Structure in Dichotomous Preferences

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Abstract

Many hard computational social choice problems are known to become tractable when voters' preferences belong to a restricted domain, such as those of single-peaked or single-crossing preferences. However, to date, all algorithmic results of this type have been obtained for the setting where each voter's preference list is a total order of candidates. The goal of this paper is to extend this line of research to the setting where voters' preferences are *dichotomous*, i.e., each voter approves a subset of candidates and disapproves the remaining candidates. We propose several analogues of the notions of single-peaked and single-crossing preferences for dichotomous profiles and investigate the relationships among them. We then demonstrate that for some of these notions the respective restricted domains admit efficient algorithms for computationally hard approval-based multi-winner rules such as Proportional Approval Voting (PAV) and Maximin Approval Voting (MAV).

1 Introduction

Preference aggregation is a fundamental problem in social choice, which has recently received a considerable amount of attention from the AI community. In particular, an important research question in *computational social choice* [9] is the complexity of computing the output of various preference aggregation procedures. While for most common single-winner rules winner determination is easy, many attractive rules that output a committee (a fixed-size set of winners) or a ranking of the candidates are known to be computationally hard.

There are several ways to circumvent these hardness results, such as using approximate and parameterized algorithms. These standard algorithmic approaches are complemented by an active stream of research that analyzes the computational complexity of voting rules on *restricted preference domains*, such as the classic domains of single-peaked [5] or single-crossing [32] preferences. This research direction was popularized by Walsh [37] and Faliszewski et al. [22], and has lead to a number of efficient algorithms for winner determination under prominent voting rules as well as for manipulation and control, which can be used when voters' preferences belong to one of these restricted domains [4, 8, 22, 23, 31, 36, 37].

To the best of our knowledge, this line of work only considers settings where voters' preferences are given by total orders over the set of candidates; indeed, this is perhaps the most widely studied setting in the area of computational social choice. However, computationally complex preference aggregation problems may also arise when voters' preferences are *dichotomous*, i.e., each voter approves a subset of the candidates and disapproves the remaining candidates. Committee selection rules for voters with dichotomous preferences, or *approval-based* rules, have recently attracted some attention from the computational social choice community, and for two prominent such rules (specifically, Proportional Approval Voting (PAV) [26] and Maximin Approval Voting (MAV) [7]) computing the winning committee is known to be NP-hard [2, 29]. It is therefore natural to ask if one could identify a suitable analogue of single-peaked/single-crossing preferences for the the dichotomous setting, and design efficient algorithms for approval-based rules over such restricted dichotomous preference domains.

To address this challenge, in this paper we propose and explore a number of domain restrictions for dichotomous preferences that build on the same intuition as the concepts of single-peakedness and single-crossingness. Some of our restricted domains are defined by embedding voters or candidates into the real line, and requiring that the voters' preferences over the candidates "respect" this embedding; others are obtained by viewing dichotomous preferences as weak orders and requiring them to admit a refinement that has a desirable structural property. Surprisingly, these approaches lead to a large number of concepts that are pairwise non-equivalent and capture different aspects of our intuition about what it means for preferences to be "one-dimensional". We analyze the relationships among these restricted preference domains, (see Figure 5 for a summary), and provide polynomial-time algorithms for detecting whether a given dichotomous profile belongs to one of these domains. We then demonstrate that considering these domains is useful from the perspective of algorithm design, by providing polynomial-time and FPT algorithms for *PAV* and *MAV* under some of these domain restrictions.

A preliminary version of this paper appeared in the proceedings of IJCAI 2015 [18].

2 Basic Definitions

Let $C = \{c_1, \dots, c_m\}$ be a finite set of candidates. A (strict) *partial order* $\succ$ over C is a antireflexive, antisymmetric and transitive binary relation on C; a (strict) *total order* is a partial order that satisfies either $c \succ d$ or $d \succ c$ for every $c, d \in C$. We say that a partial order $\succ$ over C is a *dichotomous weak order* if C can be partitioned into two disjoint sets C^+ and C^- (one of which may be empty) so that $c \succ d$ for each $c \in C^+, d \in C^-$ and the candidates within C^+ and C^- are incomparable under $\succ$.

An *approval vote* on C is an arbitrary subset of C. We say that an approval vote v is *trivial* if $v = \emptyset$ or $v = C$. A *dichotomous profile* $\mathcal{P} = (v_1, \dots, v_n)$ is a list of n approval votes; we will refer to v_i as the vote of voter i. We write $\overline{v_i} = C \setminus v_i$. We associate an approval vote v_i with the dichotomous weak order $\succ_{v_i}$ that satisfies $c \succ_{v_i} d$ if and only if $c \in v_i$, $d \in \overline{v_i}$. Note that $v_i = \emptyset$ and $v_i = C$ correspond to the same dichotomous weak order, namely the empty one.

A partial order $\succ'$ over C is a *refinement* of a partial order $\succ$ over C if for every $c, d \in C$ it holds that $c \succ d$ implies $c \succ' d$. A profile $\mathcal{P}' = (\succ_1, \dots, \succ_n)$ of total orders is a *refinement* of a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ if $\succ_i$ is a refinement of $\succ_{v_i}$ for each $i = 1, \dots, n$.

Let $\lhd$ be a total order over C. A total order $\succ$ over C is said to be *single-peaked with respect to* $\lhd$ if for any triple of candidates $a, b, c \in C$ with $a \lhd b \lhd c$ or $c \lhd b \lhd a$ it holds that $a \succ b$ implies $b \succ c$. A profile $\mathcal{P}$ of total orders over C is said to be *single-peaked* if there exists a total order $\lhd$ over C such that all orders in $\mathcal{P}$ are single-peaked with respect to $\lhd$.

A profile $\mathcal{P} = (\succ_1, \dots, \succ_n)$ of total orders over C is said to be *single-crossing with respect to the given order of votes* if for every pair of candidates $a, b \in C$ such that $a \succ_1 b$ all votes where a is preferred to b precede all votes where b is preferred to a; $\mathcal{P}$ is *single-crossing* if the votes in $\mathcal{P}$ can be permuted so that it becomes single-crossing with respect to the resulting order of votes.

