How Hard Is It to Control a Group?

Yongjie Yang and Dinko Dimitrov

Abstract

Group identification models situations where a set of individuals are asked to determine who among themselves are socially qualified. In this paper, we study the complexity of three group controlling problems, namely Group Control by Adding Individuals, Group Control by Deleting Individuals and Group Control by Partition of Individuals for various social rules. In the controlling problems, an external agent has an incentive to make a given set of individuals socially qualified by adding, deleting or partition of individuals. We achieve both polynomial-time solvability results and NP-hardness results. In addition, we prove that some social rules are immune to these forms of control.

1 Introduction

Decision making plays an important role in multi-agent systems. For instance, a set of agents (or robots) need to complete a task cooperatively. Due to some reasons (e.g., in order to minimize the cost of the resources), only a few agents can take the job. In this case, all agents need to make a joint decision of which agents are going to take the job. In this paper, we study such a decision making model, in which a set N of individuals desire to select a subset of individuals of N. In particular, each individual qualifies or disqualifies every other individual, and then a social rule is applied to select the socially qualified individuals. This model has been widely studied under the name *group identification* in economics [5, 6, 16, 17]. In particular, the liberal rule, the consent rules, the Consensus-Start-Respecting rule (CSR for short) and the Liberal-Start-Respecting rule (LSR for short) have been extensively studied in the literature [5, 15, 17]. Due to the liberal rule, an individual is socially qualified if and only if this individual qualifies himself. Consent rules are a class of social rules, where each of them is characterized by two positive integers s and t. Moreover, if an individual qualifies himself, then this individual is socially qualified if and only if there are at least $s-1$ other individuals who also qualify him. On the other hand, if the individual disqualifies himself, then this individual is not socially qualified if and only if there are at least $t-1$ other individuals who also disqualify him. The CSR and the LSR rules recursively determine the socially qualified individuals. In the beginning, a set of individuals each of whom qualifies himself are considered LSR socially qualified, while the set of individuals each of whom is qualified by all individuals are considered CSR socially qualified. Then, in each iteration for both the CSR and the LSR rules, an individual is considered socially qualified if there is at least one currently socially qualified individual who qualifies this individual. The iteration terminates until no new individual can be added to the socially qualified set.

In this paper, we consider the problems where an external agent (e.g., the Chair of the committee, a powerful man/woman) has an incentive to control the results by adding, deleting or partition of individuals. In particular, in each problem the external agent has a set S of objective individuals whom he wants to make socially qualified. Though it is possible for the external agent to change the result in many cases, the external agent might give up controlling the group identification procedure if he realizes that it would take quite a long time to find out how to change the result. Motivated by this argument, we study the complexity of these problems for the liberal rule, the consent rules, the CSR and the LSR rules. We achieve both polynomial-time solvability results and NP-hardness results for these problems. Our main results are summarized in Table 1.

1.1 Preliminaries

Social Rule. Let N be a set of individuals. A *profile* $\varphi : N \times N \to \{0, 1\}$ over N is a mapping such that $\varphi(a, a') = 1$ means that individual $a \in N$ *qualifies* individual $a' \in N$. A *social rule* is a function f which associates a pair (φ, T) of each profile φ over N and a subset $T \subseteq N$ of individuals with a subset $f(\varphi, T) \subseteq T$. We call the individuals in $f(\varphi, T)$ the *socially qualified individuals* of T with respect to f and φ. In this paper, we mainly study the following social rules.

Liberal Rule f^L. An individual is socially qualified if and only if this individual qualifies himself. That is, for every $T \subseteq N$ and every individual $a \in T$, $a \in f^L(\varphi, T)$ if and only if $\varphi(a, a) = 1$.

Consent Rule $f^{(s,t)}$: Each consent rule $f^{(s,t)}$ is specified by two positive integers s and t such that for every $T \subseteq N$ and every individual $a \in T$,
(1) if $\varphi(a, a) = 1$, then $a \in f^{(s,t)}(\varphi, T)$ if and only if $|\{a' \in T \mid \varphi(a', a) = 1\}| \geq s$, and
(2) if $\varphi(a, a) = 0$, then $a \notin f^{(s,t)}(\varphi, T)$ if and only if $|\{a' \in T \mid \varphi(a', a) = 0\}| \geq t$.

The two positive integers s and t are referred to as the *consent quotas* of the consent rule $f^{(s,t)}$. It is easy to see that the consent rule with consent quotas $s = t = 1$ is exactly the liberal rule [17].

We remark that in the original definition of consent rules by Samet and Schmeidler [17], there is an additional condition $s + t \leq n + 2$ for consent quotas s and t to satisfy, where n is the number of individuals. Indeed, Samet and Schmeidler studied the consent rules for a fixed set of individuals, and the condition $s + t \leq n + 2$ is crucial for the consent rules to satisfy the *monotonicity property*. Roughly, a social rule is *monotonic* if a socially qualified individual a is still socially qualified when someone who disqualifies a turns to qualify a. We refer to [17] for further details. Since our paper is mainly concerned with complexity of strategic group control problems, we drop this condition from the definition of the consent rules (we indeed achieve results for a more general class of social rules that encapsules the original consent rules defined in [17]). In the following, when we study a strategic group control problem for the consent rules $f^{(s,t)}$, the consent quotas s and t preserve the same during the study. That is, the values of s and t do not change in the profile after deleting or adding individuals, or during partition of individuals.

Consensus-Start-Respecting Rule (f^{CSR} for short). For every $T \subseteq N$, this rule determines the socially qualified individuals iteratively. First, all individuals who are qualified by all individuals are considered socially qualified. Then, in each iteration, all individuals which are qualified by at least one of the currently socially qualified individuals are added to the set of socially qualified individuals. The iteration terminates until no new individual is added. Precisely, for every $T \subseteq N$ let

$$K_0^C(\varphi, T) = \{a \in T \mid \forall a' \in T,\ \varphi(a', a) = 1\}.$$

For each positive integer ℓ, let

$$K_\ell^C(\varphi, T) = K_{\ell-1}^C(\varphi, T) \cup \{a \in T \mid \exists a' \in K_{\ell-1}^C(\varphi, T),\ \varphi(a', a) = 1\}.$$

Then $f^{CSR}(\varphi, T) = K_\ell^C(\varphi, T)$ for some ℓ such that $K_\ell^C(\varphi, T) = K_{\ell-1}^C(\varphi, T)$.

Liberal-Start-Respecting Rule (f^{LSR} for short). This rule is similar to f^{CSR} with the only difference that the initial socially qualified individuals are those who qualify themselves. In particular, for every $T \subseteq N$, let

$$K_0^L(\varphi, T) = \{a \in T \mid \varphi(a, a) = 1\}.$$

For each positive integer ℓ, let

$$K_\ell^L(\varphi, T) = K_{\ell-1}^L(\varphi, T) \cup \{a \in T \mid \exists a' \in K_{\ell-1}^L(\varphi, T),\ \varphi(a', a) = 1\}.$$

Then $f^{LSR}(\varphi, T) = K_\ell^L(\varphi, T)$ for some ℓ such that $K_\ell^L(\varphi, T) = K_{\ell-1}^L(\varphi, T)$.

Problem Definition. In this paper, we mainly study the complexity of the following problems.

