Finding a Collective Set of Items: From Proportional Multirepresentation to Group Recommendation

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Abstract

We consider the following problem: There is a set of items (e.g., movies) and a group of agents (e.g., passengers on a plane); each agent has some intrinsic utility for each of the items. Our goal is to pick a set of K items that maximize the total derived utility of all the agents (i.e., in our example we are to pick K movies that we put on the plane's entertainment system). However, the actual utility that an agent derives from a given item is only a fraction of its intrinsic one, and this fraction depends on how the agent ranks the item among available ones (in the movie example, the perceived value of a movie depends on the values of the other ones available). Extreme examples of our model include the setting where each agent derives utility from his or her most preferred item only (e.g., an agent will watch his or her favorite movie only), from his or her least preferred item only (e.g., the agent worries that he or she will be somehow forced to watch the worst available movie), or derives $1/K$ of the utility from each of the available items (e.g., the agent will pick a movie at random). Formally, to model this process of adjusting the derived utility, we use the mechanism of ordered weighted average (OWA) operators. Our contribution is twofold: First, we provide a formal specification of the model and provide concrete examples and settings where particular OWA operators are applicable. Second, we show that, in general, our problem is NP-hard but—depending on the OWA operator and the nature of agents' utilities—there exist good, efficient approximation algorithms (sometimes even polynomial time approximation schemes).

1 Introduction

A number of real-world problems consist of selecting a set of items for a group of agents to jointly use. Examples of such activities include picking a set of movies to put on a plane's entertainment system, deciding which journals a university library should subscribe to, deciding what common facilities to build, or even voting for a parliament (or other assembly of representatives). These examples have a number of common features:

1. There is a set of items[1] and a set of agents, where each agent has some intrinsic utility for each of the items (e.g., this utility can be the level of appreciation for a movie, the average number of articles one reads from a given issue of a journal, expected benefit from building a particular facility, the feeling—measured in some way—of being represented by a particular politician).

2. Typically, it is not possible to provide all the items to the agents and we can only pick some K of them, say (a plane's entertainment system can fit only a handful of movies, the library has a limited budget, only several sites for the facilities-to-be-built are available, the parliament has a fixed size).

[1] We use the term 'item' in the most neutral possible way. Items may be candidates running for an election or movies or possible facilities, and so on.

3. The intrinsic utilities for items extend to the sets of items in such a way that the impact of each selected item on the utility of an agent may depend on the *rank* of this item (from the agent's point of view) among the selected ones. Extreme examples include the case where each agent derives utility from his or her most preferred item only (e.g., an agent will watch his or her favorite movie only, will read / use the favorite journal / favorite facility only, will feel represented by the most appropriate politician only), from his or her least preferred item only (say, the agent worries that he or she will be somehow forced to stick to the worst item; e.g., he or she worries the family will force him or her to watch the worst available movie), or derives $1/K$ of the utility from each of the available items (e.g., the agent chooses the item—say, a movie—at random). However, in practice one should expect much more complicated schemes (e.g., an agent watches the top movie certainly, the second one probably, the third one perhaps, etc.; an agent is interested in having at least some T interesting journals in the library; an agent feels represented by some top T members of the parliament, etc.).

The goal of this paper is to formally define a model that captures all the above-described scenarios, provide a set of examples where the model is applicable, and to provide an initial set of computational results for it in terms of efficient algorithms (exact or approximate) and computational hardness results (NP-hardness and inapproximability results).

Our work builds upon, generalizes, and extends quite a number of settings that have already been studied in the literature. We provide a deeper overview of this research throughout the paper. Here we just mention the direct connection to the study of Chamberlin–Courant's voting rule [4,11,24] (and to so-called budgeted social choice [21]), and to some settings where agents enjoy equally the items they get (e.g., the Santa Claus problem [3] and the problem of designing optimal picking sequences [7,10,18]).

All the proofs omitted from this paper are available in the full technical report [27].

2 The Model

In this section we give a formal description of our model. However, before we move to the mathematical details, let us explain and justify some high-level assumptions and choices that we have made.

First, we assume that the agents have separable preferences. This means that the *intrinsic utility* of an object does not depend on what other objects are selected. This is very different from, for example, the case of combinatorial auctions. However, in our model the *impact* of an object on the global utility of an agent does depend on its rank (according to that agent) among the selected items. This distinction between the intrinsic value of an item and its value distorted by its rank are also considered in several other research fields, especially decision theory ("rank-dependent utility theory") and multicriteria decision making, from which we borrow one of the main ingredients of our approach, so-called *ordered weighted average (OWA) operators* [29] (see the following formal model; the reader might want to consult the work of Kacprzyk et al. [17] as well).[2]

Second, we assume that the agents' intrinsic utilities are provided explicitly in the input, as numerical values and that these values are comparable between agents (if one agent has twice as high a utility for some item than the other one, we take it to mean that this agent likes this item twice as much). Yet, we make no further assumptions about the nature of agents' utilities: they do not need to be normalized, they do not need to come from any

[2] Surprisingly, social choice has made very little use of OWAs; a recent exception is [16], that generalizes positional scoring voting rules by aggregating the scores of an alternative by taking into account the ranks of each score in the ordered list of scores obtained by this alternative from the votes.

particular range of values, etc. However, we often consider two special cases that can be seen as extreme ends of a spectrum of possibilities: so-called approval-based utilities (each agent's utility for each item is either 0 or 1), and so-called Borda utilities (each agent ranks the items from the most to the least desired one and an item's utility is the number of items that the agent likes less). In other words, in the approval-utilities case the agents only have extreme views regarding the items (they like them or not), and in the Borda-utilities case they have a full linear spectrum of appreciation of the items.

