Bounds on Manipulation by Merging in Weighted Voting Games[1]

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Abstract

Manipulation by merging in weighted voting games (WVGs) is a voluntary action of would-be *strategic agents* who come together to form a bloc in anticipation of receiving more payoff over the outcomes of games. The inability to limit (or understand) the effects of this menace may undermine the confidence agents have in decisions made via WVGs. If the results are not seen as fair, agents may refuse to abide by decisions made in this manner. We propose four non-trivial bounds to characterize the effects of merging in WVGs using the well-known *Shapley-Shubik* and *Banzhaf* indices. The bounds for the Shapley-Shubik index are shown to be *asymptotically tight* while those of the Banzhaf index are found to be *within constant factors*.

1 Introduction

Autonomous agents in complex environments may need to work together to achieve desired goals. This is an important feature of many multiagent environments where individual agents lack *all* the required capabilities, skills, and knowledge to complete tasks alone. Agents may thus resort to *cooperation* such as coalition formation, to complete tasks while being compensated with payoffs. One way of modeling such cooperation is via *weighted voting games* (WVGs). See Chalkiadakis, Elkind, and Wooldridge [1] for examples. WVGs are important in multiagent systems and human societies because of their usage in automated decision-making. In a WVG, a *quota* is given and each agent has an associated weight. A subset of agents whose total weight meets or exceeds the quota is said to be *winning*. Agents' power in such games is measured using *power indices*. The power of an agent reflects its ability to influence or affect the outcomes of decision-making processes. The *Shapley-Shubik* [2] and *Banzhaf* [3, 4] indices are two prominent indices for computing agents' power.

Even though WVGs are useful in modeling cooperation, they are not immune from the vulnerability of *manipulation* (i.e., dishonest behavior) by some players called *manipulators*, or referred to as being *strategic*, that may be present in the games. With the possibility of manipulation, it becomes challenging to establish or maintain trust in such games. This problem of insincere and manipulative behaviors among agents in WVGs has received attention of researchers in recent years. See the work of [5, 6, 8, 9, 10]. Manipulation by merging in WVGs involves voluntary coordinated action of strategic agents who come together to form a *bloc* by merging their weights into a single weight [6, 13]. The agents in the bloc are assumed to be assimilated voters since they can no more vote as individual voters in the new game, rather as a bloc. The new game consists of the previous agents in the original game that are not assimilated, as well as the bloc formed by the assimilated voters.

Strategic agents merge their weights in anticipation of gaining more power over the outcomes of games. In a beneficial merge, merged agents are compensated commensurate with their share of the power gained by the bloc. Agents in the bloc are assumed to be working cooperatively and have transferable utility. Thus, proceeds can easily be distributed among the manipulators without bickering. Common settings that may be vulnerable to

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such attack are online elections, rating systems, electronic negotiation, and auctions. See for example, Yokoo, Sakurai, and Matsubara [11], where the effects of false-name bids in combinatorial auction as a form of Internet fraud was studied.

A motivation for this problem can be found in decision-making, e.g., in negotiation settings. Consider a set of agents, $A = \{a_1, a_2, \ldots, a_n\}$, negotiating on how to allocate some budgets, B. Let a payoff method, such as the Shapley-Shubik or Banzhaf index allocate B as say, $P = \{p_1, p_2, \ldots, p_n\}$, to agents A, respectively, based on their contributions to the coalition. Suppose some strategic agents, $S \subset A$, merge their weights to form a single bloc, they may be able to increase their share of the budget.

We motivate this problem further using a real-world example from social choice domain. Consider a parliament consisting of five political parties, A, B, C, D, and E, which have 20, 30, 40, 50, and, 50 representatives (i.e., weights), respectively. This parliament is to vote on a \$100 million spending bill and how much of this amount should be controlled by each party. Furthermore, the bill requires a quota, $q \in [111, 120]$, i.e., the number of votes to pass. Assuming that all members of a political party votes in the same direction on a bill, the Banzhaf index allocates the amount of the spending bill to be controlled by each party as follows: $A =$ \$4m, $B =$ \$20m, $C =$ \$20m, $D =$ \$28m, and $E =$ \$28m. Now, suppose political parties A, B, and D, merge their weights to form a bloc with weight 100. The sum of the initial allocation to each party in the bloc is \$4m + \$20m + \$28m = \$52m. However, the new allocation of the amount by Banzhaf index to the manipulators' bloc in the altered game is \$60m, which is more than \$52m. This indicates a beneficial merge with a payoff increase of \$8m for the manipulators' bloc.

The scenarios described above obviously raise the following important questions that we seek to answer in this research: *What is the extent of budgets, payoffs, or power (depending on the settings under consideration) that manipulators may gain? Analogously, what is the amount of damage that is caused to the non-manipulating agents in the games?*

This research is thus primarily motivated by the need to provide insights into understanding the details of the problem of this insincere and manipulative behavior in WVGs. We are concern that the inability to limit (or understand) the effects of this manipulation may undermine the confidence agents have in decisions made via WVGs. If the results from this decision-making process are not seen as fair, agents may refuse to abide by decisions made in this manner. The present work is limited to the case when the number of strategic agents in the games is 2. Our main results are as follows.

- We propose *four* non-trivial bounds to characterize the effects of manipulation by merging in WVGs using the well-known Shapley-Shubik and Banzhaf power indices.
- The bounds for the Shapley-Shubik index are shown to be asymptotically tight, i.e., there exists at least a game in which strategic agents achieve the proposed bounds by merging their weights to form a bloc.
- Also, the bounds for the Banzhaf index are found to be within constant factors.