Group Control by Adding Individuals (GCAI)

Input: A 6-tuple (f, N, φ, S, T, k) of a social rule f, a set N of individuals, a profile φ over N, two nonempty subsets $S, T \subseteq N$ such that $S \subseteq T$ and $S \nsubseteq f(\varphi, T)$, and a positive integer k.

Question: Is there a subset $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S \subseteq f(\varphi, V)$ with $V = T \cup U$?

Group Control by Deleting Individuals (GCDI)

Input: A 5-tuple (f, N, φ, S, k) of a social rule f, a set N of individuals, a profile φ over N, a nonempty subset $S \subseteq N$ such that $S \nsubseteq f(\varphi, N)$, and a positive integer k.

Question: Is there a subset $U \subseteq N \setminus S$ such that $|U| \leq k$ and $S \subseteq f(\varphi, V)$ with $V = N \setminus U$?

Group Control by Partition of Individuals (GCPI)

Input: A 4-tuple (f, N, φ, S) of a social rule f, a set N of individuals, a profile φ over N, and a nonempty subset $S \subseteq N$ such that $S \nsubseteq f(\varphi, N)$.

Question: Is there a subset $U \subseteq N$ such that $S \subseteq f(\varphi, V)$ with $V = f(\varphi, U) \cup f(\varphi, N \setminus U)$?

A social rule is *immune* to a problem defined as above if it is impossible to make a non-socially qualified individual $a \in S$ socially qualified by carrying out the operations (adding/deleting/partition of individuals) in the problems, that is, there are only No-instances. If a social rule is not immune to a problem defined above, we say it is *susceptible* to the problem.

Graph. An *undirected graph* is a tuple (W, E) where W is the *vertex set* and E is the *edge set*. A vertex v *dominates* a vertex u if there is an edge between v and u. A vertex subset A dominates another vertex subset B, if for every vertex $u \in B$ there is a vertex $v \in A$ that dominates u. A *directed graph* is a tuple (W, A) where W is the vertex set and A is the *arc set*. A(n) *undirected (directed) bipartite graph* is a(n) undirected (directed) graph whose vertex set can be partitioned into two sets L and R such that there are no edges (arcs) between every two vertices in M for both $M = L$ and $M = R$. A *directed path* in a directed graph $G = (W, A)$ is a vertex sequence $(v_1, v_2, ..., v_t)$ such that $(v_i, v_{i+1}) \in A$ for every $i = 1, 2, ..., t-1$. We say that this is a path from v_1 to v_t, or simply a $(v_1 \rightarrow v_t)$-path. We refer to [2, 4] for further details on undirected and directed graphs. Unless stated otherwise, in this paper we simply use "graph" for "undirected graph".

Two NP-hard Problems. We assume familiarity with basic notation in complexity theory such as NP-hardness. Our NP-hardness results are shown by reductions from the following NP-hard problems.

Exact 3 Set Cover (X3C)

Input: A universal set X with $|X| = 3\kappa$ for some positive integer κ and a collection $\mathcal{C}$ of 3-subsets of X.

Question: Is there a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| = \kappa$ and each $x \in X$ appears in exactly one set of $\mathcal{C}'$?

The NP-hardness of the X3C problem was given in [10]. In this paper, we assume that each element $x \in X$ appears in exactly three different 3-subsets of X in $\mathcal{C}$. Therefore, we have that $|\mathcal{C}| = 3\kappa$. This assumption does not change the NP-hardness of the X3C problem [11].

Labeled Red-Blue Dominating Set (LRBDS)

Input: A bipartite graph $B = (R \cup B, E)$, where each vertex in R has a label from $\{1, 2, ..., k\}$.

<table>
<tr><th rowspan="3"></th><th rowspan="3">L</th><th colspan="8">consent rules $f^{(s,t)}$</th><th rowspan="3">CSR</th><th rowspan="3">LSR</th></tr>
<tr><th colspan="2">$s = 1$</th><th colspan="3">$s = 2$</th><th colspan="3">$s \geq 3$</th></tr>
<tr><th>$t = 2$</th><th>$t \geq 3$</th><th>$t = 1$</th><th>$t = 2$</th><th>$t \geq 3$</th><th>$t = 1$</th><th>$t = 2$</th><th>$t \geq 3$</th></tr>
<tr><td>GCAI</td><td>I</td><td>I</td><td>I</td><td colspan="6">NP</td><td>NP</td><td>NP</td></tr>
<tr><td>GCDI</td><td>I</td><td>P</td><td>NP</td><td>I</td><td>P</td><td>NP</td><td>I</td><td>P</td><td>NP</td><td>I</td><td>I</td></tr>
<tr><td>GCPI</td><td>I</td><td>?</td><td>?</td><td>I</td><td>?</td><td>?</td><td>I</td><td>NP</td><td>?</td><td>I</td><td>I</td></tr>
</table>

Table 1: A summary of our results. In the table, "NP" means "NP-hard", "P" means "polynomial-time solvable", and "I" means "immune". Moreover, "L" stands for the liberal rule. Recall that the consent rule $f^{(1,1)}$ is exactly the liberal rule. Thus, we don't explicitly give a column for the consent rule $f^{(1,1)}$. The entries filled with "?" mean that the corresponding problems are open. The results for the liberal rule are from Theorem 1. The immunity results for the consent rules are from Theorem 2. The polynomial-time solvability results for the consent rules are from Theorem 3. The NP-hardness results of the GCAI and GCDI problems for the consent rules are from Theorem 4. The NP-hardness result of the GCPI for the consent rules is from Theorem 5. The NP-hardness results for the CSR and the LSR rules are from Theorem 6. The immunity results for the CSR and the LSR rules are from Theorem 7.

Question: Is there a subset $W \subseteq R$ such that $|W \cap R_i| \leq 1$ for every $i \in \{1, 2, ...k\}$ and W dominates B, where R_i is the set of all vertices in R that has label i?

Lemma 1. *The* LRBDS *problem is NP-hard.*

The proof for the above lemma is deferred to Appendix.

1.2 Related Work

To the best of our knowledge, group identification as a classic model for identifying socially qualified individuals has not been studied from the complexity point of view. The words "control by adding/deleleing/partition of" in the problem names is reminiscent of many strategic voting problems, such as control by adding/deleting/partition of voters/candidates, which have been extensively studied in the literature [7, 8, 12, 19, 20]. In a voting system, we have a set of candidates and a set of voters. Each voter casts a vote, and a voting rule is carried out to select a set of candidates. From this standpoint, group identification can be considered as a voting system where the individuals are both voters and candidates. Nevertheless, group identification differs from voting systems in many significant aspects. First, the goal of a voting system is to select a subset of candidates, which are often called winners since they are considered as more competitive or outstanding compared with the remaining candidates for some specific purpose. Despite that the goal of group identification is also to identify a set of individuals (socially qualified individuals) from the whole individuals, it does not imply that socially qualified individuals must be more competitive or outstanding than the remaining individuals. For instance, in situations where we want to identify left-wing party members among a group of people, the model of group identification is more suitable. In other words, group identification is more close to a classification model. Second, as voting systems aim to select a set of competitive candidates for some special purpose, more often than not, the number of winners are pre-decided (e.g., in a single-winner voting, exactly one candidate is selected as the winner). As a consequence, many voting systems need to adopt a certain tie-breaking method to break the tie when many candidates are considered equally competitive. However, group identification does not need a tie breaking method, since there is no size bound of the number of socially qualified individuals.