Third, we take the utilitarian view and measure the social welfare of the agents as the sum of their perceived utilities. Other choices are of course possible. In particular, it is tempting to define the social welfare for the group as the minimum of the utilities for all agents (an *egalitarian* model). We have considered this approach and obtained some results for it. However, although the model is worth studying, our computational results were uniformly negative (computational hardness and inapproximability results), and due to space limitations, we have kept these results out of the current paper.

2.1 The Formal Setting

Let $N = [n]$ be a set of n agents and let $A = \{a_1, \dots, a_m\}$ be a set of m items. The goal is to pick a set W of some K items (where K will be part of our input) that, in some sense, are most satisfying for the agents. To this end, (1) for each agent $i \in N$ and for each item $a_j \in A$, we have an intrinsic utility u_{i,a_j} that agent i derives from a_j; (2) the utility that each agent derives from a set of K items is an ordered weighted average [29] of this agent's intrinsic utilities for these items.

A weighted ordered average (OWA) over K numbers is a function defined through a vector $\alpha^{(K)} = \langle \alpha_1, \dots, \alpha_K \rangle$ of K (nonnegative) numbers[3] as follows: Let $\vec{x} = \langle x_1, \dots, x_K \rangle$ be a vector of K numbers and let $\vec{x}^{\downarrow} = \langle x_1^{\downarrow}, \dots, x_K^{\downarrow} \rangle$ be the nonincreasing rearrangement of $\vec{x}$, that is, $x_i^{\downarrow} = x_{\sigma(i)}$, where σ is any permutation of $\{1, \dots, K\}$ such that $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \dots \geq x_{\sigma(K)}$. Then we set:

$$\mathrm{OWA}_{\alpha^{(K)}}(\vec{x}) = \sum_{i=1}^{K} \alpha_i x_i^{\downarrow}$$

To make the notation lighter, we will write $\alpha^{(K)}(x_1, \dots, x_K)$, instead of $\mathrm{OWA}_{\alpha^{(K)}}(x_1, \dots, x_K)$.

We will provide a more detailed discussion of OWA operators useful in our context later; for the time being let us note that they can be used, for example, to express the arithmetic average (through the size-K vector $(\frac{1}{K}, \dots, \frac{1}{K})$), the maximum and minimum operators (through vectors $(1, 0, \dots, 0)$, and $(0, \dots, 0, 1)$, respectively) and the median (through the vector that has 0s everywhere, except for the middle position, where it has 1).

Given the above setup, we formalize our problem of computing "the most satisfying set of K items" in the following way.

Definition 1. *In the* OWA-Winner *problem we are given a set $N = [n]$ of agents with utilities over m items (alternatives) from the set $A = \{a_1, \dots, a_m\}$, a positive integer K ($K \leq m$), and a K-number OWA $\alpha^{(K)}$. The task is to compute a subset $W = \{w_1, \dots, w_K\}$ of A such that $u_{\mathrm{ut}}^{\alpha^{(K)}}(W) = \sum_{i=1}^{n} \alpha^{(K)}(u_{i,w_1}, \dots, u_{i,w_K})$ is maximized.*[4]

[3]The standard definition of OWAs assumes normalization, that is, $\sum_{i=1}^{K} \alpha_i = 1$. We do not make this assumption here, for the sake of convenience; note that whether OWA vectors are normalized or not is irrelevant to all notions and results of this paper.

[4]Formally, what we define here should be called a utilitarian OWA-Winner problem because we are interested in maximizing the sum of agents' utilities. As said before, it is also natural to consider egalitarian OWA-Winner problem, where we maximize the utility of the worst-off agent.

We sometimes treat the OWA-Winner problem as a decision problem (e.g., to speak of its NP-hardness). In this case, we assume that we are additionally given a value T and we ask if there is a subset W such that $u_{\mathrm{ut}}^{\alpha^{(K)}}(W) \geq T$.

For a family $(\alpha^{(K)})_{K=1}^{\infty}$ of OWAs, we write α-OWA-Winner to denote the variant of the OWA-Winner problem where, for a given solution size K, we use OWA $\alpha^{(K)}$. From now on we will not mention the size of the OWA vector explicitly and it will always be clear from context. We implicitly assume that OWAs in our families are polynomial-time computable.

Finally, we will often speak of variants of OWA-Winner where agents' utilities are somehow restricted. In particular, by approval-based utilities we mean that each agent's utilities come from the set $\{0, 1\}$, and by Borda-based utilities we mean the case where for each agent i the set of his or her utilities for all the items, that is, $\{u_{i,a_1}, \dots, u_{i,a_m}\}$ is equal to $\{0, \dots, m-1\}$. The term "Borda-based utilities" comes from the fact that such utilities translate to the scores under Borda voting rule (see, e.g., the text of Brams and Fishburn [8] for a discussion of voting procedures). Indeed, one of the most convenient ways to represent Borda-based utilities is through preference orders. For example, if for some agent i it holds that for each $a_j \in A$ we have $u_{i,a_j} = j - 1$, then we would say that this agent has preference order $a_m \succ a_{m-1} \succ \cdots \succ a_1$. That is, the agent puts the item with utility $m-1$ on the top position in the preference order, then the item with utility $m-2$, then the one with utility $m-3$, down to the item with utility 0. In fact, one way in which Borda-based utilities could arise is if the agents originally have the preference orders and not the utility values. One may derive the utilities using Borda's rule of assigning utility ℓ to an item if the agent prefers it to ℓ items (or, of course, one may use one of many other ways of deriving the utility values). Borda-based utilities are also used in several works on fair division (e.g., by Brams and King [9]).