A profile $\mathcal{P} = (\succ_1, \dots, \succ_n)$ of total orders over C is said to be 1-*Euclidean* if there is a mapping ρ of voters and candidates into the real line such that $c \succ_i d$ if and only if $|\rho(i) - \rho(c)| < |\rho(i) - \rho(d)|$. A 1-Euclidean profile is both single-peaked and single-crossing.

3 Preference Restrictions

We will now define a number of constraints that a dichotomous profile may satisfy. Most of these constraints can be divided into two basic groups: those that are based on ordering voters and/or candidates on the line and requiring the votes to respect this order (this includes VEI, VI, CEI, CI, DE, and DUE), and those that are based on viewing votes as weak orders and asking if there is a single-peaked/single-crossing/1-Euclidean profile of

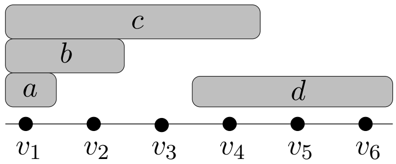

Figure 1: Voter Extremal Interval

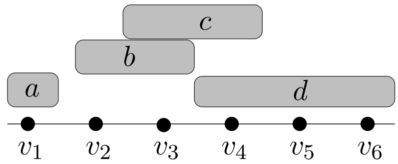

Figure 2: Voter Interval

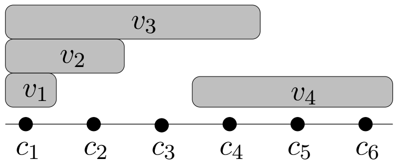

Figure 3: Candidate Extremal Interval

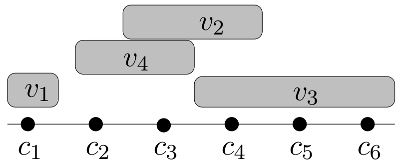

Figure 4: Candidate Interval

total orders that refines the given profile (this includes PSP, PSC, and PE). Some of these constraints have been studied before under other names: for instance, the CI domain was discussed in the context of judgement aggregation [15, 30] and manipulation [22]. The study of the latter type of constraints was initiated by Lackner [28]. We will also consider constraints that are based on partitioning voters/candidates (2PART and PART), as well as two constraints (WSC and SSC) that have been introduced in a recent paper of Elkind et al. [20] in order to understand the best way of extending the single-crossing property to weak orders.

Fix a profile $\mathcal{P} = (v_1, \dots, v_n)$ over C.

1. *2-partition (2PART)*: We say that $\mathcal{P}$ satisfies 2PART if $\mathcal{P}$ contains only two distinct votes v, v', and $v \cap v' = \emptyset$, $v \cup v' = C$.

2. *Partition (PART)*: We say that $\mathcal{P}$ satisfies PART if C can be partitioned into pairwise disjoint subsets $C_1, \dots, C_\ell$ such that $\{v_1, \dots, v_n\} = \{C_1, \dots, C_\ell\}$ (i.e., each voter in $\mathcal{P}$ approves one of the sets $C_1, \dots, C_\ell$). Note that this constraint contains as a special case profiles where every voter approves of exactly one candidate.

3. *Voter Extremal Interval (VEI)*: We say that $\mathcal{P}$ satisfies VEI if the voters in $\mathcal{P}$ can be reordered so that for every candidate c the voters that approve c form a prefix or a suffix of the ordering. Equivalently, both the voters who approve c and the voters who disapprove c form an interval of that ordering (Figure 1).

4. *Voter Interval (VI)*: We say that $\mathcal{P}$ satisfies VI if the voters in $\mathcal{P}$ can be reordered so that for every candidate c the voters that approve c form an interval of that ordering (Figure 2).

5. *Candidate Extremal Interval (CEI)*: We say that $\mathcal{P}$ satisfies CEI if candidates in C can be ordered so that each of the sets v_i forms a prefix or a suffix of that ordering. Equivalently, both v_i and $\overline{v_i}$ form an interval of that ordering (Figure 3).

6. *Candidate Interval (CI)*: We say that $\mathcal{P}$ satisfies CI if candidates in C can be ordered so that each of the sets v_i forms an interval of that ordering (Figure 4).

7. *Dichotomous Uniformly Euclidean (DUE)*: We say that $\mathcal{P}$ satisfies DUE if there is a mapping ρ of voters and candidates into the real line and a radius r such that for every voter i it holds that $v_i = \{c : |\rho(i) - \rho(c)| \le r\}$.

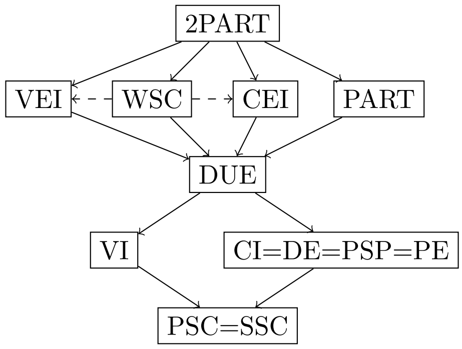

Figure 5: Relations between notions of structure. Dashed lines indicate that the respective containment holds only subject to additional conditions.

8. *Dichotomous Euclidean (DE)*: We say that $\mathcal{P}$ satisfies DE if there is a mapping ρ of voters and candidates into the real line such that for every voter i there exists a radius r_i with $v_i = \{c : |\rho(i) - \rho(c)| \leq r_i\}$.

9. *Possibly single-peaked (PSP)*: We say that $\mathcal{P}$ satisfies PSP if there is a single-peaked profile of total orders $\mathcal{P}'$ that is a refinement of $\mathcal{P}$.

10. *Possibly single-crossing (PSC)*: We say that $\mathcal{P}$ satisfies PSC if there is a single-crossing profile of total orders $\mathcal{P}'$ that is a refinement of $\mathcal{P}$.