It is also worth pointing out that the classic voting system Approval, which has been widely studied in the literature [3, 9, 13, 14, 18], has the flavor of group identification. In an Approval voting, each voter approves or disapproves each candidate. Thus, each voter's vote is represented by an 1-0 vector, where the entries with 1s (resp. 0s) mean that the voter approves (resp. disapproves) the corresponding candidate. The winners are among the candidates which get the most approvals. If the voters and candidates are the same group of individuals, then it seems that Approval voting is a social rule. Nevertheless, as discussed above, Approval voting is more often considered as a single-winner voting system and thus need to utilize a tie breaking method. Recently, several variants of Approval voting have been studied as multi-winner voting systems. However, the number of winners is bounded by (or exactly equals to) an integer k [1]. Moreover, to the best of our knowledge, complexity of control by adding/deleting/partition of voters/candidates has not been studied for Approval voting when the voters and candidates coincide, though it is fairly easy to check that many complexity results in this case can be directly obtained from the results in the general case.

2 Complexity Results

In this section, we investigate the GCAI, GCDI and GCPI problems for the liberal rule, consent rules, CSR rule and LSR rule. For each social rule, we study first if it is immune or susceptible to the problem under consideration. If it is susceptible, we further explore the complexity of the problem for the social rule.

2.1 Liberal Rule

The intrinsic property of the liberal rule is that it completely leaves to each individual to determine whether himself is socially qualified or not. Put it another way, whether an individual is socially qualified is independent of the opinions of any other individuals. As a consequence, the answer to the question whether an individual is socially qualified before and after adding/deleting/partition of individuals are the same, as implied by the following theorem.

Theorem 1. *The liberal rule is immune to GCAI, GCDI, and GCPI.*

Proof. Consider instances of GCAI, GCDI, and GCPI with the liberal rule as their social rule, i.e., $f \equiv f^L$. As assumptions of instances, $S \not\subseteq f(\varphi, T)$ with $S \subseteq T$ being imposed in GCAI and $S \not\subseteq f(\varphi, N)$ being imposed in both GCDI and GCPI. According to the definition of the liberal rule, each of the above assumptions implies that there exists an individual $a \in S$ such that $\varphi(a, a) = 0$, and hence, $a \notin f^L(\varphi, V)$ for every $V \subseteq N$. It follows that $S \not\subseteq f^L(\varphi, V)$ for every $S \subseteq V \subseteq N$. Therefore, for all instances with the liberal rule as the social rule, the answers to GCAI, GCDI, and GCPI are always "No". This completes the proof. □

2.2 Consent Rules

In the following, we study the GCAI and GCDI problems for consent rules with different consent quotas. In particular, we achieve dichotomy results with respect to the consent quotas for the consent rules, as summarized in Table 1.

We first consider the consent rules $f^{(1,t)}$ and $f^{(s,1)}$. These two consent rules have some flavor of the liberal rule (or equivalently the consent rule $f^{(1,1)}$). In particular, the consent rule $f^{(1,t)}$ (resp. $f^{(s,1)}$) is positive liberal (resp. negative liberal), in the sense that an individual's own qualification (resp. disqualification) is sufficient to determine his social qualification, regardless of the opinions of any other individuals. We shall see that similar to the liberal rule, the consent rules $f^{(1,t)}$ and $f^{(s,1)}$ are correspondingly immune to some problems studied in this paper, as summarized in the following theorem.

Theorem 2. *The consent rule $f^{(s,1)}$ is immune to GCDI and GCPI for every possible integer s, and the consent rule $f^{(1,t)}$ is immune to GCAI for every positive integer t.*

Proof. We first consider the consent rule $f^{(s,1)}$. Let $a \in S$ be an individual which is not socially qualified, i.e., $a \notin f^{(s,1)}(\varphi, N)$. We distinguish between two cases.

Case $\varphi(a,a) = 1$**:** There are at most $s-1$ individuals in N qualifying individual a, i.e., $|\{a' \in N \mid \varphi(a', a) = 1\}| < s$, and thus $|\{a' \in V \mid \varphi(a', a) = 1\}| < s$ for every $V \subseteq N$. Therefore, it is impossible to make individual $a \in S$ socially qualified by deleting or partition of individuals.

Case $\varphi(a,a) = 0$**:** By definition, each social rule f satisfies $f(\varphi, V) \subseteq V$, and hence, $S \not\subseteq f^{(s,1)}(\varphi, V)$ if $S \not\subseteq V$; otherwise, when $S \subseteq V$, we have $|\{a' \in V \mid \varphi(a', a) = 0\}| \geq 1$ from $\varphi(a,a) = 0$, which implies that $a \notin f^{(s,1)}(\varphi, V)$. Hence, it is impossible to make individual a to be socially qualified by deleting or partition of individuals, i.e., $S \not\subseteq f^{(s,1)}(\varphi, V)$ for any $V \subseteq N$.

Therefore, for each instance with $f^{(s,1)}$ as its social rule, the answers to GCDI and GCPI are always "No".

Now we come to the consent rule $f^{1,t}$. Let $a \in S$ be an individual which is not socially qualified, that is $a \notin f^{(1,t)}(\varphi, T)$. This implies that $\varphi(a,a) = 0$ and, moreover, there are at least t individuals a' (including a) in T such that $\varphi(a', a) = 0$. Therefore, no matter which individuals the set U includes, there will be still at least t individuals $a' \in T \cup U$ such that $\varphi(a', a) = 0$, implying that a is still not socially qualified. □

Now we study consent rules where none of the consent quotas is equal to 1. We shall see that these rules are susceptible to all the three problems GCAI, GCDI and GCPI. In addition, we investigate the complexity of the GCAI, GCDI and GCPI problems for the consent rules $f^{(s\geq 2, t\geq 2)}$. We first study the GCDI problem for the consent rule $f^{(s,2)}$. To show that the consent rule $f^{(s,2)}$ is not immune to the GCDI problem, we need only to give an instance where we can make all individuals in S socially qualified by deleting a limited number of individuals, given that not all individuals in S are socially qualified in advance. To this end, consider an instance $(f^{(s,2)}, N = \{a, b\}, \varphi, S = \{a\}, k = 1)$ where $\varphi(a,a) = \varphi(b,a) = 0$. It is clear that we can make a socially qualified by deleting b from the instance. Now we study the complexity of the problem.

Theorem 3. *The GCDI problem for the consent rule $f^{(s,2)}$ is polynomial-time solvable for every positive integer s.*

Proof. Let $L = \{a \in S \mid \varphi(a,a) = 1\}$ and $\bar{L} = S \setminus L = \{a \in S \mid \varphi(a,a) = 0\}$. For each $a \in \bar{L}$, let $U_a \subseteq N$ be the set of individuals each of which is outside of S and disqualifies a, i.e., $U_a = \{a' \in N \setminus S \mid \varphi(a', a) = 0\}$. Moreover, let $U = \bigcup_{a \in \bar{L}} U_a$. Then, the algorithm returns "No" if $S \not\subseteq f^{(s,2)}(\varphi, N \setminus U)$ or $|U| > k$, and otherwise returns "Yes".

The correctness of the algorithm is shown based on the following observations. According to the consent rule $f^{(s,2)}$, $a \in \bar{L}$ is socially qualified if there is no further individual $a' \neq a$ such that $\varphi(a', a) = 0$. Therefore, in order to make $a \in \bar{L}$ socially qualified, all individuals $a' \in N \setminus S$ with $\varphi(a', a) = 0$ have to be deleted. This directly implies that all individuals in U, as defined above, have to be deleted.