2 Related Work

WVGs are widely studied [14, 15, 16, 17, 18]. They have found applications in many real-world environments, including the United Nations Security Council, the Electoral College of the United States, the International Monetary Fund [19, 20], the Council of Ministers, and the European Community [15]. The issue of WVGs design has also recently received attention of many researchers in the field [21, 22, 23]. The *Shapley value* [24], its variant, Shapley-Shubik [2], and the Banzhaf [3, 4] indices are the well-known power indices used in measuring power of agents in WVGs. Other lesser known power indices are the Deegan-Packel [25], Johnston [26], and Holler-Packel [27] indices.

WVGs are vulnerable to various forms of dishonest behaviors, referred to as *manipulations*. These manipulations are due to strategic players that may be present in the games. Prominent among these forms of behaviors are manipulations by *splitting* and *merging* [5, 7, 6, 8, 9, 10]. Unlike in merging where two or more strategic agents merge their weights to form a single bloc, manipulation by splitting involves a strategic agent splitting its weight among two or more false agents in anticipation of gaining more power. These two forms of manipulations have received attention of many researchers for the cases when the number, k, of strategic agents involved is either $k = 2$ or $k > 2$. *However, none of these works has considered the bounds on the extent of power that strategic agents may gain when they merge their weights using the Shapley-Shubik and Banzhaf power indices.* It is important to point out that the effects of manipulation by splitting are well studied. Table 1 provides a summary on the state of the arts on manipulation by splitting in WVGs.

Bounds	# Strategic agents	Shapely-Shubik Index	Banzhaf Index
Upper	$k = 2$	Bachrach & Elkind [5]	Aziz & Peterson [28]
	$k > 2$	Lasisi & Allan [10]	Lasisi & Allan [10]
Lower	$k = 2$	Bachrach & Elkind [5]	Aziz et al. [6]
	$k > 2$	Lasisi & Allan [10]	Lasisi & Allan [10]

Table 1: Summary of bounds for manipulation by splitting in weighted voting games

Previous work [6] has shown that the problem of finding beneficial merge is NP-hard for both the Shapley-Shubik and Banzhaf indices. This complexity result seems sufficient to discourage would-be strategic agents from merging. However, NP-hardness is a worst case measure, and only shows that at least an instance of the problem requires such complexity. Thus, the real-life instances of WVGs that we care about may be easy to manipulate [9]. Furthermore, Felsenthal and Machover [13] characterize situations when it is advantageous or disadvantageous for agents to merge, and show that using the Penrose-Banzhaf measure, merging can be advantageous or disadvantageous. Also, Lasisi and Allan [12] consider empirical evaluation of the extent of susceptibility of three power indices, namely, Shapley-Shubik, Banzhaf, and Deegan-Packel, to merging. Their results show that the Shapley-Shubik index is the most susceptible to merging among the three indices. However, none of these works provide bounds on the extent of power that manipulators may gain in the case that the merging is advantageous.

In contrast to these works, we propose new bounds on the extent of power that strategic agents may gain with respect to merging in WVGs using the Shapley-Shubik and Banzhaf indices to compute agents' power.

3 Preliminaries

We present some preliminaries in this section, including definitions and notation, formal problem definition, and illustrative examples needed to provide necessary backgrounds in WVGs.

3.1 Definitions and Notation

Let $I = \{1, \ldots, n\}$ be a set of $n \in \mathbb{N}$ agents. Let $\{w_1, \ldots, w_n\}$ be the corresponding weights of these agents. The non-empty subsets, $S \subseteq I$, are called *coalitions*.

Definition 1. *Simple Game*

A simple game is a coalitional game, (I, v), where $v : 2^I \to \{0, 1\}$. A coalition $S \subseteq I$ *wins* if $v(S) = 1$ and *loses* if $v(S) = 0$.

Definition 2. *Weighted Voting Game*

A weighted voting game is a simple game which has a weighted form, (W, q), where $W = (w_1, \ldots, w_n) \in (\mathbb{R}^+)^n$ corresponds to the weights of agents in I, and $q \in \mathbb{R}^+$ is the quota of the game. A coalition S wins if the total weight of S, $w(S) = \sum_{i \in S} w_i \geq q$, which implies that $v(S) = 1$. A WVG G of n agents with quota q is denoted by $G = [q; w_1, \ldots, w_n]$. Note also that $\frac{1}{2}w(I) < q \leq w(I)$.

Definition 3. *Critical Agent*

An agent $i \in S$ is critical to a coalition S if $w(S) \geq q$ and $w(S \backslash \{i\}) < q$.

Definition 4. *Shapley-Shubik Power Index*

The Shapley-Shubik index quantifies the marginal contribution of an agent to the grand coalition. Each permutation of the agents is considered. We term an agent *pivotal* in a permutation if the agents preceding it do not form a winning coalition, but by including this agent, a winning coalition is formed. We specify the computation of the index using notation of [6]. Denote by π, a permutation of the agents, so $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, and by Π the set of all possible permutations. Denote by $S_\pi(i)$ the predecessors of agent i in π, i.e., $S_\pi(i) = \{j : \pi(j) < \pi(i)\}$. The Shapley-Shubik index, $\varphi_i(G)$, for each agent i in a WVG G:

$$\varphi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]. \tag{1}$$

Definition 5. *Banzhaf Power Index*

The Banzhaf power index computation for an agent i is the proportion of the number of coalitions i is critical compared to the total number of coalitions any agent in the game is critical. The Banzhaf index, $\beta_i(G)$, for each agent i in a WVG G is given by

$$\beta_i(G) = \frac{\eta_i(G)}{\sum_{j \in I} \eta_j(G)} \tag{2}$$

where $\eta_i(G)$ is the number of winning coalitions in which agent i is critical in game G.