11. *Possibly Euclidean (PE)*: We say that $\mathcal{P}$ satisfies PE if there is a 1-Euclidean profile of total orders $\mathcal{P}'$ that is a refinement of $\mathcal{P}$.

12. *Seemingly single-crossing (SSC)*: We say that $\mathcal{P}$ satisfies SSC if the voters in $\mathcal{P}$ can be reordered so that for each pair of candidates $a, b \in C$ it holds that either all votes v_i with $a \in v_i$, $b \notin v_i$ precede all votes v_j with $a \notin v_j$, $b \in v_j$ or vice versa.

13. *Weakly single-crossing (WSC)*: We say that $\mathcal{P}$ satisfies WSC if the voters in $\mathcal{P}$ can be reordered so that for each pair of candidates $a, b \in C$ it holds that each of the vote sets $V_1 = \{v_i : a \in v_i, b \notin v_i\}$, $V_2 = \{v_i : a \notin v_i, b \in v_i\}$, $V_3 = \{v \in \mathcal{P} : v \notin V_1 \cup V_2\}$ forms an interval of this ordering, with V_3 appearing between V_1 and V_2.

3.1 Relations

The relationships among the properties defined above are depicted in Figure 5, where arrows indicate containment, i.e., more restrictive notions are at the top. All these containments are strict.

The four arrows at the top level of the diagram are immediate: any profile with at most two distinct votes where each candidate is approved in at least one of these votes satisfies VEI, CEI and WSC, and by definition 2PART is a special case of PART.

To understand the arrows in the next level, we first characterize the dichotomous profiles that are weakly single-crossing.

Lemma 1. *A dichotomous profile $\mathcal{P}$ satisfies WSC if and only if there exist three votes u, v, w such that (1) for every $v_i \in \mathcal{P}$ it holds that $\succ_{v_i} \in \{\succ_u, \succ_v, \succ_w\}$, and (2) $\succ_v$ is equal to either $\succ_{u \cap w}$ or $\succ_{u \cup w}$.*

Proof sketch. It is easy to check that every profile satisfying (1)–(2) satisfies WSC. For the converse direction, assume without loss of generality that the ordering of the votes $v_1 \sqsubset v_2 \sqsubset \cdots \sqsubset v_n$ witnesses that $\mathcal{P}$ satisfies WSC. Let $u = v_1$, $w = v_n$, and set $C_1 = u \cap w$, $C_2 = u \cap \overline{w}$, $C_3 = \overline{u} \cap w$, $C_4 = \overline{u} \cap \overline{w}$. The WSC property implies that for every $\ell = 1, 2, 3, 4$, every $a, b \in C_\ell$, and every $v_i \in \mathcal{P}$ we have $a \in v_i$ if and only if $b \in v_i$, i.e., candidates in each C_ℓ occur as a block in all votes. Note that $v_1 = u = C_1 \cup C_2$, $v_n = w = C_1 \cup C_3$.

Suppose that $C_1, C_4 \neq \emptyset$. Then $C_1 \subseteq v_i$, $C_4 \subseteq \overline{v_i}$ for all $v_i \in \mathcal{P}$. Indeed, fix a pair of candidates $a \in C_1$, $b \in C_4$. Both the first and the last voter strictly prefer a to b, and therefore so do all other voters. Thus, if $\mathcal{P}$ contains a vote $v_i \neq u, w$, it has to be the case that $v_i = C_1 = u \cap w$ or $v_i = C_1 \cup C_2 \cup C_3 = u \cup w$; moreover, if both of these votes occur simultaneously and are distinct from each other and u, w (i.e., $C_2, C_3 \neq \emptyset$), the WSC property is violated. Indeed, suppose that $v_i = C_1$, $v_j = C_1 \cup C_2 \cup C_3$. Fix candidates $a \in C_1$, $b \in C_4$. If v_i appears before v_j, consider a candidate $c \in C_2$: we get a contradiction as voters v_1 and v_j are indifferent between a and c, but v_i strictly prefers a to c. If v_i appears after v_j, consider a candidate $d \in C_3$: we get a contradiction as voters v_1 and v_i are indifferent between d and b, but v_j strictly prefers d to b. When C_1 or C_4 is empty, the analysis is similar; note, however, that trivial votes ($v_i = C$ and $v_i = \emptyset$) may alternate arbitrarily without violating the WSC property (this is why the lemma is stated in terms of weak orders rather than approval votes). □

We can now show that under mild additional conditions (no trivial voters/candidates) WSC implies VEI and CEI.

Proposition 2. *Let $\mathcal{P}$ be a dichotomous profile that either contains only two distinct votes or contains no vote v_i with $v_i = \emptyset$. If $\mathcal{P}$ satisfies WSC, then it satisfies VEI.*

Proof. Assume without loss of generality that $\mathcal{P}$ satisfies WSC with respect to an ordering of voters $v_1 \sqsubset \cdots \sqsubset v_n$, and let $u = v_1$, $w = v_n$. We will show that $\mathcal{P}$ satisfies VEI with respect to $\sqsubset$. If $\mathcal{P}$ only contains two distinct votes, this claim is immediate, so assume that $\emptyset \notin \mathcal{P}$. Consider a vote $v \in \mathcal{P}$ that is distinct from u and w. Since $\emptyset \notin \mathcal{P}$, by Lemma 1 there exist i, j with $1 < i < j < n$ such that $v_k = u$ for $k < i$, $v_k = v$ for $k = i, \ldots, j$, $v_k = w$ for $k > j$, and $v \in \{u \cup w, u \cap w\}$. Suppose first that $v = u \cap w$. Then candidates in $u \cap w$ are approved by all voters, candidates in $u \setminus w$ are approved by the first $i - 1$ voters, candidates in $w \setminus u$ are approved by the last $n - j$ voters, and the remaining candidates are not approved by anyone. On the other hand, if $v = u \cup w$, then candidates in $u \cap w$ are approved by all voters, candidates in $u \setminus w$ are approved by the first j voters, candidates in $w \setminus u$ are approved by the last $n - i + 1$ voters, and the remaining candidates are not approved by anyone. □

The condition that the profile must not contain $\emptyset$ is necessary: the profile $(\{a, b\}, \emptyset, \{b, c\})$ satisfies WSC, but not VEI.

