We are now in position to define the positional social decision scheme induced by a scoring vector and an aggregation function.

Definition 1. *Let $\boldsymbol{s}$ be a scoring vector and W an aggregation function. The social decision scheme $F_{\boldsymbol{s},W}$ is defined as follows: for every profile $\succ$,*

$$F_{\boldsymbol{s},W}(\succ) = \{p \in \Delta(X) \mid \boldsymbol{u}(p) \text{ is maximal in } >_W\}$$

where $>_W$ is considered over $\{\boldsymbol{u}(p') \mid p' \in \Delta(X)\}$.

Note for any W such that $>_W$ is a total preorder the above equality makes sense, thus the definition can also be applied to leximin even though this is not an aggregation function.

The following example shows that the three most obvious choices of W, namely sum, minimum and product, lead to radically different outcomes.

Example 1. *Let $X = \{a, b\}$, $n = 3$, and $\succ = (ab, ab, ba)$. Let $\boldsymbol{s} = (1, 0)$.*

1. $F_{\boldsymbol{s},\Sigma}(\succ) = \{(a : 1,\ b : 0)\}$.
2. $F_{\boldsymbol{s},\min}(\succ) = \{(a : \frac{1}{2},\ b : \frac{1}{2})\}$.
3. $F_{\boldsymbol{s},\Pi}(\succ) = \{(a : \frac{2}{3},\ b : \frac{1}{3})\}$. *(See Section 4.)*

Note that in Example 1 $F_{\boldsymbol{s},\text{leximin}}(\succ)$ coincides with $F_{\boldsymbol{s},\min}(\succ)$; see Section 3 for examples on which they differ. The following observation shows that using $\sum(\cdot)$ leads to a well-known place.

Observation 1. *$F_{\boldsymbol{s},\Sigma}(\succ)$ consists of all lotteries whose support is contained in $R_{\boldsymbol{s}}(\succ)$, where $R_{\boldsymbol{s}}$ is the irresolute scoring rule associated with scoring vector $\boldsymbol{s}$. This is thus a superset of the output of uniform $R_{\boldsymbol{s}}$.*

3 Egalitarianism

The simplest form of egalitarianism is obtained with $W = \min$, and a finer form with leximin. As we see in Section 5, leximin offers the guarantee of ex-post efficiency, while this is not the case for min. On the other hand, $F_{\boldsymbol{s},\min}$ is simpler to compute. We first show that $F_{\boldsymbol{s},\min}$ can be computed in polynomial time—more precisely, that one optimal lottery can be computed in polynomial time. (This observation is not really new; see for instance [16] for a similar LP resolution of egalitarian allocation problems.)

Proposition 1. *For every $\boldsymbol{s}$, an optimal lottery for $F_{\boldsymbol{s},\min}$ can be computed in polynomial time.*

Proof. $F_{\boldsymbol{s},\min}$ can be computed by the resolution of a linear program (LP) with $m+1$ variables:

$$\begin{aligned}&\text{Maximize } t \text{ s.t.}\\ &\textstyle\sum_{j=1}^{m} u_i^j p_j \geq t \text{ for } i = 1, \dots, n\\ &\textstyle\sum_{j=1}^{m} p_j = 1\\ &p_j \geq 0 \text{ for } j = 1, \dots, m\end{aligned}$$

□

On the 1000-voter example of the Introduction, $F_{\mathbf{bor},\min}$ outputs the lottery where b has probability 1, while if we consider the additional 200 voters with preferences *caedb*, we get $(b : \frac{1}{3},\ c : \frac{2}{3})$.

When the scoring vector is the plurality vector, we obtain this easy characterization:

Observation 2. *$F_{\mathbf{plu},\min}$ outputs all alternatives ranked first in some vote, with uniform probability, and thus coincides with egalitarian simultaneous reservation.*

We now look at $F_{\boldsymbol{s},\text{leximin}}$. Although $F_{\boldsymbol{s},\text{leximin}}$ can output several lotteries, it is *essentially single valued*, that is, each voter gets the same expected utility (derived from $\boldsymbol{s}$) in each possible lottery outcome of the rule:

Proposition 2. *For any $\boldsymbol{s}$, $F_{\boldsymbol{s},\text{leximin}}$ is essentially single-valued.*

Proof. Assume that there are two different lotteries p and p' with different utility vectors u and u', both leximin-optimal, which implies that for all $i \in \{1, \dots, n\}$, $u_i^{\uparrow} = u_i'^{\uparrow}$. Now, consider the lottery $p^* = (p + p')/2$ and denote by w the corresponding utility vector. We can easily check that w leximin-dominates u (and u'), contradicting the leximin-optimality of u. □