Example 2. Let $n = 6$, $m = 6$, $K = 3$, $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $\alpha = (2, 1, 0)$, and Borda-based utilities derived from the following rankings:

$$\begin{aligned}
\texttt{3 agents: } & a_1 \succ a_2 \succ a_3 \succ a_5 \succ a_6 \succ a_4 \\
\texttt{2 agents: } & a_6 \succ a_1 \succ a_4 \succ a_3 \succ a_5 \succ a_2 \\
\texttt{1 agent: } & a_5 \succ a_4 \succ a_2 \succ a_3 \succ a_6 \succ a_1
\end{aligned}$$

Let us compute the score of $\{a_1, a_2, a_6\}$. The first three agents get utility $2 \times 5 + 4 = 14$ each, the next two get $2 \times 5 + 4 = 14$ each, and the last one gets $2 \times 3 + 1 = 7$. Therefore the score of $\{a_1, a_2, a_6\}$ is $42 + 28 + 7 = 77$. It can be checked that this is the optimal set. (The next best ones are $\{a_1, a_2, a_4\}$, $\{a_1, a_2, a_5\}$ and $\{a_1, a_5, a_6\}$, all with score 75.) Note that 3-Borda (in our terms, the rule defined by 3-best OWA, $\alpha' = (1, 1, 1)$) would choose $\{a_1, a_2, a_3\}$ and that Chamberlin and Courant's rule (in our terms, the rule defined by 1-best OWA, $\alpha'' = (1, 0, 0)$) would choose $\{a_1, a_5, a_6\}$.

2.2 A Dictionary of Useful OWA Families

OWA-Winner is a remarkably general problem and we will usually focus on some special cases, for particular families of OWAs. Below we give a catalog of particularly useful OWA families (in the description below we take K to be the dimension of the vectors to which we apply a given OWA).

1. ***k*-median OWA.** For each $k \in \{1, \dots, K\}$, k-med$^{(K)}$ is the OWA defined by the vector of $k-1$ zeros, followed by a single one, followed by $K-k$ zeros. It is easy to see that k-med$^{(K)}(x_1, \dots, x_K)$ is the k-th largest number in the set $\{x_1, \dots, x_K\}$ and is known as

the k-median of $\vec{x}$. In particular, 1-med$^{(K)}(\vec{x})$ is the maximum operator, K-med$^{(K)}(\vec{x})$ is the minimum operator, and if K is odd, $\frac{K+1}{2}$-med$^{(K)}(\vec{x})$ is the median operator.

2. **k-best OWA.** For each $k \in \{1, \dots, K\}$, k-best$^{(K)}$ OWA is defined through the vector of k ones followed by $K-k$ zeros. That is, k-best$^{(K)}(\vec{x})$ is the sum of the top k values in $\vec{x}$ (with appropriate scaling, this means an arithmetic average of the top k numbers). K-best$_K^{(K)}$ is simply the sum of all the numbers in $\vec{x}$ (after scaling, the arithmetic average).
3. **Arithmetic progression OWA.** This OWA is defined through the vector aprog[a]$^{(K)} = \langle a + (K-1)b, a + (K-2)b, \dots, a \rangle$, where $a \geq 0$ and $b > 0$. (One can easily check that the choice of b has no impact on the outcome of OWA-Winner; this is not the case for a, though.)
4. **Geometric progression OWA.** This OWA is defined through the vector gprog$[p]^{(K)} = \langle p^{K-1}, p^{K-2}, \dots, 1 \rangle$, where $p > 1$. (This is without loss of generality, because multiplying the vector by a constant factor has no impact on the outcome of OWA-Winner; but the choice of p matters.)
5. **Hurwicz OWA.** This OWA is defined through a vector $(\lambda, 0, \dots, 0, 1-\lambda)$, where λ, $0 \leq \lambda \leq 1$, is a parameter.

Naturally, all sorts of middle-ground OWAs are possible between these particular cases, and can be tailored for a specific application. As our natural assumption is that highly ranked items have more impact than lower-ranked objects, we often make the assumption that OWA vectors are *nonincreasing*, that is, $\alpha_1 \geq \ldots \geq \alpha_K$. While most OWA operators we consider in the paper are indeed nonincreasing, this is not the case for k-medians (except for 1-median) and Hurwicz (except for $\lambda = 1$).

3 Applications of the Model

We give here four different scenarios where our model is applicable. The first one is a generalization of Chamberlin and Courant's proportional representation rule. The common feature of the latter three scenarios is that they each focus on some form of uncertainty about the final outcome; the impact of a selected item is the probability that this item will be actually used by the agent.

3.1 Generalizing Chamberlin–Courant's Voting Rule

Our research started as an attempt to generalize the Chamberlin–Courant's voting rule for electing sets of representatives [11]. For this rule, voters (agents) have Borda utilities over a set of candidates and we wish to elect a K-member committee (for instance, a parliament), such that each voter is represented by one member of the committee. In other words, if we select K candidates, then a voter is "represented" by the selected candidate that he or she ranks highest among the chosen ones. The idea is that then, in the parliament, each selected candidate would have voting power proportional to the number of voters that he or she represents. It is easy to see that winner determination under Chamberlin–Courant's voting rule corresponds exactly to solving 1-best-OWA-Winner for the case of Borda utilities.

3.2 Malfunctioning Items or Unavailable Candidates

In this model, we assume that, as in classical group recommendation setting and budgeted social choice setting [21], each user only benefits from one item, but that the items may not be working properly: if we select (off-line) a set of items S, then (on-line) there will be a subset

S^+ of items that can be used, and a set $S^- = S \setminus S^+$ of objects that are 'malfunctioning' or are 'unavailable' and cannot be used. For instance, items are radio channels that can be unreachable, or items are candidates running in an election and these candidates may finally decide to not take a position in the elected committee, or items are parking lots that are to be built but that can sometimes be full (see [20] for further examples of social choice with possibly unavailable candidates). Moreover, we have a prior probability distribution about the (un)availability of items: as in [20], we assume that each item is available with probability p (i.i.d.). The utility an agent gets from a set of selected items S is the value of the best available object in S, that is, of the best object in S^+. Therefore, it is the value of the item ranked in position i in S if the first $i-1$ items are unavailable and the ith item is available. The expected contribution of an item to the utility of a user is therefore proportional to $p(1-p)^{i-1}$, which leads to the OWA defined by $\alpha_i = p(1-p)^{i-1}$, which is a geometric progression with initial value p and coefficient $1-p$.