Proposition 3. *Let $\mathcal{P}$ be a dichotomous profile that either contains only two distinct votes or in which every candidate is approved in at least one vote and disapproved in at least one vote. If $\mathcal{P}$ satisfies WSC, then it satisfies CEI.*

Proof. Suppose that $\mathcal{P}$ is WSC with respect to some ordering of voters; let u and w be, respectively, the first and the last vote in this ordering. If $\mathcal{P}$ contains a trivial vote, it contains at most two non-trivial votes, in which case the claim is obvious. Thus, assume that it contains no trivial votes. Then we have $u \cap w = \emptyset$ (any candidate in $u \cap w$ would be approved by all voters) and $\overline{u} \cap \overline{w} = \emptyset$ (any candidate in $\overline{u} \cap \overline{w}$ would be disapproved by all voters). It is now easy to see that ordering the candidates so that all candidates approved by u precede all candidates approved by w witnesses that $\mathcal{P}$ is CEI. □

To see that conditions of Proposition 3 are necessary, consider the profile $(\{a, b\}, \{b, c\})$ over $\{a, b, c, d\}$ and the profile $(\{a, b\}, \{b\}, \{b, c\})$ over $\{a, b, c\}$: both of these profiles satisfy WSC, but not CEI.

Interestingly, requiring a dichotomous profile to satisy WSC, CEI and VEI simultaneously, turns out to be very demanding: we obtain 2-partition profiles.

Proposition 4. *A dichotomous profile is WSC, CEI and VEI if and only if it is a 2-partition.*

Proof. It is immediate that a 2-partition profile is WSC, CEI, and VEI. For the converse direction, let $\mathcal{P}$ be a CEI, VEI and WSC profile. By Lemma 1, $\mathcal{P}$ contains at most three distinct votes u, v, w with $v = u \cap w$ or $v = u \cup w$. Since $\mathcal{P}$ is CEI, we know from Proposition 3 that every candidate is approved at least once. Hence $u \cup w = C$. Furthermore, every candidate is disapproved at least once. Thus, $u \cap w = \emptyset$, since this intersection is also approved by v. Thus, v is a trivial vote. This is possible because of Lemma 2 and hence v does not appear in $\mathcal{P}$. We have shown that $\mathcal{P}$ is a 2-partition profile. □

Next, we will relate CEI and VEI to DUE.

Proposition 5. *If a dichotomous profile $\mathcal{P}$ satisfies CEI or VEI, then it satisfies DUE.*

Proof. Suppose first that $\mathcal{P}$ satisfies CEI with respect to the ordering $c_1 \lhd \cdots \lhd c_m$ of candidates. Map the candidates into the real line by setting $\rho(c_i) = i$, and let $r = m$. We can now place each voter i to the left or to the right of all candidates at an appropriate distance so that the set of candidates within distance r from him coincides with v_i. For VEI the argument is similar: if $\mathcal{P}$ satisfies VEI with respect to the ordering $v_1 \sqsubset \cdots \sqsubset v_n$ of voters, we place voters on the real line according to $\rho(i) = i$, let $r = n$, and place each candidate to the left or to the right of all voters at an appropriate distance. □

The proof that WSC implies DUE is also based on our characterization of WSC preferences.

Proposition 6. *If a dichotomous profile $\mathcal{P}$ satisfies WSC, then it satisfies DUE.*

Proof. Clearly empty votes can be ignored when checking whether a profile satisfies DUE, so assume $\mathcal{P}$ contains no empty votes. Then it contains at most three distinct votes u, v, w with $v = u \cap w$ or $v = u \cup w$. Set $\rho(c) = 1$ for $c \in u \setminus w$, $\rho(c) = 2$ for $c \in u \cap w$, $\rho(c) = 3$ for $c \in w \setminus u$, $\rho(c) = 10$ for $c \notin u \cup w$. We set $r = 1$ if $v = u \cap w$ and $r = 2$ if $v = u \cup w$, and position the voters accordingly. □

The last arrow on this level is from PART to DUE: here, the containment is straightforward, as the candidates approved by each voter can be placed as a block on the axis, with the respective voter(s) placed in the center of this block.

Proposition 7. *If a dichotomous profile $\mathcal{P}$ satisfies DUE then it satisfies both VI and CI. The converse direction does not hold: there are profiles that satisfy VI and CI but not DUE.*

Proof. Since $\mathcal{P}$ satisfies DUE, we have an embedding ρ of voters and candidates into the real line. For VI, we order voters as induced by the ρ mapping; the voters approving some candidate form an interval on this induced order. For CI, we order candidates as induced by the ρ mapping; voters always approve a single interval on this ordering.

For showing that the converse direction does not hold, consider the profile $(\{a, b, c\}, \{b, c, d\}, \{b\}, \{c\})$. Towards a contradiction assume that ρ is a mapping of voters and candidates into the real line that witnesses the DUE property for a fixed radius r. The given profile satisfies CI only with respect to the orders $a \lhd b \lhd c \lhd d$, $a \lhd c \lhd b \lhd d$ and their reverses. Since the profile is symmetric with respect to a and d and with respect to b and c, we can assume without loss of generality that ρ orders candidates as the order $a \lhd b \lhd c \lhd d$ does. Then it has to hold that $|\rho(a) - \rho(c)| \leq 2r$ since a and c appear in the same vote. However, due to the vote $\{b\}$, it also has to hold that $|\rho(a) - \rho(c)| > 2r$; this is a contradiction. □

We see that similar to total orders, where the intersection of the single-peaked and the single-crossing domain is a strict subset of the 1-Euclidean domain (see discussion in [16, 19]), for dichotomous preferences the intersection of VI and CI does not yield DUE. The next results show that the classes of CI, DE, PSP and PE preferences coincide. We remark that the equivalence between CI and DE was observed by Faliszewski et al. [22] in the conference version of their paper.