On the other hand, there are vectors $\boldsymbol{s}$ for which $F_{\boldsymbol{s},\min}$ is not essentially single-valued, see the following example:

Example 2. *Let $n = 3$, $m = 4$, $\succ = (abcd, acbd, bdac)$, and $\boldsymbol{s} = (1, 1, 0, 0)$. $F_{\boldsymbol{s},\min}$ contains all lotteries of the form*

$$(a : p_a,\ b : p_b,\ c : \frac{1}{2} - p_a,\ d : \frac{1}{2} - p_b),$$

where $p_a + p_b \geq \frac{1}{2}$. The expected utility of the first voter varies between $\frac{1}{2}$ and 1. On the other hand, $F_{\boldsymbol{s},\text{leximin}}$ contains the unique lottery $(a : \frac{1}{2},\ b : \frac{1}{2})$.

Clearly, $F_{\boldsymbol{s},\text{leximin}}(\succ) \subseteq F_{\boldsymbol{s},\min}(\succ)$. It turns out that $F_{\boldsymbol{s},\text{leximin}}$ is polynomial-time computable as well, but instead of one LP, we solve $O(n^2)$ LPs.

Proposition 3. *For every $\boldsymbol{s}$, $F_{\boldsymbol{s},\text{leximin}}$ is polynomial time computable.*

Proof. The algorithm is specified as Algorithm 1 that requires running at most $n(n+1)/2$ LPs. The algorithm maintains a set N' of voters whose utilities are fixed. In each iteration of the while loop, one more voter is added to N'. The claim is that each time we run the main LP in the while loop, there exists at least one voter who cannot get utility strictly more than t^* and who can be added to set N'. Assume for contradiction that for each voter i in $N \setminus N'$, there exists at least one lottery which satisfied the guarantees of the other voters and which gives i utility strictly more than t^*. But in that case a convex combination of all such lotteries corresponding to each voter in $N \setminus N'$ will give each such voter utility strictly more than t^* which contradicts the optimality of the main LP. Now that we know that at

Input: (N, X, u) where u_i^j is the utility of voter i for the j-th alternative.
Output: $p = (p_1, \ldots, p_m)$.

$N' \longleftarrow \emptyset$
while $N' \neq N$ **do**
solve the following "main" linear program (LP):

$$\begin{aligned} &\text{Maximize} \quad t \quad \text{s.t.} \\ &\sum_{j=1}^{m} u_i^j p_j \geq t \text{ for } i \in N \setminus N' \\ &\sum_{j=1}^{m} u_i^j p_j = t_i \text{ for } i \in N' \\ &\sum_{j=1}^{m} p_j = 1 \quad \text{and } p_i \geq 0 \;\; \forall i \in \{1, \ldots, m\} \end{aligned}$$

Let the solution to the previous LP be t^*.
Find some $i' \in N \setminus N'$ for which the following corresponding LP has $\epsilon = 0$ as the optimal solution.

$$\begin{aligned} &\text{Maximize} \quad \epsilon \quad \text{s.t.} \\ &\sum_{j=1}^{m} u_i^j p_j \geq t^* \text{ for } i \in N \setminus (N' \cup \{i'\}) \\ &\sum_{j=1}^{m} u_{i'}^j p_j \geq t^* + \epsilon \\ &\sum_{j=1}^{m} u_i^j p_j = t_i \text{ for } i \in N' \\ &\sum_{j=1}^{m} p_j = 1 \quad \text{and } p_i \geq 0 \;\; \forall i \in \{1, \ldots, m\}, \epsilon \geq 0. \end{aligned}$$

For such an i', $N' \longleftarrow N' \cup \{i'\}$ and $t_{i'} \longleftarrow t^*$
end while
return $p = (p_1, \ldots, p_m)$

Algorithm 1: Computing a leximin lottery

least one voter i in $N \setminus N'$ cannot get more than t^* without violating the minimal guarantees of other voters, we can identify such an voter i' by running the second LP corresponding to i'. If $\epsilon = 0$ for the second LP, voter i''s utility is now fixed to t^* and cannot be increased. Hence i' is correctly added to N'. □

4 Nash social welfare

While egalitarianism is an appealing way of generalizing scoring rules to randomized social choice, it is not always desirable. Especially when the number of voters is large, maximizing the expected utility of the worst-off voter may lead to a high loss for a large fraction of the population. On the other hand, we have seen that utilitarian social welfare can be highly unfair to the minority. It it thus tempting to look for an aggregation function that lies in between; the most obvious choice is then the *Nash social welfare function* [14], where social welfare is the product of individual utilities. We start by showing that essential single-valuedness holds.