Now let us consider $f^{(s,2)}(\varphi, N \setminus U)$. Suppose $S \not\subseteq f^{(s,2)}(\varphi, N \setminus U)$, and let $a \in S \setminus f^{(s,2)}(\varphi, N \setminus U)$. We distinguish between the following two cases.

Case $a \in L$**:** According to the consent rule $f^{(s,2)}$, there are at most $s-1$ individuals $a' \in N \setminus U$ such that $\varphi(a', a) = 1$. Since deleting individuals does not increase the number of individuals that qualify a, the individual a cannot be socially qualified; and thus, the given instance is a No-instance.

Case $a \in \bar{L}$**:** In this case, there is an individual $a' \in S$ such that $a' \neq a$ and $\varphi(a', a) = 0$. Since we cannot delete individuals in S due to the definition of the problem, individual a cannot be socially qualified; and thus, the given instance is a No-instance.

Due to the above analysis, if $S \nsubseteq f^{(s,2)}(\varphi, N \setminus U)$, we can safely return "Yes". Since we are allowed to delete at most k individuals, and according to the above analysis all individuals in U must be deleted, if $|U| > k$, we can safely return "No" too. On the other hand, if $S \subseteq f^{(s,2)}(\varphi, N \setminus U)$ and $|U| \leq k$, U itself is an evidence for answering "Yes".

Finally, observe that construction of the set U, and the decisions of $S \subseteq f^{(s,2)}(\varphi, N \setminus U)$ and $|U| \leq k$ can be done in $\mathcal{O}(|N|^2)$ time. This completes the proof. □

Theorem 2 reveals that it is practically tractable for an external agent to control a group identification procedure if the external agent is allowed to delete individuals and if the procedure adopts the consent rule $f^{(s,2)}$ to identify the socially qualified individuals.

Now we study the GCAI problem for consent rules with consent quota $s \geq 2$, and the GCDI problem for consent rules with consent quota $t \geq 3$. In contrast to the polynomial-time solvability of the GCDI problem for consent rules with consent quota $t = 2$, as stated in Theorem 3, we prove that the same problem for consent rules with quota $t \geq 3$ becomes NP-hard. In addition, we prove that the GCAI problem for consent rules with quota $s \geq 2$ is also NP-hard. Our results are summarized in the following theorem. It should be noted that the instances in our NP-hardness reductions directly imply the nonimmunity of the consent rules $f^{(s,t)}$ where $s \geq 2$ to GCAI, and the nonimmunity of the consent rules $f^{(s,t)}$ where $t \geq 3$ to GCDI.

Theorem 4. *GCAI for every consent rule $f^{(s,t)}$ where $s \geq 2$ and $t \geq 1$, and GCDI for every consent rule $f^{(s,t)}$ where $t \geq 3$ and $s \geq 1$ are NP-hard.*

Proof. We prove the theorem by reductions from the X3C problem. Let's first consider the GCAI problem for the consent rule $f^{(2,t)}$. Given an instance $\mathcal{I} = (X, \mathcal{C})$ with $|X| = 3\kappa$, we create an instance $\mathcal{E}_{\mathcal{I}} = (f^{(s,t)}, N, \varphi, S, T, k)$ for GCAI as follows.

There are $|X| + |\mathcal{C}|$ individuals in $N = \{a_x \mid x \in X\} \cup \{a_c \mid c \in \mathcal{C}\}$. The first $|X|$ individuals $\{a_x \mid x \in X\}$ one-to-one correspond to the elements in X, and the last $|\mathcal{C}|$ individuals $\{a_c \mid c \in \mathcal{C}\}$ one-to-one correspond to elements in $\mathcal{C}$. We define $S = T = \{a_x \in N \mid x \in X\}$. In addition, we set $k = \kappa$. Now we define the profile φ .

- For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 1$ if and only if $x = x'$.
- For each $x \in X$ and for each $c \in \mathcal{C}$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.
- For each $c, c' \in \mathcal{C}$, $\varphi(a_c, a_{c'}) = 0$.

For the proof, the values of $\varphi(a_x, a_c)$ where $x \in X$ and $c \in \mathcal{C}$ are not essential. Obviously, the construction of $\mathcal{E}_{\mathcal{I}}$ can be done in polynomial time, namely $\mathcal{O}((|X| + |\mathcal{C}|)^2)$ time.

Now we prove the correctness of the reduction, i.e., we show that $\mathcal{I}$ is a Yes-instance for X3C if and only if $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance for GCAI.

($\Rightarrow$:) Suppose $\mathcal{I}$ is a Yes-instance for X3C, and let $\mathcal{C}' \subseteq \mathcal{C}$ be an exact 3-set cover, i.e., $|\mathcal{C}'| = k$ and for every $x \in X$ there exists a $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \in N \mid c \in \mathcal{C}'\}$. Then, according to the definition of φ, for each $a_x \in S$, there exists an $a_c \in U$ such that $\varphi(a_c, a_x) = 1$. Moreover, each $a_x \in S$ qualifies himself (i.e., $\varphi(a_x, a_x) = 1$). Therefore, according to the definition of the consent rule $f^{(2,t)}$, $a_x \in f^{(2,t)}(\varphi, T \cup U)$ for every $a_x \in S$, i.e., $S \subseteq f^{(2,t)}(\varphi, T \cup U)$. By definition, we have $|U| = |\mathcal{C}'| = k = \kappa$. Therefore, $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance for GCAI.

(:$\Leftarrow$) Suppose $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance for GCAI, and let $U \subseteq N \setminus T$ be a set of individuals such that $|U| \leq k = \kappa$ and $S \subseteq f^{(2,t)}(\varphi, T \cup U)$. From $S \subseteq f^{(2,t)}(\varphi, T \cup U)$ and, for all $a_x, a_{x'} \in S = T$, $\varphi(a_x, a_{x'}) = 1$ if and only $x = x'$, it follows that, for each $a_x \in S$, there is an $a_c \in U$ such that $\varphi(a_c, a_x) = 1$. Then, according to the definition of the profile φ, for each $x \in X$, there exists $c \in \mathcal{C}$ such that $a_c \in U$ and $x \in c$. This implies that $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$ is an exact 3-set cover of $\mathcal{I}$. Thus, $\mathcal{I}$ is a Yes-instance.

The NP-hardness reduction for the problem GCAI for any $s > 2$ can be adapted from the above reduction for $s = 2$. Precisely, we introduce further $s - 2$ dummy individuals in T, and let all these dummy individuals qualify every individual in $S = \{a_x \in N \mid x \in X\}$. The opinions of a dummy individual over any other individual in N and the other way around do not matter in the proof, and thus can be set arbitrarily. Now for each individual $a_x \in S$, there are exactly $s-1$ individuals in T who qualify a_x. Moreover, in order to make each $a_x \in S$ be socially qualified, we need one more individual in $N \setminus T$ who qualifies a_x.

Now let's consider the GCDI problem for the consent rule $f^{(s,t\geq 3)}$. We first consider $t = 3$. The reduction for this problem is similar to the above reduction for the GCAI problem for the consent rule $f^{(2,t)}$ with the following differences.