3.3 Uncertainty About the Number of Items Enjoyed by a User

We assume now that there is some uncertainty about the number of items that a user will enjoy. A first possible reason is that users may have a limited capacity to enjoy items. For instance, items are movies or books and each user has a time constraint that will prevent him or her from enjoying all selected items. A second possible reason is that users are reluctant to use items they don't like enough: they will watch only the films whose value reaches a given subjective threshold. We give here two possible models for the choice of the OWA vectors:

- We first assume a uniform probability distribution on the number of items that a user enjoys. That is, a user will enjoy exactly his or her first i items in S with probability $P(i) = \frac{1}{K+1}$. Thus, she will enjoy the item ranked i if she enjoys at least i items, which occurs with probability $\frac{K-i+1}{K+1}$. This leads to the OWA vector defined by $\alpha_i = K-i+1$ (we disregard the normalizing constant), which is an arithmetic progression.

- Second, we assume that the values given by each user to each item are distributed uniformly, i.i.d., on $[0,1]$ and that each user uses only the items that have a value at least θ, where θ is a fixed (user-independent) threshold. Therefore, a user enjoys the item in S ranked in position i if he or she values at least i items at least θ, which occurs with probability $\sum_{j=i}^{K} \begin{pmatrix} K \\ i \end{pmatrix} (1-\theta)^i \theta^{K-i}$, thus leading to the following OWA vector defined by $\alpha_i = \sum_{j=i}^{K} \begin{pmatrix} K \\ i \end{pmatrix} (1-\theta)^i \theta^{K-i}$. For instance, if $K=4$ and $\theta = \frac{3}{4}$, the OWA (omitting the denominators) is $\alpha = (175, 67, 13, 1)$; for $K = 4$ and $\theta = \frac{1}{2}$, we get $\alpha = (14, 12, 5, 1)$; and for $K = 4$ and $\theta = \frac{1}{4}$, we get $\alpha = (252, 243, 189, 81)$.

3.4 Ignorance About Which Item Will Be Assigned to a User

We now assume that a matching mechanism will be used posterior to the selection of the K items. The matching mechanism used is not specified; it may also be a randomized mechanism.

If users have a *complete ignorance about the mechanism used*, then it makes sense to use known criteria for decision under complete uncertainty (see, e.g., the book of Luce and Raiffa [22]):

- the *Wald* criterion assumes that agents are extremely risk-averse, and corresponds to $\alpha = K\text{-med}^{(K)}$: we, therefore, seek to maximize $\sum_{i=1}^{n} \min_{w_j \in W} u_{i,w_j}$.

- the *Hurwicz* criterion is a linear combination between the worst and the best outcomes, and corresponds to $\alpha = (\lambda, 0, \dots, 0, 1-\lambda)$ for some fixed $\lambda \in (0, 1)$.

If users still have a complete ignorance about the mechanism used *except that they know that they are guaranteed to get one of their best i items*, then the Wald and Hurwicz criteria now lead, respectively, to the OWAs $\alpha = i\text{-med}^{(K)}$ and $\alpha = (\lambda, 0, \dots, 0, 1-\lambda, 0, \dots, 0)$, with $1-\lambda$ in position i.

If users know that the mechanism used is a random mechanism with a uniform distribution among the items ranked in positions 1 to i, then the choice of i-best OWA makes sense. More generally, the matching mechanism may assign items to agents with a probability that depends on the rank and that decreases when the rank increases.

4 Computational Results

We start our analysis by discussing worst-case results in Section 4.1. Then we move on to approximability results, in Section 4.2 for the case of general utilities (but with some focus on approval-based ones) and in Section 4.3 for the case of Borda-based ones. It turns out that while in general the problem is NP-hard and good approximation algorithms are rare, for the case of Borda-based utilities it is possible to obtain polynomial-time approximation schemes (PTASes) for a relatively large, interesting family of OWAs.

4.1 Computing Exact Solutions

In general, it seems that OWA-Winner is a rather difficult problem. However, as long as we seek a size-K winner set where K is fixed constant, then the problem is in P.

Proposition 3. *For each fixed constant K (the size of the winner set),* OWA-Winner *is in* P.

By results of Procaccia, Rosenschein and Zohar [24] and Lu and Boutilier [21], we know that the 1-best-OWA-Winner problem is NP-hard both for approval and for Borda-based utilities (in this case the problem is equivalent to winner determination under appropriate variants of Chamberlin–Courant voting rule; in effect, many results regarding the complexity of this rule are applicable for this variant of the problem [4,26,28,30]). A simple reduction shows that this result carries over to each family of k-best OWAs and of k-med OWAs, where k is a fixed positive integer.

Proposition 4. *For each fixed k,* k-best-OWA-Winner *and* k-med-OWA-Winner *are* NP*-complete, even if the utility profiles are restricted to be approval-based or Borda-based.*

On the other hand, it is immediate to see that K-best-OWA-Winner is in P.

Proposition 5. K-best-OWA-Winner *is in* P*.*

Indeed, if the agents' utilities are either approval-based or Borda-based, K-best-OWA-Winner boils down to (polynomial-time) winner determination for K-best approval rule and for K-Borda rule [12], respectively (see also the work of Elkind et al. [13] for a general discussion of multiwinner rules). Given this result, it is quite interesting that already $(K-1)$-best-OWA-Winner is NP-hard, both for approval-based and for Borda-based utilities. In our proof we give a reduction from the standard VertexCover problem (below we also define its version for cubic graphs, which will be useful a bit later).