Proposition 4. *For all $\boldsymbol{s}$, $F_{\boldsymbol{s},\Pi}$ is essentially single-valued.*

Proof. Let us assume that there are two different Nash lotteries p and p' that yield a Nash social welfare of $U > 0$, with different utility vectors u and u' (i.e., $u_i \neq u'_i$ for at least some voter i). Now, the lottery p^* defined as $p^*_j = (p_j + p'_j)/2$ for every alternative j yields a utility vector u^* with Nash social welfare

$$\prod_{i\in N} u^*_i = \prod_{i\in N} \frac{u_i + u'_i}{2} > \prod_{i\in N} \sqrt{u_i \cdot u'_i} = U,$$

which contradicts the optimality of U. The inequality follows by the relation between the arithmetic and geometric means and since all utilities are positive. □

Now we consider the cases of plurality and veto, in which the optimal lottery is not only polynomial-time computable, but it also has a nice closed-form expression.

Proposition 5. *$F_{\mathbf{plu},\Pi}$ is the lottery p defined as follows: for every x, $p(x)$ is the proportion of voters who rank x first.*

Therefore, $F_{\mathbf{plu},\Pi}$ coincides with random dictatorship. This result is in fact not new: see Example 3.6a, page 79, in [17].

Things are considerably more complicated with veto. In any profile $\succ$ where there are alternatives that are not vetoed by any voter, $F_{\mathbf{vet},\Pi}(P)$ can be any probability distribution over these alternatives. We will show how to define $F_{\mathbf{vet},\Pi}$ for profiles in which every alternative is vetoed at least once. First, sort the alternatives in non-decreasing order in terms of their number of vetoes and rename them as $a_1, a_2, ..., a_m$ (i.e., a_1 and a_m are the least and most vetoed alternatives, respectively). For an alternative a, we denote by $\mathrm{vt}(a)$ the number of vetoes it receives in the profile.

Proposition 6. *Let k^* be the maximum integer such that $(k^* - 1)\mathrm{vt}(a_{k^*}) < \sum_{j=1}^{k^*} \mathrm{vt}(a_j)$. Then, there exists $k \in \{2, ..., k^*\}$ such that $F_{\mathbf{vet},\Pi}$ is the lottery p defined as*

$$p_{a_i} = 1 - \frac{(k-1)\mathrm{vt}(a_i)}{\sum_{j=1}^{k} \mathrm{vt}(a_j)},$$

if $i = 1, 2, ..., k$, and as $p_{a_i} = 0$ otherwise.

Proof. It can be easily seen that p, as defined, is indeed a lottery for every value of k, i.e., $\sum_{j=1}^{m} p_{a_j} = 1$. Since we consider only the case in which every alternative is vetoed by some voter, no alternative is picked with probability 1; this would make the Nash product equal to zero.

Let p be a lottery corresponding to $F_{\mathbf{vet},\Pi}$ and K be the set of alternatives in its support. Without loss of generality, we can assume that the lottery is monotonic in the ordering defined above for the alternatives, i.e., $p_{a_j} \geq p_{a_{j+1}}$. Indeed, it is not possible to have two alternatives a and b with $\mathrm{vt}(a) > \mathrm{vt}(b)$ such that $a \in K$ and $b \notin K$. If this was the case, the lottery p' with $p'_x = p_x$ for every alternative different than a and b, $p'_a = 0$, and $p'_b = p_a$ would have strictly higher Nash product compared to p. Also, $p_{a_j} = 0$ implies that $p_{a_{j+1}} = 0$. Indeed, if this is not the case and $p_{a_{j+1}} > 0$, then it must be that $\mathrm{vt}(a_{j+1}) = \mathrm{vt}(a_j)$; hence, by considering the lottery p' with $p'_{a_j} = p_{a_{j+1}}$ and $p'_{a_{j+1}} = 0$, we get a lottery with the same Nash social welfare with p.