1. There is no T in this reduction; but keeping $S = \{a_x \in N \mid x \in X\}$;
2. The values of $\varphi(a, b)$ for every $a, b \in N$ is reversed. That is, we have $\varphi(a, b) = 1$ in the current reduction if and only if $\varphi(a, b) = 0$ in the above reduction for GCAI; and
3. $k = 2\kappa$.

Now we prove the correctness of the reduction.

($\Rightarrow$:) Suppose that there is an exact 3-set cover $\mathcal{C}' \subset \mathcal{C}$ for $\mathcal{I}$, i.e., $|\mathcal{C}'| = k$ and for every $x \in X$ there exists exactly one $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \mid c \in \mathcal{C} \setminus \mathcal{C}'\}$ and $U' = \{a_c \mid c \in \mathcal{C}'\}$. Clearly, $S \cap U = \emptyset$. Moreover, $N \setminus U = S \cup U'$. Let a_x be an individual in S where $x \in X$. Then, according to the construction, there is exactly one $a_c \in U'$ such that $\varphi(a_c, a_x) = 0$. Since $\varphi(a_{x'}, a_x) = 1$ for all $a_{x'} \in S \setminus \{a_x\}$, according to the consent rule $f^{(s,3)}$, $a_x \in f^{(s,3)}(\varphi, N \setminus U)$. Since this holds for every $a_x \in S$, we can conclude that $S \subseteq f^{(s,3)}(\varphi, N \setminus U)$.

($\Leftarrow$:) Suppose that there is a $U \subseteq N \setminus S$ such that $|U| \leq 2\kappa$ and $S \subseteq f^{(s,3)}(\varphi, N \setminus U)$. Let $U' = N \setminus (S \cup U)$, and $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U'\}$. Thus, $N \setminus U = S \cup U'$. Due to the fact $\varphi(a_x, a_x) = 0$ for every $a_x \in S$ where $x \in X$ and the definition of φ, it holds that for every $a_x \in S$, there is at most one $a_c \in U'$ such that $\varphi(a_c, a_x) = 0$ and $x \in c$. As a result, there is no $x \in X$ which belongs to two distinct sets in U'. Moreover, since $|U'| = 3\kappa - |U| \geq \kappa$, and $|S| = 3\kappa$, it follows that $|U'| = \kappa = k$ and $|\mathcal{C}'|$ is an exact 3-set cover of $\mathcal{I}$.

The NP-hardness of the problem for any integer $t > 3$ can be adapted from the above reduction by introducing some dummy individuals. In particular, we introduce further $t - 3$ individuals in S. Let S' denote the set of the $t - 3$ dummy individuals. Thus, $S = \{a_x \in N \mid x \in X\} \cup S'$. We want each dummy individual in S' to be a robust socially qualified individual, that is, every $d \in S'$ is socially qualified regardless of which individuals (at most $k = 2\kappa$) would be deleted. To this end, for every $d \in S'$, we let d disqualify himself, and let all the other individuals qualify d. We set $\varphi(d, a_x) = 0$ for every $d \in S'$ and $a_x \in \{a_{x'} \mid x' \in X\}$. Thus, for every $a_x \in S$ where $x \in X$, there are in total $t+1$ individuals in N who disqualify a_x. The other entries in the profile not defined above can be set arbitrarily. In order to make each $a_x \in S$ where $x \in X$ socially qualified, we need to delete exactly two individuals in $N \setminus S$ who disqualify a_x. This happens if and only if there is an exact 3-set cover for $\mathcal{I}$, as we discussed in the proof for the consent rule $f^{(s,3)}$. □

Even though the consent rule $f^{(s\geq 2,t)}$ (resp. $f^{(s,t\geq 3)}$) is susceptible to the GCAI (resp. GCDI) problem, Theorem 4 reveals that it is unpractical for an external agent to successfully control a group identification procedure with the consent rule $f^{(s\geq 2,t)}$ (resp. $f^{(s,t\geq 3)}$) as the social rule to identify the socially qualified individuals, by adding (resp. deleting) individuals.

Now we study the GCPI problem. We have shown in Theorem 2 that all consent rules $f^{(s,t)}$ where $t = 1$ are immune to GCPI. We show now that if the consent quota $t > 1$, then the consent rule is susceptible to GCPI. Consider an instance $(f^{(s,t)}, N, \varphi, S)$ where $t \geq 2$, $N = \{a_1, a_2, ..., a_{t+1}\}$ and $S = \{a_1\}$. Moreover, $\varphi(a_i, a_j) = 0$ for every $i, j \in \{1, 2, ..., t+1\}$. Clearly, $f^{(s,t)}(\varphi, N) = \emptyset$. Now consider the partition $(U = S, N \setminus U)$ of N. Then, $f^{(s,t)}(\varphi, U) = S$. Moreover, for every individual $a_i \in N \setminus U$, at least t individuals in $N \setminus U$ disqualifying a_i, implying that $f^{(s,t)}(\varphi, N \setminus$

$U) = \emptyset$. In summary, $S = \{a_1\} = f^{(s,t)}(\varphi, f^{(s,t)}(\varphi, U) \cup f^{(s,t)}(\varphi, N \setminus U))$. Now it is of particular interest to study the complexity of the GCPI problem for the consent rules.

Theorem 5. *GCPI is NP-hard for consent rules $f^{(s,2)}$ such that $s \geq 3$, even when $|S| = 1$.*

Proof. We prove the theorem by a reduction from the LRBDS problem. Let $I = (G = (R \cup B, E), \{1, 2, ..., k\})$ be an instance of the LRBDS problem. Let $s \geq 3$. We create an instance $\mathcal{E}_{\mathcal{I}} = (f^{(s,2)}, N, \varphi, S)$ for the GCPI problem for the consent rule $f^{(s,2)}$ as follows. We create $k + s - 2 + |B| + |R|$ individuals in total. Let $(R_1, R_2, ..., R_k)$ be the partition of R with respect to the labels of the vertices. That is, R_i where $1 \leq i \leq k$, is the set of vertices in R with label i. For each vertex $v \in R_i$ where $1 \leq i \leq k$, we create an individual $a_i(v)$. Let $A_i = \{a_i(v) \mid v \in R_i\}$. Moreover, for every vertex $u \in B$, we create an individual $a(u)$. Let $A(B) = \{a(u) \mid u \in B\}$. In addition, we create $k + 1$ individuals $C = \{c_1, c_2, ..., c_k\} \cup \{w\}$, where each $c_i, 1 \leq i \leq k$, correspondes to the label i and $S = \{w\}$. Fianlly, we create $s - 3$ dummy individuals $A_{dummy} = \{d_1, d_2, ..., d_{s-3}\}$. Hence, $N = \bigcup_{1 \leq i \leq k} A_i \cup A(B) \cup C \cup A_{dummy}$. The profile φ is defined as follows.