3.2 Formal Problem Definition

Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. Let $k \in \mathbb{N}$, $2 \leq k < n$. Consider a manipulators' coalition S of k agents which is a k-subset of the n-set I. We assume that S contains k distinct elements chosen from I. Suppose the manipulators in S merge their weights to form a bloc denoted by $\&S$, i.e., agents $i \in S$ have been assimilated into the bloc $\&S$, then, we have a new set of agents in the game after merging. Thus, the initial game G of n agents has been altered by the manipulators to give a new game G' of $n - k + 1$ agents consisting of the bloc, $\&S$, and other agents not in the bloc, i.e., $I \backslash S$.

Let ϕ be either the Shapley-Shubik or Banzhaf index. Denote by $(\phi_1(G), \ldots, \phi_n(G)) \in [0, 1]^n$ the power of agents in a WVG G of n agents. Thus, for the strategic agents $i \in S$ with power $\phi_i(G)$ in game G, the sum of the power of the k manipulators in S is $\sum_{i \in S} \phi_i(G)$, while that of the bloc formed by the manipulators in the altered game G' is $\phi_{\&S}(G')$. The ratio $\tau = \frac{\phi_{\&S}(G')}{\sum_{i \in S} \phi_i(G)}$ compares the power of the bloc in G' to the sum of the original power of the agents in the merged bloc. τ gives a factor of the power gained or lost when strategic agents $i \in S$ alter G to give G'. We say that ϕ is susceptible to manipulation if there exists a game G' such that $\tau > 1$; the merging is termed *advantageous*. If $\tau < 1$, then the merging is *disadvantageous*, while the merging is *neutral* when $\tau = 1$.

3.3 Examples of Manipulation by Merging

We provide next illustration of manipulation by merging in WVGs using the Banzhaf power index to compute agents' power. The strategic agents in each game are shown in bold.

Example 1. *Advantageous Merge*

Let $G = [28; \mathbf{8}, 8, 8, 6, \mathbf{5}, 5, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{2}]$ be a WVG, i.e., a game with quota, $q = 28$, and ten agents, $1, 2, \ldots, 10$. The power of the strategic agents are, $\beta_1(G) \approx 0.1784$,$\beta_5(G) \approx 0.0843, \beta_7(G) = \beta_8(G) = \beta_9(G) = \beta_{10}(G) \approx 0.0412$. Their cumulative power is ≈ 0.4275. Suppose the manipulators form a bloc and alter G by merging their weights into a single weight as follows; $G' = [28; \mathbf{23}, 8, 8, 6, 5]$. The power of this bloc is $\beta_{\&S}(G') = \beta_1(G') = 0.7895 > 0.4275$. The factor by which the bloc gains is $\frac{0.7895}{0.4275} \approx 1.85$

Example 2. *Disadvantageous Merge*

Let $G = [56; 10, 9, 9, \mathbf{9}, 8, \mathbf{7}, 6, \mathbf{6}, \mathbf{2}, \mathbf{1}]$ be a WVG of ten agents. The power of the strategic agents are, $\beta_4(G) \approx 0.1238$, $\beta_6(G) = \beta_8(G) \approx 0.1139$, $\beta_9(G) \approx 0.0248$, and $\beta_{10}(G) \approx 0.0149$. Their cumulative power is ≈ 0.3913. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [56; \mathbf{25}, 10, 9, 9, 8, 6]$. The power of this bloc is $\beta_{\&S}(G') = \beta_1(G') \approx 0.2308 < 0.3913$. The factor by which the bloc loses is $\frac{0.2308}{0.3913} \approx 0.58$

Example 3. *Neutral Merge*

Let $G = [3; 2, \mathbf{1}, \mathbf{1}, 1]$ be a WVG of four agents. The power of the strategic agents are, $\beta_2(G) = \beta_3(G) \approx 0.1666667$. Their cumulative power is ≈ 0.3333334. Suppose the manipulators form a bloc $\&S$ and alter G as follows; $G' = [3; 2, \mathbf{2}, 1]$. The power of this bloc is $\beta_{\&S}(G') = \beta_2(G') \approx 0.3333333$. Rounding the cumulative power of the manipulators (in G) and that of the bloc (in G') to 0.3333 shows that the strategic agents neither gain nor lose power by merging their weights in this case.

We have shown that strategic agents may gain power, lose power, or their power may remain the same when they engage in manipulation by merging using the Banzhaf index.

4 Shapley-Shubik Index Bounds

This section proposes upper and lower bounds to characterize the effect of manipulation by merging in WVGs using the Shapley-Shubik power index. The proposed bounds are shown to be asymptotically tight.

4.1 Upper Bound

Theorem 1. *(Upper Bound) Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Shapley-Shubik power, $\varphi_{\&S}(G')$, of the bloc in the new game, $\varphi_{\&S}(G') \leq \frac{n}{2}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. Moreover, this bound is asymptotically tight.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let Π_G be the set of all permutations of the n agents in game G. Also, let Π_{G-2} be the set of all permutations of the remaining $n-2$ non-manipulating agents in G, i.e., not including m_1 and m_2. Again, for any permutation $\pi \in \Pi_{G-2}$, let $r \in \mathbb{N}$ be the possible positions in π for insertion of m_1 or m_2 within the non-manipulating agents. Thus, $1 \leq r \leq n-1$.