Definition 6. *In the* VertexCover *problem we are given an undirected graph $G = (V, E)$, where $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$, and a positive integer K. We ask if there is a set C of up to K vertices such that each edge is incident to at least one vertex from C. The* CubicVertexCover *problem is identical to the standard* VertexCover *problem, except each vertex in the input graph is required to have degree exactly three.*

VertexCover is well-known to be NP-hard [15]; NP-hardness for CubicVertexCover was shown by Alimonti and Kann [1].

Theorem 7. $(K-1)$-best-OWA-Winner *is* NP-*complete even for approval-based utilities.*

Proof. Membership in NP is clear. We show a reduction from the VertexCover problem. Let I be an instance of VertexCover with graph $G = (V, E)$, where $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$, and with a positive integer K (without loss of generality, we assume that $K \geq 3$ and $K < m$).

We construct an instance I' of $(K-1)$-best-OWA-Winner in the following way. We let the set of items be $A = V$ and we form $2n$ agents, two for each edge. Specifically, if e_i is an edge connecting two vertices, call them $v_{i,1}$ and $v_{i,2}$, then we introduce two agents, e_i^1 and e_i^2, with the following utilities: e_i^1 has utility 1 for $v_{i,1}$ and for $v_{i,2}$, and has utility 0 for all the other items; e_i^2 has opposite utilities—it has utility 0 for $v_{i,1}$ and for $v_{i,2}$, and has utility 1 for all the remaining ones.

Let W be some set of K items (i.e., vertices) and consider the sum of the utilities derived by the two agents e_i^1 and e_i^2 from W under $(K-1)$-best-OWA. If neither $v_{i,1}$ nor $v_{i,2}$ belong to W, then the total utility of e_i^1 and e_i^2 is equal to $K-1$ (the former agent gets utility 0 and the latter one gets $K-1$). If only one of the items, i.e., either $v_{i,1}$ or $v_{i,2}$, belongs to W, then the total utility of e_i^1 and e_i^2 is equal to K (the former agent gets utility 1 and the latter one still gets $K-1$). Finally, if both items $v_{i,1}, v_{i,2}$ belong to W, then the total utility of e_i^1 and e_i^2 is also equal to K (the former gets utility 2 and the latter gets utility $K-2$). Thus the total utility of all agents is equal to $K \cdot n$ if and only if the answer to the instance I is "yes". This shows that the reduction is correct and, since the reduction is computable in polynomial time, the proof is complete. □

A variant of this result for Borda-based utilities follows by an application of a similar idea, but the restriction to Borda-based utilities requires a much more technical proof.

Theorem 8. $(K-1)$-best-OWA-Winner *is* NP-*hard even for Borda-based utilities.*

Using a proof that combines the ideas of the proof of Theorems 4 and 7, we show that indeed OWA-Winner is NP-hard for a large class of natural OWAs. This time, for the sake of simplicity, we give a proof for the approval-based utilities only.

Theorem 9. *Fix an OWA family α, such that there exists p, such that for every $\alpha^{(K)}$ we have $\alpha_p^{(K)} > \alpha_{p+1}^{(K)}$; α-*OWA-Winner *is* NP-*hard for approval-based utilities.*

Proof. We give a reduction from CubicVertexCover problem. Let I be an instance of CubicVertexCover with graph $G = (V, E)$, where $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$, and positive integer K. W.l.o.g., we assume that $n > 3$.

We construct an instance I' of α-OWA-Winner. In I' we set $N = E$ (the agents correspond to the edges), $A = V \cup \{b_1, b_2, \ldots b_{p-1}\}$ (there are $(p-1)$ dummy items; other items correspond to the vertices), and we seek a collection of items of size $K + p - 1$. Each agent e_i, $e_i \in E$, has utility 1 exactly for all the dummy items and for two vertices that e_i connects and for each of the dummy items (for the remaining items e_i has utility 0). In effect, each agent has utility 1 for exactly $p + 1$ items.

We claim that I is a yes-instance of CubicVertexCover if and only if there exists a solution for I' with the total utility at least $n\sum_{i=1}^{p}\alpha_i + (3K-n)\alpha_{p+1}$.

($\Rightarrow$) If there is a vertex cover C of size K for G, then by selecting the items $W = C \cup \{b_1, b_2, \ldots b_{p-1}\}$ we obtain the required utility of the agents. Indeed, for every agent e_i there are at least p items in W for which i gives value 1 (the $p-1$ dummy items and at least one vertex incident to e_i). These items contribute the value $n\sum_{i=1}^{p}\alpha_i$ to the total agents' utility. Additionally, since every non-dummy item has value 1 for exactly 3 agents, and since every agent has at most $(p+1)$ items with value 1, there are exactly $(3K-n)$ agents that have exactly $(p+1)$ items in W with values 1. These $(p+1)$'th additional utility-1 items of the $(3k-n)$ agents contribute $(3K-n)\alpha_{p+1}$ to the total utility. Altogether, the agents' utility is $n\sum_{i=1}^{p}\alpha_i + (3K-n)\alpha_{p+1}$, as claimed.