Proposition 8. *Let $\mathcal{P}$ be a dichotomous profile. Then the following conditions are equivalent: (a) $\mathcal{P}$ satisfies PE (b) $\mathcal{P}$ satisfies PSP (c) $\mathcal{P}$ satisfies CI (d) $\mathcal{P}$ satisfies DE.*

Proof sketch. Suppose $\mathcal{P}$ satisfies PE, and let $\mathcal{P}'$ be a refinement of $\mathcal{P}$ that, together with a mapping ρ, witnesses this. Then $\mathcal{P}'$ is single-peaked and therefore $\mathcal{P}$ satisfies PSP. If $\mathcal{P}$ satisfies PSP, as witnessed by a refinement $\mathcal{P}'$ and an axis $\lhd$, then $\mathcal{P}$ satisfies CI with respect to $\lhd$. If $\mathcal{P}$ satisfies CI with respect to an order $\lhd$ of candidates, we can map the candidates into the real axis in the order suggested by $\lhd$ so that the distance between every two adjacent candidates is 1. We can then choose an appropriate approval radius and position for each voter. Finally, if $\mathcal{P}$ satisfies DE, as witnessed by a mapping ρ, we can use this mapping to construct a refinement of $\mathcal{P}$; by construction, this refinement is 1-Euclidean (we may have to modify ρ slightly to avoid ties). □

Also, every PE profile is PSC since every 1-Euclidean refinement is also single-crossing. Interestingly, the converse is not true.

Example 1. Consider the profile $\mathcal{P} = (\{a, b\}, \{a, c\}, \{b, c\})$ over $C = \{a, b, c\}$. It satisfies PSC, as witnessed by the single-crossing refinement $(a \succ b \succ c, c \succ a \succ b, c \succ b \succ a)$. However, in every refinement of $\mathcal{P}$ the first voter ranks c last, the second voter ranks b last, and the third voter ranks a last. Thus, no such refinement can be single-peaked, and, consequently, no such refinement can be 1-Euclidean.

The equivalence between PSC and SSC is not entirely obvious: while it is clear that a profile that violates SSC also violates PSC, to prove the converse one needs to use an argument similar to the proof of Theorem 4 in [20]. This has been shown in the extended version of [20].

Proposition 9. *If a dichotomous profile $\mathcal{P}$ satisfies VI, it also satisfies SSC.*

Proof. Assume that an VI profile is not SSC. Since it is not SSC, for every ordering of votes $\sqsubset$ there are two candidates a, b and votes $v_i \sqsubset v_j \sqsubset v_k$ such that $v_i : a \succ b$, $v_j : b \succ a$ and $v_k : a \succ b$. This implies, however, that for every $\sqsubset$ there is a candidate a and votes $v_i \sqsubset v_j \sqsubset v_k$ such that v_i and v_k approve of a and v_j disapproves a. This contradicts our assumption that the given profile is VI. □

We omit all remaining examples showing non-containment due to space constraints.

3.2 Detection

To exploit the constraints defined in Section 3, we have developed algorithms that can decide whether a given profile belongs to one of the restricted domains defined by these constraints.

In the following we present polynomial-time detection algorithms for all constraints under consideration.

Clearly, verifying whether a given profile satisfies 2PART or PART is straightforward. For most of the remaining problems, we can proceed by a reduction to the classic Consecutive 1s problem [6] (for CI, this was shown by Faliszewski et al. [22]; subsequently, Faliszewski (personal communication) observed that this result extends to VI). This problem asks if the columns of a given 0-1 matrix can be permuted in such a way that in each row of the resulting matrix the 1s are consecutive, i.e., the 1s form an interval in each row; it admits a linear-time algorithm [6].

Theorem 10. *Detecting whether a dichotomous profile satisfies CEI, CI, VI or VEI is possible in $\mathcal{O}(m \cdot n)$ time.*

Proof. Let $C = \{c_1, c_2, \dots, c_m\}$ and $\mathcal{P} = (v_1, v_2, \dots, v_n)$. We construct an instance of Consecutive 1s in slightly different ways, depending on the property we want to detect. In all cases, we obtain a "yes"-instance if and only if the given profile has the desired property.

Let us start with CI. For each vote, we create one row of the matrix: for each $i \in [n]$ and $j \in [m]$, the j-th entry of the i-th row is 1 if $c_j \in v_i$ and 0 otherwise. In this way, we obtain an m-by-n matrix. Permuting the columns of this matrix so that 1s form an interval in each row is equivalent to permuting candidates so that the set of candidates approved by each voter forms an interval. For CEI, we combine the matrix for CI with its complement, i.e., we add a second row for each vote v_i, so that the j-th entry of that row is 0 of $c_j \in v_i$ and 1 otherwise. A column permutation of the resulting m-by-$2n$ matrix such that 1s form an interval in each row corresponds to permuting candidates so that for each voter both the set of her approved candidates and the set of her disapproved candidates form an interval; this is equivalent to the CEI property. For VI it suffices to transpose the matrix constructed for CI, and for VEI this matrix has to be combined with its complement. □

The following lemma shows how to verify the DUE property[1].

Theorem 11. *Detecting whether a dichotomous profile satisfies DUE is possible in $\mathcal{O}(m \cdot n)$ time.*

Proof. Let $C = \{c_1, c_2, \dots, c_m\}$ and $\mathcal{P} = (v_1, v_2, \dots, v_n)$. The profile $\mathcal{P}$ satisfies DUE if there is a mapping ρ of voters and candidates into the real line and a radius r such that for every voter i it holds that $v_i = \{c : |\rho(i) - \rho(c)| \le r\}$. This is equivalent to requiring that for every voter and candidate there exists an interval on the real line of length r such that the interval of a voter and a candidate overlap if and only if the voter approves the candidate. (Given an interval, the function ρ maps to the center of the corresponding interval; given ρ, intervals are centred around the corresponding values of ρ.) This is exactly the characterization of *unit interval bigraphs*, i.e., a bipartite graph (bigraph) (V, E) with partition $V = X \cup Y$ is a unit interval bigraph if for every vertex $v \in V$ there exists an interval $I(v)$ on the real line with $|I(v)| = 1$ such that $\{x, y\} \in E$ if and only if $x \in X$, $y \in Y$, and $I(x) \cap I(y) \neq 0$. Unit interval bigraphs are equivalent to *proper interval bigraphs* [34, 38]. A bigraph (V, E) with partition $V = X \cup Y$ is a proper interval bigraphs if for every vertex $v \in V$ there exists an interval $I(v)$ on the real line such that no interval contains another and $\{x, y\} \in E$ if and only if $x \in X$, $y \in Y$, and $I(x) \cap I(y) \neq 0$.