1. $\varphi(w, w) = 0$;
2. $\varphi(a(u), a(u')) = 0$ for every $u, u' \in B$ if and only if $u = u'$;
3. $\varphi(c_i, c_j) = 1$ for every $c_i, c_j \in C$ if and only if $i = j$;
4. $\varphi(x, w) = 0$ for every $x \in C \cup A(B)$;
5. $\varphi(a_i(v), w) = 1$ for every $v \in R_i$ where $1 \leq i \leq k$;
6. $\varphi(c_i, a(u)) = 1$ for every $c_i \in C$ and $a(u) \in A(B)$;
7. $\varphi(a(u), c_i) = 0$ for every $a(u) \in A(B)$ and $c_i \in C$;
8. $\varphi(d_i, d_{i'}) = 0$ for every dummy individual $d_i, d_{i'} \in A_{dummy}$;
9. $\varphi(d_i, w) = 0$ for every dummy individual $d_i \in A_{dummy}$;
10. $\varphi(d_i, x) = 1$ for every dummy individual $d_i \in A_{dummy}$ and every $x \in N \setminus (A_{dummy} \cup \{w\})$;
11. $\varphi(x, d_i) = 0$ for every dummy individual $d_i \in A_{dummy}$ and every $x \in N \setminus (A_{dummy} \cup \{w\})$;
12. $\varphi(a_i(v), a(u)) = 0$ for every $v \in R_i$ where $1 \leq i \leq k$ and every $a(u) \in A(B)$ if and only if $(v, u) \in E$;
13. $\varphi(a_i(v), c_j) = 1$ for every $a_i(v) \in R_i$ and $c_j \in C$ if and only if $i = j$; and
14. $\varphi(x, y)$ which is not defined above can be set arbitrarily.

Now we show the correctness of the reduction.

($\Rightarrow$:) Let W be a labeled red-blue dominating set of G. We shall show that $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance. Let $U \subseteq N$ be the set consisting of the individual w and all individuals that correspond to $R \setminus W$. That is, $U = S \cup \{a_i(v) \mid v \in R_i \setminus W, 1 \leq i \leq k\}$. Since $\varphi(w, w) = 0$, and every individual corresponding to some vertex in R qualifies w (see 5), it holds that $w \in f^{(s,2)}(\varphi, U)$. Now, let's consider the profile restricted to $N \setminus U$. Observe that $(N \setminus U) \cap (\bigcup_{1 \leq i \leq k} A_i) = \{a_i(v) \mid v \in R_i \cap W, 1 \leq i \leq k\}$. Let $a(u)$ be a candidate in $A(B)$ where $u \in B$. According to the construction of φ and the fact that W dominates B, there is at least one individual $a_i(v)$, corresponding to a vertex $v \in W$ dominating u, that disqualifies $a(u)$ (see 12). Since $\varphi(a(u), a(u)) = 0$ (see 2), it holds that $a(u) \notin f^{(s,2)}(\varphi, N \setminus U)$. Since this holds for every $a(u) \in A(B)$, we have that $A(B) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. On the other hand, for every $1 \leq i \leq k$, since $|W \cap R_i| \leq 1$,

$N \setminus U$ contains at most one individual $a_i(v) \in A_i$. According to the construction of φ, for every $c_i \in C$ only the following $s-2$ individuals in $N \setminus U$ qualifies c_i: (1) c_i himself; (2) $a_i(v) \in A_i$ where $v \in W$ (see 13); and (3) all $s-3$ dummy individuals (see 10). It directly follows that $c_i \notin f^{(s,2)}(\varphi, N \setminus U)$ for every $c_i \in C$. Finally, since $\varphi(d_i, d_{i'}) = 0$ for every $d_i, d_{i'} \in A_{dummy}$ and all individuals in $N \setminus U$ disqualify all dummy individuals, it holds that $d_i \notin f^{(s,2)}(\varphi, N \setminus U)$ for every $d_i \in A_{dummy}$. In conclusion, $(A(B) \cup C \cup A_{dummy}) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. Now, it is easy to verify that $\varphi(x, w) = 1$ for every $x \in (f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U) \setminus \{w\})$. As a result, $w \in f^{(s,2)}(\varphi, f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U))$.

($:\Leftarrow$). Let $U \subseteq N$ such that $w \in f^{(s,2)}(\varphi, f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U))$. Due to symmetry, assume that $w \in U$. Since $\varphi(w, w) = 0$, all the other individuals that disqualify w must be in $N \setminus U$. That is, $A(B) \cup C \cup A_{dummy} \subseteq N \setminus U$. Moreover, all individuals in $A(B) \cup C \cup A_{dummy}$ must be eliminated in the profile restricted to $N \setminus U$, i.e., $(A(B) \cup C \cup A_{dummy}) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. Let $a(u)$ be a vertex in $A(B)$ where $u \in B$. Since $\varphi(a(u), a(u)) = 0$, to eliminate $a(u)$, at least one individual that disqualifies $a(u)$ must be in $N \setminus U$. Due to the construction of the profile, all individuals in $N \setminus U$ that disqualify $a(u)$, except $a(u)$ himself, are in $\bigcup_{1 \leq i \leq k} A_i$. Hence, at least one $a_i(v) \in A_i$ where $v \in R_i$ that disqualifies $a(u)$ must be in $N \setminus U$. According to the construction, the vertex v dominates u in the graph G. This implies that $W = \{v \in R \mid a_i(v) \in N \setminus U\}$ dominates B. Now, we show that W contains at most one vertex in each R_i where $1 \leq i \leq k$. Let c_i be an individual in C where $1 \leq i \leq k$. Since $\varphi(c_i, c_i) = 1$, in order to eliminate each c_i, at most $s-1$ individuals that qualify c_i can be in $N \setminus U$. According to the construction of the profile, all the $s-3$ dummy individuals in A_{dummy} qualify c_i. Moreover, all individuals in A_i qualify c_i. According to the above discussion, at most one of the individuals in A_i can be in $N \setminus U$. Due to the definition of φ, this implies that $|W \cap R_i| \leq 1$. Now, it is easy to see that W is a solution of the instance I. □

2.3 LSR and CSR

In this section, we study the LSR and the CSR social rules. We first prove that the GCAI problem is NP-hard for both the LSR and the CSR social rules. The instances created in the proof of the following theorem directly imply that both the LSR and the CSR social rules are susceptible to the GCAI problem.

Theorem 6. *GCAI for both f^{LSR} and f^{CSR} are NP-hard.*

Proof. We prove the theorem by reductions from the X3C problem. Let's first consider the social rule f^{LSR}. Given an instance $\mathcal{I} = (X, \mathcal{C})$ with $|X| = 3\kappa$, we create an instance $\mathcal{E}_{\mathcal{I}} = (f^{LSR}, N, \varphi, S, T, k)$ for the GCAI problem as follows.

The definitions of N, S, T and k are the same as in the NP-hardness reduction for GCAI for the consent rule $f^{(2,t)}$ in Theorem 4. The profile φ is defined as follows.

- For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 0$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_x, a_c) = 0$.
- For each $c, c' \in \mathcal{C}$, $\varphi(a_c, a_{c'}) = 1$ if and only if $c = c'$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.

Now we prove the correctness of the reduction.

($\Rightarrow$:) Suppose that there is an exact 3-set cover $\mathcal{C}' \subset \mathcal{C}$ for $\mathcal{I}$, i.e., $|\mathcal{C}'| = k$ and for every $x \in X$ there exists exactly one $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \mid c \in \mathcal{C}'\}$. According to the definition of φ, it holds that $U \subseteq f^{LSR}(\varphi, T \cup U)$. Moreover, for every $a_x \in S$ where $x \in X$, there is an $a_c \in U$ where $c \in \mathcal{C}'$ such that $\varphi(a_c, a_x) = 1$ and $x \in c$. Since $U \subseteq f^{LSR}(\varphi, T \cup U)$, according to the definition of the social rule f^{LSR}, it holds that $a_x \in f^{LSR}(\varphi, T \cup U)$ for every $a_x \in S$. Therefore, $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance since it has a solution U.