($\Leftarrow$) Let us assume that there is a set of $(K+p-1)$ items with total utility at least $n\sum_{i=1}^{p}\alpha_i + (3K-n)\alpha_{p+1}$. In I' we have $(p-1)$ items that have value 1 for each of the n agents, and every other item has value 1 for exactly 3 agents. Thus, the sum of the utilities of $K+p-1$ items (without applying the OWA operator yet) is at most $(p-1)n + 3K = pn + (3K-n)$. Thus, the total utility of the agents (now applying the OWA operator) is $n\sum_{i=1}^{p}\alpha_i + (3K-n)\alpha_{p+1}$ only if for each agent e_i the solution contains p items with utility 1. Since there are only $p-1$ dummy items, it meas that for each agent e_i there is a vertex v_j in the solution such that e_j is incident to v_j. That is, I is a yes-instance of CubicVertexCover. □

The above theorem applies directly, for example, to the families of geometric progression OWAs and arithmetic progression OWAs.

Corollary 10. *The problems* gprog[p]-OWA-Winner *(for any $p > 1$) and* aprog[a]-OWA-Winner *(for any $a > 0$) are NP-complete.*

In fact, the following theorem (whose proof builds upon the above constructions) shows an even stronger NP-hardness result.

Theorem 11. *Fix an OWA family α, such that for every K, $\alpha^{(K)}$ is nonincreasing and nonconstant; α-*OWA-Winner *is* NP*-hard for approval-based utilities.*

By the above discussion, we conjecture that the family of constant OWAs, that is, the family of K-best OWAs, is the only natural family for which α-OWA-Winner is in P. We leave this conjecture as a natural follow-up question.

Nonetheless, we still might be in a position where it is necessary to obtain an exact solution for a given α-OWA-Winner instance. In such a case, it might be possible to use an integer linear programming (ILP) formulation of the problem, given in the full version of this paper [27].

4.2 (In)Approximability Results: General Utilities and Approval Utilities

The OWA-Winner problem is particularly well-suited for applications that involve recommendation systems (see, e.g., the work of Lu and Boutilier [21] for a discussion of 1-best-OWA-Winner in this context). For recommendation systems it often suffices to find good approximate solutions instead of perfect, exact ones, especially if we only have estimates of agents' utilities. It turns out that the quality of the approximate solutions that we can produce for OWA-Winner very strongly depends on both the properties of the particular family of OWAs used and on the nature of agents' utilities.

First, we show that as long as our OWA is nonincreasing, a simple greedy algorithm achieves $\left(1-\frac{1}{e}\right)$ approximation ratio. This result follows by showing that for a nonincreasing

OWA α, the function u^{α}_{ut} (recall Definition 1) is submodular and nondecreasing, and by applying the famous result of Nemhauser et al. [23].

Recall that if A is some set and u is a function $u\colon 2^A \to \mathbb{R}_+$, then we say that: (1) u is submodular if for each W and W', $W \subseteq W' \subseteq A$, and each $a \in A \setminus W'$ it holds that $u(W \cup a) - u(W) \geq u(W' \cup a) - u(W')$, and (2) u is nondecreasing if for each $W \subseteq A$ and each $a \in A$ it holds that $u(W \cup \{a\}) \geq u(W)$.

Lemma 12. *Let I be an instance of* OWA-Winner *with a nonincreasing OWA α. The function u^{α}_{ut} is submodular and nondecreasing.*

Algorithm 1. Select K items greedily, each time choosing an item that increases the utility of the agents most (maximizes the function u^{α}_{ut}). For the pseudo-code we refer the reader to the full version [27].

Theorem 13. *For a nonincreasing OWA α, Algorithm 1 is a polynomial time $(1 - 1/e)$-approximation algorithm for the problem of finding the utilitarian set of K winners.*

Proof. The thesis follows from the results of Nemhauser et al. [23] on approximating nondecreasing submodular functions. □

Is a $(1 - \frac{1}{e})$-approximation algorithm a good result? Irrespective if one views it as sufficient or not, this is the best possible approximation ratio of a polynomial-time algorithm for (unrestricted) OWA-Winner with a nonincreasing OWA. The reason is that 1-best-OWA-Winner with approval-based utilities is, in essence, another name for the MaxCover problem, and if P $\neq$ NP, then $(1 - \frac{1}{e})$ is approximation upper bound for MaxCover [14].

It is quite plausible that there are no constant-factor approximation algorithms for many not-nonincreasing OWAs. As an example, let us consider the case of families of OWAs whose first entries are zero (but that, nonetheless, have a nonzero entry at a sufficiently early position). If there were a good approximation algorithm for winner determination under such OWAs, then there would be a good approximation algorithm for the Densest-K-Subgraph problem, which seems unlikely.

Definition 14. *In a* Densest-k-Subgraph *problem we are given an undirected graph $G = (V, E)$ and a positive integer K. We ask for a subgraph S with K vertices with the maximal number of edges.*

Theorem 15. *Fix some integer ℓ, $\ell \geq 2$. Let α be a family of OWAs such that each OWA in the family (for at least ℓ numbers) has 0s on positions 1 through $\ell - 1$, and has a nonzero value on the ℓ'th position. If there is a polynomial-time $x(n)$-approximation algorithm for α-*OWA-Winner *then there is a polynomial-time $x(n)$-approximation algorithm for the* Densest-k-Subgraph *problem.*

It seems that the Densest-k-Subgraph is not easy to approximate. Khot [19] ruled out the existence of a PTAS for the problem under standard complexity-theoretic assumptions. Bhaskara et al. [6] showed the polynomial integrality gap. Raghavendra and Steurer [25] and Alon et al. [2] proved that there is no polynomial-time constant approximation under non-standard assumptions. Finally, the best approximation algorithm for the problem that we know of, due to Bhaskara et al. [5], has approximation ratio $O(n^{1/4+\epsilon})$, where n is the number of vertices in the input graph.

As a further evidence that OWAs that are not-nonincreasing are particularly hard to deal with from the point of view of approximation algorithms, we show that for an extreme example of an OWA family, i.e., for the K-med OWAs, there is a very strong hardness-of-approximation result.