Now, define the bigraph of $\mathcal{P}$ to be (V, E) with vertices being candidates and voters. Two vertices are connected by an edge if and only if one vertex represents a voter and the other a candidate and the voter approves the candidate. Then, the profile $\mathcal{P}$ satisfies DUE if and

[1] Nederlof and Woeginger [33] have communicated to us an alternative proof for the detecting whether a profile satisfies DUE in polynomial time.

only if the bipartite graph of $\mathcal{P}$ is a unit interval graph and hence a proper interval bigraph. Checking whether a graph is a proper interval bigraph can be done in time $\mathcal{O}(|V| + |E|)$ [14, 24]. In our case we have $|V| = m + n$ and E can be trivially bounded by $m \cdot n$.

The algorithm of [24] actually outputs a proper interval representation if possible. This representation can be used to compute the mapping ρ that witnesses the DUE property. Let us briefly sketch how to obtain a unit interval representation of a graph (V, E); it is then straightforward to actually obtain ρ via the correspondence mentioned above. We assume that the intervals $I_1, \dots, I_{|V|}$ are ordered with respect to their left endpoint. First, by scaling we ensure that the first interval has length 1. Then, we iteratively apply the following step for interval I_k: let x be the rightmost point of the interval I_{k-1}. We stretch or shrink the interval $[x, +\infty)$ such that I_k has length 1. Note that $[x, +\infty)$ contains at least a part of I_k because otherwise the proper interval property would be violated. In this way intersections are preserved and we ensure that every interval has length 1. □

For WSC, Elkind et al. [20] provide an algorithm that works for any weak orders (not just dichotomous ones). They leave the complexity of detecting PSC and SSC as an open problem. The corresponding result for dichotomous preferences, i.e., detecting SSC in polynomial time, follows from a result by Beresnev and Davydov [3] (in Russian; for an English description of this result, see Klinz et al. [27]). Let us briefly summarize this result. An m-by-n matrix $A = (a_{i,j})$ is 1-connected if for every pair of rows i, i' the sequence $(a_{i,j} - a_{i',j})_{j=1\dots n}$ has exactly one sign change, i.e., there is an index $j' \in [1 : n]$ such that for all $j \le j'$, $a_{i,j} - a_{i',j} \ge 0$ and for all $j > j'$, $a_{i,j} - a_{i',j} \le 0$ or vice versa. Beresnev and Davydov [3] show that this property can be checked in $\mathcal{O}(m^2 \cdot n^2)$ time for $(0,1)$-matrices. Let $C = \{c_1, c_2, \dots, c_m\}$ and $\mathcal{P} = (v_1, v_2, \dots, v_n)$. We define A to be the m-by-n $(0,1)$-matrix corresponding to the dichotomous profile $\mathcal{P}$ by setting $a_{i,j}$ to 1 if $c_i \in v_j$ and 0 otherwise. Now observe that $\mathcal{P}$ satisfies SSC if and only if for every pair of candidates $c_i, c_{i'}$, either all votes v_j with $c_i \in v_j$, $c_{i'} \notin v_j$ precede all votes $v_{j'}$ with $c_i \notin v_{j'}$, $c_{i'} \in v_{j'}$ or vice versa. This is exactly the case if the positive entries of $(a_{i,j} - a_{i',j})_{j=1\dots n}$ precede all negative entries or vice versa, i.e., A is 1-connected. Hence we obtain the following result.

Lemma 12. *Detecting whether a dichotomous profile satisfies SSC is possible in $\mathcal{O}(m^2 \cdot n^2)$ time.*

4 Algorithms for Committee Selection

In this section, we consider two classic approval-based committee selection rules—Proportional Approval Voting (*PAV*) and Maximin Approval Voting (*MAV*)—and argue that we can design efficient algorithms for these rules when voters' preferences belong to some of the domains in our list (for some of the richer domains, we may need to place mild additional restrictions on voters' preferences). Due to space constraints we omit some of the algorithm descriptions and proofs. We start by providing formal definitions of these rules.

Definition 1. Every non-increasing infinite sequence of non-negative reals $\mathbf{w} = (w_1, w_2, \dots)$ that satisfies $w_1 = 1$ defines a *committee selection rule* $\mathbf{w}$-*PAV*. This rule takes a set of candidates C, a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ and a target committee size $k \le |C|$ as its input. For every size-k subset W of C, it computes its $\mathbf{w}$-*PAV* score as $\sum_{v_i \in \mathcal{P}} u_{\mathbf{w}}(|W \cap v_i|)$, where $u_{\mathbf{w}}(p) = \sum_{j=1}^{p} w_j$, and outputs a size-$k$ subset with the highest $\mathbf{w}$-*PAV* score, breaking ties arbitrarily. The $\mathbf{w}$-*PAV* rule with $\mathbf{w} = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ is usually referred to simply as the *PAV* rule, and we write $u(p) = 1 + \dots + \frac{1}{p}$.

Definition 2. Given a set of candidates C, a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ and a target committee size $k \le |C|$, the *MAV*-score of a size-k subset W of C is computed as

$\max_{v_i \in \mathcal{P}}(|W \setminus v_i| + |v_i \setminus W|)$. *MAV* outputs a size-$k$ subset with the lowest *MAV* score, breaking ties arbitrarily.