Hence, the set K consists of the first k alternatives in the ordering defined above. We know that $k \geq 2$; we will also show that $k \leq k^*$. Since all the probabilities associated to alternatives in K are strictly positive (and also strictly smaller than 1), this means that their values nullify the partial derivatives of the Nash product as a function of the probabilities

p uses for the alternatives in K. The partial derivative of the Nash product with respect to p_{a_i} for $i = 1, 2, ..., k-1$ is

$$\frac{\partial}{\partial p_{a_i}} \left(\sum_{j=1}^{k-1} p_{a_j}\right)^{\mathrm{vt}(a_k)} \prod_{j=1}^{k-1} \left(1 - p_{a_j}\right)^{\mathrm{vt}(a_j)}$$
$$= \left(\frac{\mathrm{vt}(a_k)}{\sum_{j=1}^{k-1} p_{a_j}} - \frac{\mathrm{vt}(a_i)}{1 - p_{a_i}}\right) \cdot \left(\sum_{j=1}^{k-1} p_{a_j}\right)^{\mathrm{vt}(a_k)}$$
$$\cdot \prod_{j=1}^{k-1} \left(1 - p_{a_j}\right)^{\mathrm{vt}(a_j)}.$$

Using $\sum_{j=1}^{k-1} p_{a_j} = 1 - p_{a_k}$ we obtain

$$p(a_i) = 1 - \frac{\mathrm{vt}(a_i)}{\mathrm{vt}(a_k)}(1 - p(a_k)) \tag{1}$$

and, summing (1) over all alternatives in K, we have

$$1 = \sum_{a \in K} p(a) = p(a_k) + \sum_{j=1}^{k-1} p(a_j)$$
$$= p(a_k) + k - 1 - \frac{\sum_{i=1}^{k-1} \mathrm{vt}(a_i)}{\mathrm{vt}(a_k)}(1 - p(a_k))$$
$$= p(a_k)\frac{\sum_{i=1}^{k} \mathrm{vt}(a_i)}{\mathrm{vt}(a_k)} + k - \frac{\sum_{i=1}^{k} \mathrm{vt}(a_i)}{\mathrm{vt}(a_k)}$$

and, hence,

$$p(a_k) = 1 - \frac{(k-1)\mathrm{vt}(a_k)}{\sum_{j=1}^{k} \mathrm{vt}(a_j)}.$$

Now, if we had $k > k^*$, then, by the definition of k^*, it would also be $p(a_k) = 0$ which contradicts our assumption that the probabilities in the support of p are strictly positive. So, $k \leq k^*$ and, by (1), for $i = 1, 2, ..., k$ we have

$$p(a_i) = 1 - \frac{(k-1)\mathrm{vt}(a_i)}{\sum_{j=1}^{k} \mathrm{vt}(a_j)}$$

□

Proposition 6 suggests a polynomial-time algorithm for computing $F_{\mathbf{vet},\Pi}$ in profiles where each alternative is vetoed at least once as follows: it first computes k^*; then, it considers the $k^* - 1$ lotteries defined according to the statement of Proposition 6 and picks the one that maximizes the Nash social welfare.

One might hope to obtain similar positive results for computing $F_{\boldsymbol{s},\Pi}$ exactly for other scoring vectors such as k-approval[6] or Borda. Unfortunately, there is a fundamental difficulty in doing so. The following examples shows that the Nash lottery for 2-approval and Borda can have irrational probabilities.

[6]The k-approval scoring vector has 1 in the first k coordinates and 0 in the rest.

Example 3. *Let* $\succ = (abcd, acbd, bcad, cdab)$ *and* $\boldsymbol{s} = (1, 1, 0, 0)$*.* $F_{\boldsymbol{s},\Pi}(\succ)$ *is the solution of the following non-LP:*

$$\begin{aligned}&\text{Maximize}\\ &((p_a + p_b)\cdot(p_a + p_c)\cdot(p_b + p_c)\cdot(p_c + p_d)) \quad \text{s.t.}\\ &p_a + p_b + p_c + p_d = 1\end{aligned}$$

As d *is Pareto dominated by* c*,* $p_d = 0$*. Permuting* a *and* b *in the voters' rankings does not change the utilities received due to 2-approval, thus* $p_a = p_b \leq 1/2$*. Thus* $p_c = 1 - 2p_a$*. Substituting into the above non-LP, we have a Nash welfare of* $2p_a(1 - 2p_a)(1 - p_a)^2$*, which is maximized for* $p_a = \frac{7-\sqrt{17}}{16}$*.*

Example 4. *Let* $\succ = (abc, acb, cab, cab)$*.* $F_{\mathbf{bor},\Pi}(\succ)$ *is the solution of the following non-LP:*

$$\begin{aligned}&\text{Maximize} \quad \left((2p_a + p_b)\cdot(2p_a + p_c)\cdot(2p_c + p_a)^2\right) \quad \text{s.t.}\\ &p_a + p_b + p_c = 1\end{aligned}$$