Theorem 16. *There exists a constant c such that there is no polynomial-time* $(2^{c\sqrt{\lg n}}/n)$*-approximation algorithm for* K-med-OWA-Winner *unless for some* ϵ *we have* 3-SAT $\in$ $DTIME(2^{n^{3/4+\epsilon}})$.

Based on the proof of the above theorem, we also obtain that Hurwicz[λ]-OWA-Winner is NP-hard. Interestingly, even though Hurwicz[λ] OWA is not nonincreasing, we do show an approximation algorithm for it with a constant approximation ratio.

Proposition 17. *(1) There is an algorithm that for* Hurwicz[λ]-OWA-Winner *with no restrictions on the utility functions achieves approximation ratio* $\lambda(1 - \frac{1}{e})$. *(2) For each positive* ϵ*, there is an algorithm that for* Hurwicz[λ]-OWA-Winner *for the case of Borda-based utilities achieves approximation ration* $\lambda(1 - \epsilon)$.

Returning to nonincreasing OWAs, we can even show an example of a PTAS for OWA-Winner for a certain family OWAs. However, to defeat the relation with the MaxCover problem, these OWAs need to be of a very special form: they need to be as similar to the K-best OWA as possible.

Theorem 18. *Consider a nonincreasing OWA* α, $\alpha = \langle \alpha_1, \dots, \alpha_K \rangle$. *Let* I *be an instance for* α-OWA-Winner *(where we seek a winner set of size* K*). An optimal solution for the same instance but with* K-best-*OWA is a* $(\sum_{i=1}^{K} \alpha_i)/(K\alpha_1)$*-approximate solution for* I.

As a consequence of this theorem, we immediately get the following result.

Theorem 19. *Let* $f : \mathbb{N} \rightarrow \mathbb{N}$ *be a function computable in polynomial-time with respect to the value of its argument, such that* $f(K)$ *is* $o(K)$. *There is a PTAS for* $(K - f(K))$-best-OWA-Winner.

Nonetheless, both Proposition 17 and Theorem 19, have a bitter-sweet taste. In essence, their proofs say that instead of using a particular OWA family (either Hurwicz[λ] or $(K - f(K))$-best OWA), we might as well use a different, simpler one (1-best OWA or K-best OWA).

Still, Theorem 19 is a very interesting result when contrasted with Theorem 15. Theorem 19 says that there is a PTAS for α-OWA-Winner for OWA family $\langle 1, \dots, 1, 0 \rangle$, whereas Theorem 15 suggests that it is unlikely that there is a constant-factor approximation algorithm for α-OWA-Winner with OWA family $\langle 0, 1, \dots, 1 \rangle$. Even though these two OWA families seem very similar, the fact that one is nonincreasing and the other one is not makes a huge difference in terms of approximability of our problem.

4.3 Polynomial Time Approximation Schemes: Borda Utilities

For the Borda-based utilities we can present much more positive results. Due to space restriction, we only outline our results here and point the reader to the full version of the paper for details.

Theorem 20. *Fix a positive integer* ℓ *and let* α *be a nonincreasing OWA where at most first* ℓ *entries are nonzero. Given an instance* I *of* α-OWA-Winner*, there exists a polynoial-time algorithm that computes a* $\left(1 - \frac{2\mathrm{W}(K/\ell)}{K/\ell}\right)$*-approximate solution for* I *in polynomial-time.*

Theorem 21. *Fix a value* ℓ *and let* α *be a family of nonincreasing OWAs that have nonzero values on top* ℓ *positions only. There is a PTAS for* α-OWA-Winner *for the case of Borda-based utilities.*

Using the above result, we can also obtain a PTAS for OWA-Winner for geometric progression OWAs, for the case of Borda utilities. This is quite a useful result: some of our scenarios from Section 3 yield OWAs of this form.

Corollary 22. *Fix a value* $p > 1$. *There is a PTAS for* gprog[p]-OWA-Winner *for the case of Borda-based utilities.*

Interestingly, Theorem 21 can be generalized to the case of arbitrary OWAs (not necessarily nonincreasing) that have nonzero entries among top ℓ positions only. We use the following lemma, which is a direct consequence of Theorem 20 (or of the algorithm for 1-best-OWA-Winner of Skowron et al. [28]).

Lemma 23. *Consider a set* N *of* n *agents and a set* A *of* m *items, where the agents rank the items from the most preferred ones to the least preferred ones. Let* K, p, *and* t *be some positive integers such that* $K \leq m$, $p \leq K$, *and* $t \leq p$. *Let* $x = \frac{\gamma}{p}m$. *There is a polynomial-time algorithm that finds a collection* C *of up to* K/p *items such that there are at least* $n\left(1 - \exp\left(-\frac{\gamma K}{p^2}\right)\right)$ *agents that each rank at least one member of* C *between positions* $(t-1)x+1$ *and* tx.

Theorem 24. *Fix a positive integer* ℓ *and let* α *be a family of OWAs that have nonzero entries on top* ℓ *positions only. There is a polynomial-time* $(1 - \frac{2\ell \mathrm{W}(K/\ell)}{K/\ell})$*-approximation algorithm for* α-OWA-Winner *for the case of Borda-based utilities.*

Proof. Consider an input instance I of α-OWA-Winner with the set $N = [n]$ of agents, with the set A of m items, and where we seek winner set of size K. Let $\alpha = \langle \alpha_1, \ldots, \alpha_\ell, 0, \ldots, 0\rangle$ be the OWA used in this instance. We set $x = \frac{\gamma}{\ell}m$.