The **w**-*PAV* rule is defined by Kilgour and Marshall [26], see also [25]. Intuitively, under this rule each voter is assumed to derive a utility of 1 from having exactly one of his approved candidates in the winning set; his marginal utility from having more of his approved candidates in the winning set is non-increasing. The goal of the rule is to maximize the sum of voters' utilities. In what follows we assume that the entries of $\mathbf{w}$ are rational and w_i can be computed in time poly(i). *PAV* is of particular interest since it is the only known approval-based committee selection rule that satisfies the *Extendend Justified Representation* property [1], which intuitively states that every large enough homogenous group has to be represented in the committee. In contrast, *MAV* [7] has an egalitarian objective: for each candidate committee, it computes the dissatisfaction of the least happy voter, and outputs a committee that minimizes the quantity.

Computing the winning committee under *MAV* and *PAV* is NP-hard, see, respectively, [29] and [2, 35]. The hardness result for *PAV* extends to **w**-*PAV* as long as $\mathbf{w}$ satisfies $w_1 > w_2$; moreover, it holds even if each voter approves of at most two candidates or if each candidate is approved by at most three voters.

We will now show that *PAV* admits an algorithm whose running time is polynomial in the number of voters and the number of candidates if the input profile satisfies CI or VI and, furthermore, each voter approves at most s candidates or each candidate is approved by at most d voters, where s and d are given constants. More specifically, we prove that *PAV* winner determination for CI and VI preferences is in FPT with respect to parameter s and in XP with respect to parameter d. For simplicity, we state our results for *PAV*; however, all of them can be extended to **w**-*PAV*.

In what follows, we write $[x : y]$ to denote the set $\{z \in \mathbb{Z} : x \le z \le y\}$.

Theorem 13. *Given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $|v_i| \le s$ for all $v_i \in \mathcal{P}$ and $\mathcal{P}$ satisfies VI, then we can find a winning committee under PAV in time $\mathcal{O}(2^{2s} \cdot k \cdot n)$.*

Proof. Assume that $\mathcal{P}$ satisfies VI with respect to the order of voters $v_1 \sqsubset \cdots \sqsubset v_n$. For each triple (i, A, ℓ), where $i \in [1 : n]$, $A \subseteq v_i$, and $\ell \in [0 : k]$, let $r(i, A, \ell)$ be the maximum utility that the first i voters can obtain from a committee W such that $W \cap v_i = A$, $|W| = \ell$, and $W \subseteq v_1 \cup \ldots \cup v_i$.

We have $r(1, A, |A|) = u(|A|)$ for every $A \subseteq v_1$ and $r(1, A, \ell) = -\infty$ for every $A \subseteq v_1$, $\ell \in [0 : k] \setminus \{|A|\}$. To compute $r(i + 1, A, \ell)$ for $i \in [1 : n - 1]$, $A \subseteq v_{i+1}$ and $\ell \in [0 : k]$, we let $p = |A \setminus v_i|$ and set

$$r(i + 1, A, \ell) = \max_{D \subseteq v_i \setminus v_{i+1}} r(i, D \cup (A \cap v_i), \ell - p) + u(|A|).$$

Indeed, every committee W with $|W| = \ell$, $W \cap v_{i+1} = A$, $W \subseteq v_1 \cup \ldots \cup v_{i+1}$ contains exactly $\ell - p$ candidates from $v_1 \cup \ldots \cup v_i$ and its intersection with v_i is of the form $D \cup (A \cap v_i)$, where candidates in D are approved by v_i, but not v_{i+1}. We output $\max_{A \subseteq v_n} r(n, A, k)$.

This dynamic program has $n \cdot 2^s \cdot (k + 1)$ states, and the value of each state is computed using $\mathcal{O}(2^s)$ arithmetic operations. Assuming that basic calculations take constant time, we obtain a total runtime of $\mathcal{O}(2^{2s} \cdot k \cdot n)$. □

A similar dynamic programming algorithm can be used if voters' preferences satisfy CI.

Theorem 14. *Given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $|v_i| \le s$ for all $v_i \in \mathcal{P}$ and $\mathcal{P}$ satisfies CI, then we can find a winning committee under PAV in time $\mathcal{O}(2^s \cdot n \cdot m)$.*

Our next two theorems also considers CI and VI preferences, and deal with the case where no candidate is approved by too many voters. Just as the algorithms in the proofs of Theorems 13 and 14, the algorithms for this case are based on dynamic programming.

Theorem 15. *Given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $|\{i \mid c \in v_i\}| \leq d$ for all $c \in C$ and $\mathcal{P}$ satisfies CI or VI, then we can find a winning committee under PAV in time* $\mathrm{poly}(d, m, n, k^d)$.

It is an open question whether the constraints on s and d in Theorems 13, 14 and 15 are necessary. The dynamic programming algorithms presented here seem to fundamentally rely on the s or d; a polynomial-time algorithm for winner determination of *PAV* under CI and VI preferences would require a substantially different approach. It is of course also possible that *PAV* remains hard under CI and VI preferences.

For "truncated" weight vectors $\mathbf{w}$ we can find $\mathbf{w}$-*PAV* winners in polynomial time. As the $(1, 0, \dots)$-*PAV* rule is essentially the classic Chamberlin–Courant rule [11] for dichotomous preferences, our next result can be seen as an extension of the results of [4] and [36] for the Chamberlin–Courant rule and single-peaked and single-crossing preferences: while we work on a less expressive domain (dichotomous preferences vs. total orders), we can handle a larger class of rules (all weight vectors with a constant number of non-zero entries rather than just $(1, 0, \dots,))$.

Theorem 16. *Consider a weight vector $\mathbf{w}$ where $w_i = 0$ for $i > i_0$ for some constant i_0. Then given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $\mathcal{P}$ satisfies VI, we can find a winning committee under $\mathbf{w}$-PAV in polynomial time.*

Proof. Assume that $\mathcal{P}$ satisfies VI with respect to the order of voters $v_1 \sqsubset \dots \sqsubset v_n$.