We first bound the number of permutations in game G, for which the manipulators m_1 and m_2 are pivotal. Consider any $\pi \in \Pi_{G-2}$. Suppose we insert m_1 and m_2 arbitrarily into π to have a resulting permutation $\pi^* \in \Pi_G$ of n agents. Let Π_G^* be the set of all permutations π^* such that one of m_1 or m_2 is pivotal for π^*. Finally, let $Q(\pi^*, r, \pi)$ be the set of all permutations π^* in which at least one of m_1 or m_2 appears on the r-th position of $\pi \in \Pi_{G-2}$ and is pivotal for π^*. For example[2], consider a WVG of six agents with quota $q = 15$. Suppose $\pi = 8\ 6\ 4\ 2$, and consider an arbitrary insertion of two manipulators, **3** and **5**, into π. Let the resulting permutation $\pi^* = 8\ \mathbf{5}\ \mathbf{3}\ 6\ 4\ 2$. The manipulators are both on the 2-nd position of π (i.e., $r = 2$). Also, the manipulator with weight 3 is pivotal for π^*.

Note that $\Pi_G^* \subseteq \Pi_G$, and every permutation in Π_G^* appears in one of the sets $Q(\pi^*, r, \pi)$ for some π and r. Thus, the Shapley-Shubik power of the manipulators in game G is:

$$\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{|\Pi_G^*|}{n!} \leq \frac{1}{n!} \sum_{\pi, r} |Q(\pi^*, r, \pi)| \tag{3}$$

Now, we bound the number of permutations in the altered game G' for which the bloc $\&S$ is pivotal. Let $\pi \in \Pi_{G-2}$. Consider a permutation $f(\pi, r)$ of agents in game G' obtained from π by inserting the bloc $\&S$ at the r-th position of π. Note that if $Q(\pi^*, r, \pi)$ is not empty, then $\&S$ is pivotal for the permutation $f(\pi, r)$. Also, all the permutations π^* in the set $Q(\pi^*, r, \pi)$ derived from a permutation $\pi \in \Pi_{G-2}$ in which at least one of the manipulators appears at the r-th position of π and is pivotal for π^* corressponds to a single permutation $f(\pi, r)$ in game G'. Furthermore, it is not difficult to see that, $|Q(\pi^*, r, \pi)| \geq 2$, for all π and r when $Q(\pi^*, r, \pi) \neq \emptyset$. This is because if one of the two manipulators, say, m_1, is pivotal at position r, we can also insert the other, i.e., m_2, immediately before or after m_1 at the same position, and there are only 2 ways for them to appear together at r. Finally, we compute the Shapley-Shubik power of the bloc $\&S$ in the altered game G' by counting all the non-empty sets $Q(\pi^*, r, \pi)$ for all π and r. Hence,

$$\begin{aligned}
\varphi_{\&S}(G') &\leq \frac{1}{(n-1)!} \sum_{\pi, r: Q(\pi^*, r, \pi) \neq \emptyset} 1 \\
&= \frac{1}{(n-1)!} \sum_{\pi, r} |Q(\pi^*, r, \pi)| \cdot \frac{1}{|Q(\pi^*, r, \pi)|} \\
&= \frac{1}{(n-1)!} \sum_{\pi, r} |Q(\pi^*, r, \pi)| \cdot \frac{1}{2} \\
&= \frac{1}{(n-1)!} \cdot \frac{n!}{2} \cdot \frac{1}{n!} \sum_{\pi, r} |Q(\pi^*, r, \pi)| \\
&= \frac{n}{2} (\varphi_{m_1}(G) + \varphi_{m_2}(G)).
\end{aligned}$$

We prove that this bound is asymptotically tight. To do so, we need only show that there exists at least one game where manipulators achieve the proposed bound. Consider

[2]The numbers in the permutations are the weights of the agents.

a WVG $G = [2n-3; 2, \ldots, 2, 1, 1]$ of n agents, and having two manipulators, say, m_1 and m_2, each with weight 1 in the game. m_1 is pivotal for any permutation of agents in G if and only if it appears at the $(n-1)$-th position and immediately followed by m_2 at the last position. Observe that the sum of the weights of all the agents before the $(n-1)$-th position is $2n-4$, which is less than the desired quota of the game. Thus, there are $(n-2)!$ ways to arrange the non-manipulating agents in G such that m_1 is pivotal at the $(n-1)$-th position. By the same argument, m_2 is pivotal for the same number of permutations. Hence, $\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{2(n-2)!}{n!}$. Suppose now that m_1 and m_2 merge their weights to form a bloc, $\&S$, of weight 2, resulting in the game, $G' = [2n-3; 2, \ldots, 2]$ of $n-1$ agents. Clearly, this game is *unanimity*[3], and requires all agents in G' to form a winning coalition. Thus, $\varphi_i(G') = \frac{1}{n-1}$ for all agents i in game G'. Finally, $\varphi_{\&S}(G') = \frac{1}{n-1} = \frac{1}{n-1} \cdot \frac{n!}{2(n-2)!} \cdot \frac{2(n-2)!}{n!} = \frac{n}{2}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. □

4.2 Lower Bound

Theorem 2. *(Lower Bound). Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Shapley-Shubik power, $\varphi_{\&S}(G')$, of the bloc in the new game, $\varphi_{\&S}(G') \geq \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. Moreover, this bound is asymptotically tight.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let $\Pi_{G'}$ be the set of all permutations of the $n-1$ agents (including the merge bloc) in game G'. Also, let $\Pi^*_{G'}$ be the set of all permutations in game G' such that the bloc $\&S$ is pivotal for the permutations. Note that $\Pi^*_{G'} \subseteq \Pi_{G'}$. Finally, let Π^*_G, $Q(\pi^*, r, \pi)$, and $f(\pi, r)$ be as defined in Theorem 3.