Our algorithm proceeds in ℓ iterations. We set $N^{(0)} = N$ and $n^{(0)} = n$. In the i-th iteration, $1 \leq i \leq \ell$, the algorithm operates as follows: Using the algorithm from Lemma 23, for $p = \ell$, we find a set $A^{(i)}$ of up to K/ℓ items such that at least $n^{(i-1)}\left(1 - \exp\left(-\frac{\gamma K}{\ell^2}\right)\right)$ of the agents from the set $N^{(i-1)}$ each rank at least one of these items among positions $(i-1)x+1, \ldots, ix$ of their preference orders. We let $N^{(i)}$ be the set of these agents and we set $n(i) = \|N^{(i)}\|$. Finally, we set $W = \bigcup_{i=1}^{\ell} A^{(i)}$ and return W as the set of winners (it is easy to see that W contains at most K items; if K contains fewer than K items then we supplement it with $K - \|W\|$ arbitrarily chosen ones).

By the construction of our algorithm, each of the agents from the set $N^{(\ell)}$ ranks at least ℓ items from the set W on positions no worse than $\ell x = \gamma m$. Thus, each such an agent assigns to each such an item utility at least equal to $\beta u_{\max}$. Consequently, the total utility that the agents from the set N derive from the solution W is at least $n^{(\ell)}\left(\sum_{i=1}^{\ell}\alpha_i\right)\beta u_{\max}$. This is so, because for each i, $1 \leq i \leq \ell$, each of the agents in the set $N^{(\ell)}$ derives utility $\alpha_i \beta u_{\max}$ from the item that she ranks as i'th best among the items from W.

By construction of our algorithm, we have $n^{(\ell)} \geq n\left(1 - \exp\left(-\frac{\gamma K}{\ell^2}\right)\right)^{\ell} \geq n\left(1 - \ell\exp\left(-\frac{\gamma K}{\ell^2}\right)\right)$. Thus, the total utility obtained by the agents is at least: $u_{ut}^{\alpha}(W) \geq n\left(1 - \ell\exp\left(-\frac{\gamma K}{\ell^2}\right)\right)\left(\sum_{i=1}^{\ell}\alpha_i\right)\beta u_{\max}$. Now, since the maximum possible total utility of all the agents is upper-bounded by $n(\sum_{i=1}^{\ell}\alpha_i)u_{\max}$, we have that our algorithm has approximation ratio $\beta\left(1 - \ell\exp\left(-\frac{\gamma K}{\ell^2}\right)\right)$. It is clear that it runs in polynomial time, and so the proof is complete. □

Theorem 25. *Fix a value* ℓ *and let* α *be a family of OWAs that have nonzero values on top* ℓ *positions only. There is a PTAS for* α-OWA-Winner *for the case of Borda-based utilities.*

Table 1: Summary of our results for the OWA families from Section 2.2. For each OWA family we provide four entries: In the first row (for a given OWA family) we give its worst case complexity (in the general case and in the Borda utilities case), and in the second row we list the best known approximation result (in the general case and in the Borda utilities case). We write K to mean the cardinality of the winner set that we seek. In the "References" column we point to the respective result in the paper/literature (for negative results we indicate simplest type of utilities where it holds, for positive results the most general type of utilities where it holds). For approximation: DkS-bounded and MEBP-bounded mean, respectively, inapproximability results derived from the Densest-k-Subgraph problem and from the Maximum Edge Biclique Problem

OWA family	general utilities	Borda utilities	References
k-median (k fixed)	NP-hard DkS-bounded	NP-hard PTAS	Proposition 4 (approval and Borda) Theorems 15 (approval) and 25 (Borda)
K-median	NP-hard MEBP-bounded	NP-hard ?	Theorems 7 (approval) and 8 (Borda) Theorem 16 (approval), open (Borda)
1-best	NP-hard $(1-\frac{1}{e})$-approx.	NP-hard PTAS	Procaccia et al. [24], Lu and Boutilier [21] Lu and Boutilier [21], Skowron et al. [28]
k-best (k fixed)	NP-hard $(1-\frac{1}{e})$-approx.	NP-hard PTAS	Proposition 4 (approval and Borda) Theorems 13 (general) and 21 (Borda)
$(K-1)$-best	NP-hard PTAS	NP-hard PTAS	Theorems 7 (approval) and 8 (Borda) Theorem 18 (general)
K-best	P	P	folk result
arithmetic progression	NP-hard $(1-\frac{1}{e})$-approx.	? $(1-\frac{1}{e})$-approx.	Corollary 10 (approval), open (Borda) Theorem 13 (general)
geometric progression	NP-hard $(1-\frac{1}{e})$-approx.	? PTAS	Corollary 10 (approval), open (Borda) Thm 13 (general), Corollary 22 (Borda)
Hurwicz[λ]	NP-hard $\lambda(1-\frac{1}{e})$-approx.	? $\lambda(1-\epsilon)$-approx. for each $\epsilon > 0$	Corollary 17 (general and Borda)

5 Summary

Our contribution is twofold. First, we have proposed a new model for the selection of a collective sets of items; this model appears to be very general, encompasses several known frameworks, and can be applied to various domains such as committee elections, group recommendation, and beyond. Second, we have investigated the computational feasibility of the model, depending on the various assumptions we can make about the agents' utilities and the choice of the OWA vector. Table 1 gives a summary of our results. We note that many of these results directly related to the OWA families that appear in the settings from Section 3, that were our motivating force.

Our research leads to many open problems. In particular, one might want to strengthen our approximation algorithms, provide algorithms for more general cases, provide more inapproximability results. Among these problems, a particularly interesting one regards the approximability of OWA-Winner for the arithmetic progression family of OWAs. For this case, our set of results is very limited. In particular, can one provide a PTAS for arithmetic-progression OWAs under Borda-based utilities? Can one provide one for $\frac{K}{2}$-best OWAs/K-median OWAs? It is also interesting to see to what extent our approximation algorithms for the case of Borda utilities can be generalized to other settings (indeed, our most recent progress shows that such a generalization is possible and frutiful).

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