Here b *is dominated by* a*, so* $p_b = 0$*, and thus* $p_c = 1 - p_a$*. Substituting these, we get a Nash social welfare of* $2p_a(1 + p_a)(2 - p_a)^2$ *which is maximized for* $p_a = \frac{1+\sqrt{33}}{8}$*.*

On the positive side, computing a lottery that approximates $F_{\boldsymbol{s},\Pi}$ for any scoring vector $\boldsymbol{s}$ is possible using techniques for solving convex programs. Recalling that by u_i^j we denote the score the j-th alternative gets from the i-th voter in a profile $\succ$, $F_{\boldsymbol{s},\Pi}(\succ)$ is the solution of the following convex program:

$$\begin{aligned}&\text{Maximize} \quad \left(\textstyle\sum_{i=1}^{n} \log u_i\right) \quad \text{s.t.}\\ &u_i = \textstyle\sum_{j=1}^{m} u_i^j p_j \text{ for } i = 1, \ldots, n\\ &\textstyle\sum_{j=1}^{m} p_j = 1\\ &p_j \geq 0, \text{ for } j = 1, \ldots, m\end{aligned}$$

Then, using standard techniques (such as the ellipsoid method; e.g., see the discussion in [18]), a lottery that approximates the objective value of this convex program within an additive term of ϵ can be computed in time that is polynomial in the size of the profile and $1/\epsilon$.

5 Normative properties

We now show that our rules enjoy a number of desirable social-choice theoretic properties. Our goal is to investigate SDSs which (due to our portioning view) should guarantee some level of fairness to the voters, be Pareto-efficient, and may allow the selection of compromise alternatives. All proofs for this section are placed in the appendix.

5.1 Positive fair share

If randomized voting for tie-breaking is used in a one-shot context, then fairness is not particularly meaningful. This is especially the case if the number of voters is large and the number of alternatives is small: we have to accept that, most likely, some voters will have their worst alternative elected (political elections being a good example). However, with a portioning view of randomized voting, fairness does count. Fairness in voting has been discussed in [2, 5, 9, 11]. Bogomolnaia *et al.* [5] define the *positive share* property [5]: an

SDS F satisfies *positive shares* if for each profile $\succ$, voter i, and $p \in F(\succ)$, if x_i is the least preferred alternative of voter i then $p(x_i) < 1$. Though positive share is a weak property, it is violated by the maximal lottery rule, by utilitarian rules $F_{\boldsymbol{s},\Sigma}$ (provided that $s_m = 0$), and by some proportional rules (however, proportional Borda and more generally proportional scoring rules satisfies it). The following result is immediate:

Proposition 7. *If $s_m = 0$ and $W \in \{\Pi, \min, \text{leximin}\}$, then $F_{\boldsymbol{s},W}$ satisfies positive shares.*

Note that $s_m = 0$ applies in particular to Borda and k-approval (including plurality and veto).

5.2 Pareto efficiency

An SDS F is *ex post efficient* if the support of any lottery in $F(\succ)$ consists only of Pareto-efficient alternatives. Other efficiency notions for SDSs need to refer to a lottery extension principle [1]. This is achieved through the notion of (first order) *stochastic dominance* (SD): a lottery p SD-dominates a lottery p' with respect to an individual's preference's $\succ_i$ if for all $i \in N$ and $x \in X$,

$$\sum_{y \in X, y \succ_i x} p'(y) \quad \geq \sum_{y \in X, y \succ_i x} p(y) \enspace .$$

Similarly, p strictly SD-dominates p' with respect to $\succ_i$ if

$$\sum_{y \in X, y \succ_i x} p'(y) \quad > \sum_{y \in X, y \succ_i x} p(y) \enspace .$$

We say F is SD-efficient if for all profiles $\succ$ there is no $p \in F(\succ)$ and $p' \in \Delta(X)$ such that p' SD-dominates p with respect to $\succ_i$ for all $i \in N$ and p' strictly SD-dominates p with respect to $\succ_i$ for some $i \in N$.

Proposition 8. *If $\boldsymbol{s}$ is a strictly decreasing scoring vector, and W a strictly monotonic aggregation function, then $F_{\boldsymbol{s},W}$ satisfies SD-efficiency.*

Since SD-efficiency implies ex-post efficiency, under the conditions of Proposition 8, $F_{\boldsymbol{s},W}$ is ex-post efficient. (When s is not strictly decreasing, and/or W is not strictly increasing, then we get the weaker result that among all lotteries in $F_{\boldsymbol{s},W}(\succ)$ there is at least one whose support consists only of Pareto-efficient alternatives.)