We first bound $|Q(\pi^*, r, \pi)|$ for any π and $1 \leq r \leq (n-1)$. There are ${}^2P_2 = 2$ permutations[4] in $Q(\pi^*, r, \pi)$ such that m_1 and m_2 appear at the r-th position of π. There are ${}^2P_1 \cdot {}^{(n-2)}P_1 = 2(n-2)$ permutations in $Q(\pi^*, r, \pi)$ such that one of the two manipulators appears at the r-th position of π and the other is elsewhere in π. In all, $|Q(\pi^*, r, \pi)| \leq 2 + 2(n-2) \leq 2(n-1)$. Again, as in Theorem 3, all the permutations π^* in the set $Q(\pi^*, r, \pi)$ derived from a permutation $\pi \in \Pi_{G-2}$ in which at least one of the manipulators appears at the r-th position of π and is pivotal for π^* corressponds to a single permutation $f(\pi, r)$ in game G'. Thus, it is clear that $|\Pi^*_{G'}| \geq \frac{1}{|Q(\pi^*, r, \pi)|} \cdot |\Pi^*_G|$. Hence,

$$\begin{aligned}
\frac{|\Pi^*_{G'}|}{(n-1)!} &\geq \frac{1}{|Q(\pi^*, r, \pi)|} \cdot \frac{1}{(n-1)!} \cdot |\Pi^*_G| \\
\varphi_{\&S}(G') &= \frac{1}{2(n-1)} \cdot \frac{n!}{(n-1)!} \cdot \frac{|\Pi^*_G|}{n!} \\
&= \frac{n}{2(n-1)} \cdot \frac{|\Pi^*_G|}{n!} \\
&= \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G)).
\end{aligned}$$

We prove that this bound is asymptotically tight. Consider a WVG $G = [n; 1, 1, \ldots, 1]$ of n agents, and having two manipulators, say, m_1 and m_2, in the game. Clearly, this game is unanimity, and requires all agents in game G to form a winning coalition. Thus,

[3] A WVG is unanimity if there is a single winning coalition in the game, and every agent in the game is critical to the coalition.

[4] $P(n, r) = \frac{n!}{(n-r)!}$.

$\varphi_i(G) = \frac{1}{n}$ for all agents i in game G, and in particular, $\varphi_{m_1}(G) + \varphi_{m_2}(G) = \frac{2}{n}$. Suppose the manipulators merge their weights to form a bloc, $\&S$, of weight 2, resulting in the game, $G' = [n; 2, 1, \ldots, 1]$ of $n-1$ agents. The game remains unanimity, and $\varphi_{\&S}(G') = \frac{1}{n-1} = \frac{1}{n-1} \cdot \frac{n}{2} \cdot \frac{2}{n} = \frac{n}{2(n-1)}(\varphi_{m_1}(G) + \varphi_{m_2}(G))$. □

5 Banzhaf Index Bounds

We propose upper and lower bounds in this section to characterize the effect of manipulation by merging in WVGs using the Banzhaf power index. The proposed bounds are found to be within constant factors.

5.1 Upper Bound

Theorem 3. *(Upper Bound) Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Banzhaf power, $\beta_{\&S}(G')$, of the bloc in the new game, $\beta_{\&S}(G') \leq 3(\beta_{m_1}(G) + \beta_{m_2}(G))$.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2 (with weights w_1 and w_2, respectively), from the original game G that would like to merge their weights, and form a bloc $\&S$ in an altered game G'. Assume without loss of generality that $w_1 \leq w_2$. Recall that $\eta_i(G)$ is the number of winning coalitions for which an agent i is critical in WVG G.

We first bound the number of winning coalitions in G for which the manipulators are critical. Let Γ_G be the set of all possible coalitions of the n agents in G. Also, let Γ_{G-2} be the set of all possible coalitions of the remaining $n-2$ non-manipulating agents in G, i.e., not including agents m_1 and m_2. Consider any coalition $c \in \Gamma_{G-2}$ such that $w(c) < q$. Suppose we add strategic agents m_1 and/or m_2 to c and have a resulting winning coalition $c^* \in \Gamma_G$. Let Γ_G^* be the set of all possible coalitions c^* such that at least one of m_1 or m_2 is critical in G. We define the subsets $C_i \subseteq \Gamma_G^*$ as follows:

$C_1 = \{C \subseteq \Gamma_{G-2} : w(C) < q, w(C) + w_1 \geq q\}$
$C_2 = \{C \subseteq \Gamma_{G-2} : w(C) < q, w(C) + w_2 \geq q\}$
$C_3 = \{C \subseteq \Gamma_{G-2} : w(C) + w_1 < q, w(C) + w_1 + w_2 \geq q\}$
$C_4 = C_5 = \{C \subseteq \Gamma_{G-2} : w(C) + w_1 < q, w(C) + w_2 < q, w(C) + w_1 + w_2 \geq q\}$.