($\Leftarrow$:) Suppose that there is a $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S = T \subseteq f^{LSR}(\varphi, T \cup U)$. Let $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$. According to the definition of φ, $f^{LSR}(\varphi, T) = \emptyset$. Moreover, every $a_x \in S$ where $x \in X$ disqualifies all individuals in N, and every $a_c \in N \setminus T$ qualifies himself. As a result, for every $a_x \in S$ where $x \in X$, there must be at least one $a_c \in U$ where $c \in \mathcal{C}'$ such that $\varphi(a_c, a_x) = 1$. According to the definition of φ, this implies that for every $x \in X$, there is at least one $c \in \mathcal{C}'$ such that $x \in c$. Since $|\mathcal{C}'| = |U| \leq k = \kappa$, this implies that $|\mathcal{C}'| = k$ and, more precisely, $\mathcal{C}'$ is an exact 3-set cover of $\mathcal{I}$.

Now let's consider the GCAI problem for f^{CSR}. Again, the definitions of N, S, T and k are the same as in the NP-hardness reduction for GCAI for the consent rule $f^{(2,t)}$ in Theorem 4. The profile φ is defined as follows.

- For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 0$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_x, a_c) = 1$.
- For each $c, c' \in \mathcal{C}$, $\varphi(a_c, a_{c'}) = 1$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.

Now we prove the correctness of the reduction.

($\Rightarrow$:) Suppose that there is a $\mathcal{C}' \subset \mathcal{C}$ such that $|\mathcal{C}'| = k$ and for every $x \in X$ there exists exactly one $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \mid c \in \mathcal{C}'\}$. Clearly, $|U| = |\mathcal{C}'| = k$. Observe that $U \subseteq f^{CSR}(\varphi, T \cup U)$. Then, according to the definition of φ, it holds that for every $a_x \in S$ where $x \in X$, there is an $a_c \in U$ where $c \in \mathcal{C}'$ such that $x \in c$ and $\varphi(a_c, a_x) = 1$. This implies that $a_x \in f^{CSR}(\varphi, T \cup U)$ for every $a_x \in S$. Thus, $\mathcal{E}_{\mathcal{I}}$ is a Yes-instance, since it has a solution U.

($\Leftarrow$:) Suppose that there is a subset $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S = T \subseteq f^{CSR}(\varphi, T \cup U)$. Let $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$. According to the definition of φ, $f^{CSR}(\varphi, T) = \emptyset$. Moreover, every individual in S disqualifies every individual in S. Furthermore, every individual in $N \setminus T$ is qualified by all individuals in N. Therefore, for every $a_x \in S$ where $x \in X$, there must be at lest one $a_c \in U$ where $c \in \mathcal{C}'$ such that $\varphi(a_c, a_x) = 1$. According to the definition of φ, this implies that for every $x \in X$ there is at least one $c \in \mathcal{C}'$ such that $x \in c$. Since $|\mathcal{C}'| = |U| \leq k = \kappa$, this implies that $|\mathcal{C}'| = k$ and, more precisely, $\mathcal{C}'$ is an exact 3-set cover of $\mathcal{I}$. □

Now we study the GCDI and the GCPI problem for the CSR and the LSR social rules. In contrast to the susceptible of both social rules to the GCAI problem, we prove that both social rules are immune to the GCDI and the GCPI problems.

Theorem 7. *The social rules f^{CSR} and f^{LSR} are immune to GCDI and GCPI.*

Proof. We first give an alternative explanation of the social rules f^{CSR} and f^{LSR} from the graph theory point of view.

Let N be a set of individuals. For every profile φ over N, we can define a directed bipartite graph $\mathcal{B}_N^{\varphi} = (L \cup R, A)$, where $L \cap R = \emptyset$ and both L and R are independent sets in $\mathcal{B}_N^{\varphi}$. Precisely, the vertex sets L and R are each a copy of N. For each $a \in N$, let $L(a)$ and $R(a)$ denote the copies of a in L and R, respectively. The arcs in A are defined as follows: there is an arc $(L(a), R(b))$ and an arc $(R(a), L(b))$ if $\varphi(a, b) = 1$ for every $a, b \in N$. For a vertex v, let $\Gamma(v)$ be the set of all vertices u such that there is a $(v \rightarrow u)$-path in $\mathcal{B}_N^{\varphi}$. Moreover, for a vertex subset $H \subseteq L \cup R$, let $\Gamma(H) = \bigcup_{v \in H} \Gamma(v)$. Let $K^{LSR} = \{L(a) \mid a \in N, (L(a), R(a)) \in A\}$ and $K^{CSR} = \{L(a) \mid a \in N, \forall b \in N \text{ it holds that } (L(b), R(a)) \in A\}$ Then,

$$f^{CSR}(\varphi, N) = \{a \in N \mid \{L(a), R(a)\} \cap \Gamma(K^{CSR}) \neq \emptyset\};$$

$$f^{LSR}(\varphi, N) = \{a \in N \mid \{L(a), R(a)\} \cap \Gamma(K^{LSR}) \neq \emptyset\}.$$

	a_1	a_2	a_3	a_4
a_1	1	0	1	1
a_2	0	0	0	1
a_3	0	0	0	1
a_4	0	1	0	1

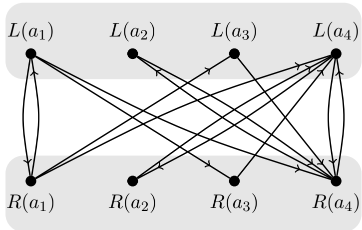

Figure 1: An illustration of the graph based explanation of the social rules f^{CSR} and f^{LSR}. The left side is a profile φ over the set $N = \{a_1, a_2, a_3, a_4\}$. The right side is the directed bipartite graph $\mathcal{B}_N^{\varphi} = (L \cup R, A)$. The vertices and arcs are as showed in the graph. It holds that $K^{CSR} = \{L(a_4)\}$ and $K^{LSR} = \{L(a_1), L(a_4)\}$. Moreover, $\Gamma(K^{CSR}) = \{R(a_4), L(a_2), L(a_4), R(a_2)\}$ and $\Gamma(K^{LSR}) = \{R(a_1), L(a_1), R(a_4), L(a_2), R(a_3), L(a_4), R(a_2)\}$. Therefore, according to the equations definition of the social rules f^{LSR} and f^{CSR} in the proof in Theorem 6, we have that $f^{CSR}(P) = \{a_2, a_4\}$ and $f^{LSR}(P) = \{a_1, a_2, a_3, a_3\}$

Notice that since for every $L(a) \in K^{CSR}$ (resp. $L(a) \in K^{LSR}$), $(L(a), R(a)) \in A$ and $(R(a), L(a)) \in A$, it holds that $K^{CSR} \subseteq \Gamma(K^{CSR})$ (resp. $K^{LSR} \subseteq \Gamma(K^{LSR})$).