The following algorithm is a refinement of Theorem 13. For each triple (i, A, ℓ), where $i \in [1 : n]$, $A \subseteq v_i$, and $\ell \in [0 : k]$, let $r(i, A, \ell)$ be the maximum utility that the first i voters can obtain from a committee W such that $|W| = \ell$, and $W \subseteq v_1 \cup \ldots \cup v_i$ and $A \subseteq W$.

We have $r(1, A, \ell) = u(\ell)$ for every $\ell \in [0 : |v_1|]$ and $A \subseteq v_1$ with $|A| = \min(i_0, \ell)$. In addition, we have $r(1, A, \ell) = -\infty$ for every other $A \subseteq v_1$ and $\ell \in [0 : k]$. To compute $r(i + 1, A, \ell)$ for $i \in [1 : n - 1]$, $A \subseteq v_{i+1}$ with $|A| \leq i_0$ and $\ell \in [|A| : k]$, we let $s = |v_{i+1} \setminus (v_i \cup A)|$, i.e., the maximal number of candidates that might have been added in the $i + 1$st step to the committee but that do not show up in A, and set

$$r(i + 1, A, \ell) = \max r(i, D \cup (A \cap v_i), \ell - |A| - r) + u(|A|),$$

where the maximum is taken over all $D \subseteq v_i \setminus v_{i+1}$ with $|D| \in [0 : i_0 - |A \cap v_i|]$ and all $r \in [0 : s]$.

This dynamic program has $n \cdot m^{i_0} \cdot (k+1)$ states, and the value of each state is computed using $\mathcal{O}(m^{i_0} + 1)$ arithmetic operations. Assuming that basic calculations take constant time, we obtain a total runtime of $\mathcal{O}(n \cdot m^{2i_0+1} \cdot k)$, which is polynomial for constant i_0. $\square$

Theorem 17. *Consider a weight vector $\mathbf{w}$ where $w_i = 0$ for $i > i_0$ for some constant i_0. Then given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $\mathcal{P}$ satisfies CI, we can find a winning committee under $\mathbf{w}$-PAV in polynomial time.*

Moreover, for the more restricted domains, such as VEI, CEI, WSC and PART we can design polynomial-time algorithms for both *MAV* and *PAV*, under no additional constraints on preferences (again, our results extend to $\mathbf{w}$-*PAV*).

Theorem 18. *Given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $\mathcal{P}$ satisfies VEI, we can find a winning committee under MAV and PAV in polynomial time.*

Proof. Assume without loss of generality that $\mathcal{P}$ satisfies VEI for voter order $v_1 \sqsubset \dots \sqsubset v_n$. Each candidate in C belongs to one of the following four groups: $C_1 = v_1 \cap v_n$, $C_2 = v_1 \setminus v_n$, $C_3 = v_n \setminus v_1$, and $C_4 = \overline{v_1} \cap \overline{v_n}$; candidates in C_1 are approved by all voters and candidates in C_4 are not approved by any of the voters.

Suppose first that $|C_1 \cup C_2 \cup C_3| < k$. Then there exists an optimal committee for both *PAV* and *MAV* that contains all candidates in $C_1 \cup C_2 \cup C_3$ and exactly $k - |C_1 \cup C_2 \cup C_3|$ candidates from C_4. Hence, we can now assume that this is not the case. Then there exist an optimal committee that contains no candidates from C_4.

Now, if $|C_1| \geq k$, an optimal committee for both *PAV* and *MAV* consists of k candidates from C_1, and if $|C_1| < k$, there exists an optimal committee that contains all candidates in C_1. It remains to decide how to allocate the remaining places among candidates in C_2 and C_3. To do so, we observe that there is a natural ordering over each of these sets: given a pair of candidates (c, c') in $C_2 \times C_2$ or $C_3 \times C_3$, we write $c \leq c'$ if $\{i : c \in v_i\} \subseteq \{i : c' \in v_i\}$. Note that every two candidates in C_2 are comparable with respect to $\leq$, and so are every two candidates in C_3. It is now easy to see that there exists an optimal committee (for *PAV* or *MAV*) that consists of candidates in C_1, top p candidates in C_2 with respect to $\leq$ and top r candidates in C_3 with respect to $\leq$ for some non-negative values of p, r with $p + r + |C_1| = k$. Thus, by considering at most k^2 possibilities for p and r, we can find an optimal committee. ☐

For CEI, we employ a dynamic programing algorithm, somewhat similar to the one used in Theorem 14. Since we consider a more constrained preferences (CEI instead of CI), we do not require to maintain an exponential number of states.

Theorem 19. *Given a dichotomous profile $\mathcal{P} = (v_1, \dots, v_n)$ over a candidate set $C = \{c_1, \dots, c_m\}$ and a target committee size k, if $\mathcal{P}$ satisfies CEI, we can find a winning committee under MAV and PAV in polynomial time.*

The same statement holds for WSC and PART as well.

5 Conclusions and Open Problems

We have initiated research on analogues of the notions of single-peakedness and single-crossingness for dichotomous preference domains. We have proposed many constraints that capture some aspects of what it means for dichotomous preferences to be single-dimensional, explored the relationship among them, and showed that these constraints can be useful for identifying efficiently solvable special cases of hard voting problems on dichotomous domains. The algorithmic results in Section 4 can be seen as a proof that our approach has merit; however, there is certainly room for improvement there, both in terms of removing restrictions on the sizes of approval sets and number of voters that approve each candidate (for *PAV*) and in terms of considering larger domains, such as PSC for *PAV* and CI or VI for *MAV*.

We have provided polynomial-time algorithms for checking whether a given dichotomous profile satisfies one of the constraint. We can also ask if it is possible to detect if a given dichotomous profile is close to satisfying a structural constraint, and whether such "almost-structured" profiles have useful algorithmic properties; similar issues for profiles of total orders have recently received a lot of attention in the literature [10, 12, 13, 17, 21, 23].

Acknowledgments

The authors would like thank Gerhard Woeginger and Dominik Peters for pointing out very helpful literature. This work was supported by the European Research Council (ERC) under grant number 639945 (ACCORD). The second author was additionally supported by the Austrian Science Foundation FWF, grant P25518-N23 and Y698.

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