5.3 Population consistency

For $\succ = (\succ_1, \dots \succ_n)$ and $\succ' = (\succ_{n+1}, \dots, \succ_p)$, we denote $\succ + \succ' = (\succ_1, \dots, \succ_p)$.

Definition 2. *An SDS F satisfies* population consistency *if whenever $F(\succ) \cap F(\succ') \neq \emptyset$ then $F(\succ + \succ') = F(\succ) \cap F(\succ')$. It is said to satisfy* weak population consistency *if for all $\succ$ and $\succ'$, $F(\succ + \succ') \supseteq F(\succ) \cap F(\succ')$.*

An aggregation function W is said to be *reinforcing* if $W(\alpha_1, \dots, \alpha_n) \geq W(\beta_1, \dots, \beta_n)$ and $W(\alpha_{n+1}, \dots, \alpha_t) \geq W(\beta_{n+1}, \dots, \beta_t)$ together imply $W(\alpha_1, \dots, \alpha_t) \geq W(\beta_1, \dots, \beta_t)$. It is further *strictly reinforcing* if it is reinforcing and if $W(\alpha_1, \dots, \alpha_n) \geq W(\beta_1, \dots, \beta_n)$ and $W(\alpha_{n+1}, \dots, \alpha_t) > W(\beta_{n+1}, \dots, \beta_t)$ together imply $W(\alpha_1, \dots, \alpha_t) > W(\beta_1, \dots, \beta_t)$.

Proposition 9.

1. *If W is reinforcing then $F_{\boldsymbol{s},W}$ satisfies weak population consistency.*

2. *If W is strictly reinforcing then $F_{\boldsymbol{s},W}$ satisfies population consistency.*

Clearly, Σ, Π and (with some abuse of notation) leximin are strictly reinforcing, and min is weakly reinforcing, therefore:

Corollary 1.

1. *$F_{\boldsymbol{s},\Sigma}$ and $F_{\boldsymbol{s},\text{leximin}}$ satisfy population consistency.*

2. *$F_{\boldsymbol{s},\min}$ satisfies weak population consistency.*

3. *$F_{\boldsymbol{s},\Pi}$ satisfies weak population consistency; if $s_m > 0$ then $F_{\boldsymbol{s},\Pi}$ satisfies population consistency.*

If W is associative[7] and monotonic (respectively strictly monotonic) then W is reinforcing (respectively strictly reinforcing). We recall that in the non-randomized setting, population consistency is the characteristic property of positional scoring rules. However, this does not carry over to positional SDSs, since the maximal lottery rule satisfies population consistency as well.

5.4 Participation

In order to define the participation property (stating that it is always in a voter's interest to participate), we first need to extend preference relations between alternatives to preference relations between lotteries. The most common way to do so is by using *stochastic dominance*: an SDS F satisfies *SD-participation* [7] if there does not exist a profile $\succ$ and an voter i for which the lottery $F(\succ_{-i})$ stochastically dominates the lottery $F(\succ)$ w.r.t. i's preferences, where $\succ_{-i}$ is the profile obtained from $\succ$ by i abstaining. This notion is defined for resolute SDSs, therefore it does not directly apply to irresolute SDSs; *however, it does apply if they are essentially single-valued*, which is the case for $W = \Pi$ or leximin.

Proposition 10. *For any strictly decreasing scoring vector s, $F_{\boldsymbol{s},\Pi}$ satisfies SD-participation.*

Proposition 11. *For any strictly decreasing scoring vector s, $F_{\boldsymbol{s},\text{leximin}}$ satisfies SD-participation.*

Stronger generalizations of participation are generally not satisfied by our rules. An SDS F satisfies *strong SD-participation* [7] if for all profiles $\succ$ and voters i the lottery $F(\succ)$ SD-dominates $F(\succ_{-i})$ with respect to $\succ_i$. Note that this is a stronger condition than SD-participation as two lotteries can be incomparable according to SD-dominance.

Proposition 12. *$F_{\mathbf{bor},\text{leximin}}$ and $F_{\mathbf{bor},\min}$ do not satisfy strong SD-participation.*

Proposition 13. *$F_{\mathbf{bor},\Pi}$ does not satisfy strong SD-participation.*

The exception here are rules that take the sum.