According to the above definition, an individual $a \in N$ is socially qualified with respect to f^{CSR} (resp. f^{LSR}) if and only if $a \in K^{CSR}$ (resp. $a \in K^{LSR}$) or there is an individual $b \in K^{CSR}$ (resp. $b \in K^{LSR}$) such that there is an $(L(a), R(b))$-path or an $(L(a), L(b))$-path. See Fig. 1 for an example. Therefore, if an individual $a \in N$ is not socially qualified with respect to f^{CSR} (resp. f^{LSR}), then $a \notin K^{CSR}$ (resp. $a \notin K^{LSR}$) and, moreover there is no directed path from a copy of some individual in K^{CSR} (resp. K^{LSR}) to a copy of a in the graph $\mathcal{B}_N^{\varphi}$. In consequence, deleting individuals cannot make a not socially qualified individual socially qualified. It follows that the social rules f^{CSR} and f^{LSR} are immune to GCDI. Since in the GCPI problem, some individuals may be deleted in the first stage but no new individuals are added, the social rules f^{CSR} and f^{LSR} are immune to GCPI too. □

3 Conclusion

We have studied the complexity of the group control by adding/deleting/partition of individuals problems for several well-studied social rules, including the liberal rule, the consent rules and the LSR and CSR rules. In particular, we achieved polynomial-time solvability results and NP-hardness results for these problems. See Table 1 for a summary of our results.

There remain several open problems fur future research. For instance, we don't know the complexity of the GCPI for the consent rules $f^{s,t}$ where $s = 2$ or $t = 1$. Exploring the complexity of the same problems for further social rules would be another interesting research direction.

Acknowledgement

We thank the anonymous reviewers of COMSOC 2016 for their constructive comments. In addition, we thank Shao-Chin Sung for very helpful comments on a draft of this paper. In particular, we thank Shao-Chin Sung for his rephrasing of the notations and proofs of some theorems which improves the readability of the paper.

References

[1] H. Aziz, S. Gaspers, J. Gudmundsson, S. Mackenzie, N. Mattei, and T. Walsh. Computational aspects of multi-winner approval voting. In *AAMAS*, pages 107–115, 2015.

[2] J. Bang-Jensen and G. Gutin. *Digraphs – Theory, Algorithms and Applications*. Springer-Verlag, London, 2008. 2nd Edition.

[3] D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. *Computational Aspects of Approval Voting*, chapter 10, pages 199–251. Handbook on Approval Voting. Springer Berlin Heidelberg, 2010.

[4] D. B.West. *Introduction to Graph Theory*. Prentice-Hall, 2st, edition, 2000.

[5] D. Dimitrov. The social choice approach to group identification. In *Consensual Processes*, pages 123–134. 2011.

[6] D. Dimitrov, S. C. Sung, and Y. Xu. Procedural group identification. *Math. Soc. Sci.*, 54(2):137–146, 2007.

[7] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. Using complexity to protect elections. *Commun. ACM.*, 53(11):74–82, 2010.

[8] P. Faliszewski and A. D. Procaccia. AI's war on manipulation: Are we winning? *AI. MAG.*, 31(4):53–64, 2010.

[9] P. C. Fishburn and S. J. Brams. Approval voting, Condorcet's principle, and runoff elections. *Public. Choice.*, 36(1):89–114, 1981.

[10] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, 1979.

[11] T. F. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.*, 38:293–306, 1985.

[12] J. J. Bartholdi III, C. A. Tovey, and M. A. Trick. How hard is it to control an election? *Math. Comput. Model.*, 16(8-9):27–40, 1992.

[13] D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In D. S. Felsenthal and M. Machover, editors, *Electoral Systems*, Studies in Choice and Welfare, pages 305–326. Springer Berlin Heidelberg, 2012.

[14] A. P. Lin. The complexity of manipulating k-Approval elections. In *ICAART (2)*, pages 212–218, 2011. http://arxiv.org/abs/1005.4159.

[15] A. D. Miller. Group identification. *Game. Econ. Behav.*, 63(1):188–202, 2008.

[16] H. Nicolas. "I want to be a J!": Liberalism in group identification problems. *Math. Soc. Sci.*, 54(1):59–70, 2007.

[17] D. Samet and D. Schmeidler. Between liberalism and democracy. *J. Economic Theory*, 110(2):213–233, 2003.

[18] Y. Yang and J. Guo. The control complexity of r-Approval: from the single-peaked case to the general case. In *AAMAS*, pages 621–628, 2014.

[19] Y. Yang and J. Guo. Controlling elections with bounded single-peaked width. In *AAMAS*, pages 629–636, 2014.

[20] Y. Yang and J. Guo. How hard is control in multi-peaked elections: A parameterized study. In *AAMAS*, pages 1729–1730, 2015.

Yongjie Yang
Universität des Saarlandes
Saarbrücken, Germany
Email: yyongjie@mmci.uni-saarland.de

Dinko Dimitrov
Universität des Saarlandes
Saarbrücken, Germany
Email: dinko.dimitrov@mx.uni-saarland.de

Appendix

The NP-hardness of the LRBDS problem.

We prove Lemma 1 by a reduction from the RED-BLUE DOMINATING SET problem which is NP-hard [10]. The definition of the problem is defined as follows.

Red-Blue Dominating Set (RBDS)

Input: A bipartite graph $B = (R \cup B, E)$ and an integer k.

Question: Is there a subset $W \subseteq R$ such that $|W| \leq k$ and W dominates B?

Let $I' = (G' = (R' \cup B', E'), k)$ be an instance of the RBDS problem. We construct an instance $I = (G = (R \cup B, E), \{1, 2, ..., k\})$ for the LRBDS as follows. The given bipartite graph G is first constructed with a copy of G', then, further $k-1$ copies of each vertex in R', each with a different label, are added to G.

The details of the construction are as follows. For each vertex $u \in B'$, we create a vertex $\bar{u} \in B$. For each vertex $v \in R'$, we create k vertices $v(1), ..., v(k) \in R$, where the vertex $v(i)$ is labeled with i. Let R_i be the set of the vertices in R that have label i. The edges of the graph G are defined as follows. If there is an edge $(v, u) \in E'$, then for every $1 \leq i \leq k$ we create an edge between $v(i)$ and $\bar{u}$. This finishes the construction. It clearly takes polynomial time.

Suppose that I' has a solution W' of size $k' \leq k$. Let $(v_{x(1)}, v_{x(2)}, ..., v_{x(k')})$ be any arbitrary order of the vertices in W'. Let $W = \{v(i) \mid v_{x(i)} \in W', 1 \leq i \leq k'\}$. It is clear that no two vertices in W have the same label, that is, $|W \cap R_i| \leq 1$ for every $1 \leq i \leq k$. We shall show that W dominates B. Let u be a vertex in B'. Since W' dominates B', there is a vertex $v_{x(i)} \in W'$ such that $(v_{x(i)}, u) \in E'$. Then, according to the construction of G, we know that $(v(i), \bar{u}) \in E$. Since this holds for every $u \in B'$, W dominates B.

Suppose that I has a solution W. We assume that for every $v \in R'$, W contains at most one of $\{v(1), v(2), ..., v(k)\}$. Indeed if W contains two vertices $v(i)$ and $v(j)$ where $1 \leq i \neq j \leq k$, then we could get a new solution $W \setminus \{v(j)\}$ for I, since according to the construction of the graph G, $v(i)$ and $v(j)$ have the same neighborhood; and thus, a vertex in B which is dominated by $v(j)$ must be dominated by $v(i)$. Let $W' = \{v \in R' \mid v(i) \in W, 1 \leq i \leq k\}$. Let u be a vertex in B'. Since W is a solution of I, there is a vertex $v(i) \in W$ which dominates $\bar{u} \in B$. Then, according to the construction of the graph G, the vertex $v \in W'$ dominates u. It follows that W' dominates B'.