Thus,

$$\eta_{m_1}(G) + \eta_{m_2}(G) = \sum_{i=1}^{5} |C_i|. \quad (4)$$

Note that C_1 are winning coalitions in G for which m_1 is critical, and which does not include m_2. Other winning coalitions, $C_2, \ldots, C_5$, of the manipulators are similarly defined.

Similarly, we bound the number of coalitions for which the non-manipulators are critical in game G. Note that there are two possibilities for an arbitrary non-manipulating agent $j \in I \setminus \{m_1, m_2\}$ to be critical in a winning coalition in game G:

$S_1 = \{S \subseteq \Gamma_{G-2} \setminus \{j\} : w(S) < q, w(S) + w_j \geq q\}$
$S_2 = \{S \subseteq I \setminus \{j\} : m_1 \in S \lor m_2 \in S, w(S) < q, w(S) + w_j \geq q\}$[5].

S_1 are the winning coalitions in G for which agent j is critical, and which does not include both m_1 and m_2. S_2 are the winning coalitions in G for which agent j is critical, and which include at least one of m_1 or m_2. Thus, we have

[5] $a \lor b$ is the *or* of a and b. At least one of the two simple parts of the compound proposition is required to be true.

$$\eta_j(G) = |S_1| + |S_2|. \tag{5}$$

Now, we bound the number of coalitions in the altered game G' for which a manipulators' bloc, $\&S$, is critical. Let $c \in \Gamma_{G-2}$ be a coalition in game G such that $w(c) < q$. We define a function f such that a coalition, $f(c) = c \cup \{\&S\}$, of agents in game G' is winning if and only if at least one of the coalitions, $c \cup \{m_1\}$, $c \cup \{m_2\}$, or $c \cup \{m_1, m_2\}$, is winning in G.

We claim that for any c such that $c \cup \{m_1\} \in C_1$, it must also be the case that $c \cup \{m_2\} \in C_2$, since $w_1 \leq w_2$. Observe that the *two* winning coalitions, $c \cup \{m_1\} \in C_1$ and $c \cup \{m_2\} \in C_2$, of the manipulators, m_1 and m_2, from G, correspond to *exactly one* winning coalition $f(c)$ of the bloc, $\&S$, in G'. Similarly, for the case when $c \cup \{m_1\} \notin C_1$ and $c \cup \{m_2\} \in C_2$, it must also be true that, $c \cup \{m_1, m_2\} \in C_3$. This is because since if m_1 is not critical in $c \cup \{m_1\}$, it also cannot be critical in the winning coalition $c \cup \{m_1, m_2\} \in C_3$, where m_2 is present. Note, however, that, only m_2 is critical for coalition $c \cup \{m_1, m_2\} \in C_3$ by definition. Thus, again, the *two* winning coalitions, $c \cup \{m_2\} \in C_2$ and $c \cup \{m_1, m_2\} \in C_3$, of m_2, from G correspond to *exactly one* winning coalition $f(c)$ of the bloc in G'. Finally, both m_1 and m_2 are critical for any coalition $c \cup \{m_1, m_2\} \in C_4$ and $c \cup \{m_1, m_2\} \in C_5$ by definition. The *two* winning coalitions, $c \cup \{m_1, m_2\} \in C_4$ and $c \cup \{m_1, m_2\} \in C_5$, of m_1, m_2, from G correspond to *exactly one* winning coalition $f(c)$ of the bloc in G'. We conclude that the number of coalitions for which a manipulators' bloc, $\&S$, is critical in G' is one-half of the number of times the manipulators, m_1 and m_2, are critical in G. Thus,

$$\eta_{\&S}(G') = \frac{\eta_{m_1}(G) + \eta_{m_2}(G)}{2}. \tag{6}$$

It remains to bound the number of coalitions in G' for which the non-manipulators are critical. Note that because the power of the manipulators' bloc is the ratio of the number of winning coalitions in which the bloc is involved divided by the total number of winning coalitions involving all agents, the most power will be obtained by the bloc when the manipulators' bloc is involved in highest number of winning coalitions and the non-manipulators are involved in the least number of winning coalitions. Let $S \in S_1$. Clearly, since $m_1 \notin S$ and $m_2 \notin S$, S remains unchanged from G to G'. Hence, for this case, the non-manipulating agent j remains critical in G' for $|S_1|$ number of winning coalitions. Similarly, let $S \in S_2$. Since at least one of m_1 or m_2 is in S, the three possible coalitions for j to be critical for S in G are: $S \cup \{m_1, j\}$, $S \cup \{m_2, j\}$, and $S \cup \{m_1, m_2, j\}$. However, these coalitions correspond to *exactly one* winning coalition, $S \cup \{\&S, j\}$, of the bloc in G'. Thus,

$$\begin{aligned}
\eta_j(G') &= |S_1| + \frac{|S_2|}{3} && (7)\\
&= \frac{3|S_1| + |S_2|}{3} && (8)\\
&= \frac{|S_1| + |S_2| + 2|S_1|}{3} && (9)\\
&= \frac{\eta_j(G)}{3} + \frac{2|S_1|}{3} && (10)\\
&\geq \frac{\eta_j(G)}{3}. && (11)
\end{aligned}$$