Proposition 14. *$F_{\boldsymbol{s},\Sigma}$ satisfies strong-SD-participation [7].*

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[7] By "associative" here we mean the natural extension of the traditional property on binary operators to functions that take sequences as input. The basic idea is the the order of the sequence is unimportant.

	Pos. fair shares	SD efficient	Polytime	Strategyproof
$F_{\mathbf{plu},\Sigma}$ (RD)	✓	✓	✓	✓
$F_{\mathbf{plu},\min}$ (ESR)	✓	✓	✓	x
$F_{\mathbf{bor},\Sigma}$	x	✓	✓	x
$F_{\mathbf{bor},\min}$	✓	x	✓	x
$F_{\mathbf{bor},\text{leximin}}$	✓	✓	✓	x
$F_{\mathbf{vet},\Pi}$	✓	x	✓	x
$F_{\mathbf{bor},\Pi}$	✓	✓	($\star$)	x
Maximal lottery	x	✓	✓	x
Proportional Borda	✓	x	✓	✓

Table 1: Summary of normative results.

5.5 Strategyproofness

Strategyproofness requires that no voter can misreport her preferences and get more ex ante utility with respect to her private cardinal utilities. Deterministic positional scoring rules are well-known to be not strategyproof. It follows that $F_{\boldsymbol{s},\Sigma}$ rules are not strategyproof. Similarly, simple examples can be constructed to show that $F_{\boldsymbol{s},\min}$, $F_{\boldsymbol{s},\text{leximin}}$ and $F_{\boldsymbol{s},\Pi}$ rules are not strategyproof as well. Note that the lack of strategyproofness of the class of rules we consider is not their design flaw but simply a consequence of imposing anonymity and ex post efficiency. It follows from [12] that for strict preferences, the only anonymous, ex post efficient, and strategyproof rule is random dictatorship. If we forego strategyproofness, the world of SDSs becomes much more interesting.

6 Discussion

Table 1 gives a summary of properties satisfied by some typical SDSs that belong to the family defined in this paper (the first seven), together with two other SDSs that are not part of the family (the last two). The first two rules have already been defined: RD stands for random dictatorship, ESR for egalitarian simultaneous reservation; the five rules after these are novel. SD participation and reinforcement are omitted from the table because they are less discriminating than the other properties. The ($\star$) indicates that no polynomial-time algorithm is known and NP-hardness is conjectured.

Coming back to our initial discussion about the possible uses of social decision schemes, it is now clearer that they should not be evaluated along the same criteria whether they are used for *randomized voting* or for *portioning*. Positive fair share, for instance, is not necessary desirable for randomized voting (it would exclude the Borda rule with uniform tie-breaking, which, in many classical contexts, is a very good rule). Our family of rules can be used to pick a desirable rule in function of the context. If all we care about is strategyproofness and efficiency, then the good old random dictatorship is probably the best solution. If we care about strategyproofness and fair share, but not so much about efficiency, then proportional Borda can be a good solution (it could for instance be used for deciding the apportionment between parties at a parliament election). New rules, in particular those induced by the min, leximin and Nash aggregation functions, reconcile fairness and efficiency, at the price of losing strategyproofness.

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A Proofs

Proposition 8. *If $\boldsymbol{s}$ is a strictly decreasing scoring vector, and W a strictly monotonic aggregation function, then $F_{\boldsymbol{s},W}$ satisfies SD-efficiency.*

Proof. Assume $\boldsymbol{s}$ is strictly decreasing, W is strictly increasing, and that p' SD-dominates p w.r.t. P. Then for every $i \in N$ and $x \in X$, $\sum_{y \in X, y \succ_i x} p'(y) \geq \sum_{y \in X, y \succ_i x} p(y)$ with one inequality being strict. Let $i \in N$ and σ such that $x_{\sigma(1)} \succ_i \ldots \succ_i x_{\sigma(m)}$. Now,

$$\begin{aligned} u_i(p') - u_i(p) &= \sum_{j=1}^{m} s_j \left(p'(x_{\sigma(j)} - p(x_{\sigma(j)}) \right) \\ &= \sum_{j=1}^{m-1} (s_j - s_{j+1}) \\ &\geq \sum_{k=1}^{j} p'(x_{\sigma(j)}) - p(x_{\sigma(j)}) \geq 0 \ , \end{aligned}$$

with one inequality being strict because $s_j > s_{j+1}$ for all j. By strict monotonicity of W, we have $W(u_1(p'), \ldots, u_n(p')) > W(u_1(p), \ldots, u_n(p))$, therefore, $p \notin F_{s,W}(P)$. □

Proposition 9.

1. *If W is reinforcing then $F_{\boldsymbol{s},W}$ satisfies weak population consistency.*

2. *If W is strictly reinforcing then $F_{\boldsymbol{s},W}$ satisfies population consistency.*

Proof. Assume W is reinforcing and let $\succ$ and $\succ'$ be as above. Let $p \in F(\succ) \cap F(\succ')$; this means that for all p', $W(u_1(p), \ldots, u_n(p)) \geq W(u_1(p'), \ldots, u_n(p'))$ and $W(u_{n+1}(p), \ldots, u_t(p)) \geq W(u_{n+1}(p'), \ldots, u_t(p'))$. For each $i \leq p$ let $u_i = (s_{r(x_k, \succ_i)} | 1 \leq k \leq p)$. Because W is reinforcing, for all p' we have $W(u_1(p), \ldots, u_t(p)) \geq W(u_1(p'), \ldots, u_t(p'))$ from which $p \in F(\succ + \succ')$ follows, which proves I.

For II, the strict inequality follows in a similar way, but we also need the converse inclusion. Let $p' \notin F(\succ) \cap F(\succ')$: without loss of generality, $p' \notin F(\succ)$. Then $W(u_1(p), \ldots, u_n(p)) > u_1 W((p'), \ldots, u_n(p'))$ and $W(u_{n+1}(p), \ldots, u_t(p)) \geq W(u_{n+1}(p'), \ldots, u_t(p'))$. Because W is strictly reinforcing, we have $W(u_1(p), \ldots, u_t(p)) > W(u_1(p'), \ldots, u_t(p'))$ therefore, $p' \notin F(\succ + \succ')$. □

Proposition 10. *For any strictly decreasing scoring vector s, $F_{\boldsymbol{s},\Pi}$ satisfies SD-participation.*

Proof. If $F_{\boldsymbol{s},\Pi}$ does not satisfy SD-participation then let $\succ$ and i be s.t.

$$F_{\boldsymbol{s},\Pi}(\succ_{-i}) = p' \text{ SD-dominates } F_{\boldsymbol{s},\Pi}(\succ) = p. \tag{2}$$

As $\boldsymbol{s}$ is strictly decreasing, (2) implies

$$u_i(p') > u_i(p). \tag{3}$$

Since $F_{\boldsymbol{s},\Pi}(\succ) = p$, we have

$$\prod_{j\in N} u_i(p) \geq \prod_{j\in N} u_i'(p). \tag{4}$$

From (3) and (4), $\prod_{j\in N\setminus\{i\}} u_i(p) \geq \prod_{j\in N\setminus\{i\}} u_i'(p)$, which contradicts $F_{\boldsymbol{s},\Pi}(\succ_{-i}) = p'$. ☐

Proposition 12. *$F_{\mathbf{bor},\text{leximin}}$ and $F_{\mathbf{bor},\min}$ do not satisfy strong SD-participation.*

Proof. Consider the profile where voter 1 has preferences *abcde*, voter 2 preferences *dbcae* and voter 3 preferences *aebdc*. When 3 does not vote, the (unique) outcome is the lottery where b has probability 1 which gives utility of 3 to voters in $\{1,2\}$. Now if 3 participates, the maximum possible minimum welfare is 2.5 which is achieved for all voters by lottery $(a : \frac{1}{2},\ d : \frac{1}{2})$. Since the second lottery does not SD-dominate the first according to voter 3's preferences, it follows that $F_{\mathbf{bor},\text{leximin}}$ and $F_{\mathbf{bor},\min}$ do not satisfy strong SD-participation. ☐

Proposition 13. *$F_{\mathbf{bor},\Pi}$ does not satisfy strong SD-participation.*

Proof. Consider the profile where voter 1 has preferences *abcdefgh*, voter 2 has preferences *fgchabde* and voter 3 has preferences *fghcabde*. The (unique) outcome is the lottery where c has probability 1 if only the first two voters participate, whereas for all three voters we have a mixture between a and f, which is not SD-dominant for voter 3. ☐

Proposition 14. *$F_{\boldsymbol{s},\Sigma}$ satisfies strong-SD-participation [7].*

Proof. It has recently been proved [7] that social welfare maximizing lotteries satisfy strong SD-participation, which implies that $F_{\boldsymbol{s},\Sigma}$ satisfies strong SD-participation. ☐

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