We compute the Banzhaf power index of the bloc $\&S$ in game G' using (6) and (9):

$$\beta_{\&S}(G') = \frac{\eta_{\&S}(G')}{\eta_{\&S}(G') + \sum_{j \in I \setminus \{m_1, m_2\}} \eta_j(G')}$$

$$
\begin{aligned}
&= \frac{\frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{2}}{\frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{2} + \sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G')} \\
&= \frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{\eta_{m_1}(G)+\eta_{m_2}(G) + 2\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G')} \\
&\leq \frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{\eta_{m_1}(G)+\eta_{m_2}(G) + \frac{2}{3}\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\leq \frac{3(\eta_{m_1}(G)+\eta_{m_2}(G))}{3(\eta_{m_1}(G)+\eta_{m_2}(G)) + 2\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\leq \frac{3(\eta_{m_1}(G)+\eta_{m_2}(G))}{\eta_{m_1}(G)+\eta_{m_2}(G) + \sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\leq 3(\beta_{m_1}(G)+\beta_{m_2}(G))
\end{aligned}
$$

□

5.2 Lower Bound

Theorem 4. *(Lower Bound) Let $G = [q; w_1, \ldots, w_n]$ be a WVG of n agents. If two manipulators, m_1 and m_2, merge their weights to form a bloc, $\&S$, in an altered game G', then, the Banzhaf power, $\beta_{\&S}(G')$, of the bloc in the new game, $\beta_{\&S}(G') \geq \frac{\beta_{m_1}(G)+\beta_{m_2}(G)}{2}$.*

Proof. Let $S \subset I$ be a coalition of two distinct manipulators, m_1 and m_2, from the original game G that would like to merge into a bloc $\&S$ in an altered game G'. Let S_1 and S_2 be as defined in Theorem 3. We are interested in finding the minimum factor of power that can be gained by a merged bloc $\&S$ in game G'. Note again that, because the power of the bloc is the ratio of the number of winning coalitions in which the bloc is involved in divided by the total number of winning coalitions involving all agents, the least power will be obtained by the bloc when the manipulators' bloc is involved in the fewest number of winning coalitions and the non-manipulating agents are involved in the highest number of winning coalitions. Thus, we seek to find the maximum number of winning coalitions in which the non-manipulating agents can participate. The case which yields the maximum number of winning coalitions in game G is that agent j is critical for the three coalitions, $S \cup \{m_1, j\}$, $S \cup \{m_2, j\}$, and $S \cup \{m_1, m_2, j\}$ involving the manipulators. This is so because coalitions, S_1, contributing to the overall total remains the same from G to G'. Thus, as before:

$$
\begin{aligned}
\eta_j(G') &= |S_1| + \frac{|S_2|}{3} && (12) \\
&\leq |S_1| + \frac{|S_2|}{3} + \frac{2|S_2|}{3} && (13) \\
&\leq |S_1| + |S_2| && (14) \\
&\leq \eta_j(G). && (15)
\end{aligned}
$$

We compute the Banzhaf power index of the bloc $\&S$ in game G' using (6) and (13):

$$
\beta_{\&S}(G') = \frac{\eta_{\&S}(G')}{\eta_{\&S}(G') + \sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G')}
$$

$$
\begin{aligned}
&= \frac{\frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{2}}{\frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{2} + \sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G')} \\
&= \frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{\eta_{m_1}(G)+\eta_{m_2}(G) + 2\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G')} \\
&\geq \frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{\eta_{m_1}(G)+\eta_{m_2}(G) + 2\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\geq \frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{2(\eta_{m_1}(G)+\eta_{m_2}(G)) + 2\sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\geq \frac{1}{2}\cdot\frac{\eta_{m_1}(G)+\eta_{m_2}(G)}{\eta_{m_1}(G)+\eta_{m_2}(G) + \sum_{j\in I\setminus\{m_1,m_2\}} \eta_j(G)} \\
&\geq \frac{\beta_{m_1}(G)+\beta_{m_2}(G)}{2}.
\end{aligned}
$$

□

Although these bounds are within constant factors, we are not certain if they are asymptotically tight. A tighter analysis may further improve the bounds.

6 Conclusions

This paper investigates the effects of manipulation by merging in weighted voting games. Manipulation by merging refers to a dishonest behavior where two or more strategic agents merge their weights to form a single bloc in anticipation of power increase. Our focus is on the characterization of the extent to which agents may gain in such manipulation. We consider two prominent payoff concepts, the Shapley-Shubik and Banzhaf power indices, that are used in evaluating agents' power in such games.

The concern of this research is based on the assumption that the inability to limit (or understand) the effects of this manipulation may undermine the confidence agents have in decisions made via weighted voting games. If the results from this class of games are not seen as fair, agents may refuse to abide by decisions made in this manner. Thus, specifically, we propose four new bounds for this problem when there are $k = 2$ strategic agents in the games. Two of the bounds are also shown to be asymptotically tight. Table 2 provides a summary on the state of the arts on the bounds for manipulation by merging in weighted voting games using both the Shapley-Shubik and Banzhaf power indices.

Bounds	# Strategic agents	Shapely-Shubik Index	Banzhaf Index
Upper	$k = 2$	This paper	This paper
	$k > 2$	?	?
Lower	$k = 2$	This paper	This paper
	$k > 2$	?	?

Table 2: Summary of bounds for manipulation by merging in weighted voting games

The manipulation we consider in this research is natural, and has practical applications, that motivate interests from both the game theory and artificial intelligence communities. The proposed results in the research fit under the models of deception and fraud, as well as models and mechanisms for establishing identities, which are crucial for maintaining trustworthy interactions.

There are several areas of ongoing research on this problem. Here are some directions for future work. We have considered the case when the number, k, of strategic agents in a weighted voting game is 2. As shown in Examples 1 and 2, it is also possible for the number of strategic agents to be more than 2. Thus, it will be interesting to see non-trivial upper and lower bounds for this problem for the case, $k > 2$, using both the Shapley-Shubik and Banzhaf indices to compute agents' power. Furthermore, our immediate future work is to complement these theoretical results with empirical evaluations to see the extent of the factors for beneficial merges to strategic agents in practice. Finally, developing methods to reduce the effects of manipulation by merging in weighted voting games is an interesting research problem to consider.

References

[1] G. Chalkiadakis, E. Elkind, and M. Wooldridge. *Computational Aspects of Cooperative Game Theory.* 2012.

[2] L. S. Shapley and M. Shubik. A Method for Evaluating the Distribution of Power in a Committee System. *American Political Science Review*, 48: 787–792, 1954.

[3] L. Penrose. The Elementary Statistics of Majority Voting. *Journal of the Royal Statistical Society (Blackwell Publishing)*, 109(1):5357, 1946

[4] J. Banzhaf. Weighted Voting Doesn't Work: A Mathematical Analysis. *Rutgers Law Review*, 19:317–343, 1965.

[5] Y. Bachrach and E. Elkind. Divide and Conquer: False-name Manipulations in Weighted Voting Games In *Proc. 7th AAMAS Conference*, 975–982, 2008.

[6] H. Aziz, Y. Bachrach, E. Elkind, and M. Paterson. False-name Manipulations in Weighted Voting Games *Journal of Artificial Intelligence Research*, 40:57–93, 2011.

[7] R. O. Lasisi and V. H. Allan. False Name Manipulations in Weighted Voting Games: Susceptibility of Power Indices. In *13th Workshop on Trust in Agent Societies of the AAMAS Conference*, 139–150, 2010.

[8] R. O. Lasisi and V. H. Allan. A Search-based Approach to Annexation and Merging in Weighted Voting Games. In *Proc. 4th International Conference on Agents and Artificial Intelligence*, 44–53, 2012.

[9] R. O. Lasisi and V. H. Allan. Experimental Evaluation of the Effects of Manipulation by Merging in Weighted Voting Games. In *Proc. 5th International Conference on Agents and Artificial Intelligence*, 196–203, 2013.

[10] R. O. Lasisi and V. H. Allan. New Bounds on False-Name Manipulation in Weighted Voting Games. In *Proc. 27th FLAIRS Conference*, 57–62, 2014.

[11] M. Yokoo and Y. Sakurai and S. Matsubara. The Effect of False-name Bids in Combinatorial Auctions: New Fraud in Internet Auctions. *Games and Economic Behavior*, 46(1):174–188, 2004.

[12] R. O. Lasisi and V. H. Allan. Annexations and Merging in Weighted Voting Games - The Extent of Susceptibility of Power Indices. In *Proc. 3rd International Conference on Agents and Artificial Intelligence*, 124–133, 2011.

[13] D. S. Felsenthal and M. Machover. Annexation and Alliances: When Are Blocs Advantageous A Priori? *Social Choice and Welfare*, 19(2):295–312, 2002.

[14] S. J. Brams. Game Theory and Politics. *Free Press, New York*, 1975.

[15] D S. Felsenthal and M. Machover. The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes. *Edward Elgar, Cheltenham*, 1998.

[16] A. D. Taylor and W. S. Zwicker. Simple Games: Desirability Relations, Trading, Pseudoweightings, first edition ed. *Princeton University Press*, 1999.

[17] A. Laruelle. On the Choice of a Power Index. *Instituto Valenciano de Investigaciones Economicas*, 2103:99–10, 1999.

[18] T. Matsui and Y. Matsui. A Survey of Algorithms for Calculating Power Indices of Weighted Majority Games. *Journal of the Operations Research Society of Japan*, 43(1), 2000.

[19] D. Leech. Voting Power in the Governance of the IMF. *Annals of Operations Research*, 109(1):375–397, 2002.

[20] J. M. Alonso-Meijide and C. Bowles. Generating Functions for Coalitional Power Indices:An Application to the IMF. *Annals of Operations Research*, 137:21–44, 2005.

[21] H. Aziz, M. Paterson, and D. Leech. Efficient Algorithm for designing Weighted Voting Games. *11th International Multitopic Conference*, 211-216, 2007.

[22] S. Fatima, M. Wooldridge, and N. R. Jennings. An Anytime Approximation Method for the Inverse Shapley Value Problem *Proc. 7th AAMAS Conference*, 935–942, 2008.

[23] B. Keijzer, T. Klos, and Y. Zhang. Enumeration and exact design of weighted voting games. *Proc. 9th AAMAS Conference*, 391–398, 2010.

[24] L. S. Shapley. A value for n-person games. *In Kuhn, H. W., and Tucker, A. W. (Eds.), Contributions to the Theory of Games*, II:307–317, 1953.

[25] J. Deegan and E. W. Packel. A New Index of Power for Simple n-Person Games. *International Journal of Game Theory*, 7:113–123, 1978.

[26] R. J. Johnston. On the Measurement of Power: Some Reactions to Laver. *Environment and Planning*, 10:907–914, 1978.

[27] M. J. Holler and E. W. Packel Power, Luck and the Right Index. *Journal of Economics*, 43:21–29, 1983.

[28] H. Aziz and M. Paterson. False-name Manipulations in Weighted Voting Games: splitting, merging and annexation. *Proc. 8th AAMAS Conference*, 409–416, 2009.

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