Some Results on *Adjusted Winner*

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Adjusted Winner

Adjusted winner (AW) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- www.nyu.edu/projects/adjustedwinner

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Item	Ann	Bob
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B	65	46
C	30	50
Total	100	100

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Suppose Ann and Bob are dividing three goods: A, B, and C.

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Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

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Notice that $65/46 \geq 5/4 \geq 1 \geq 30/50$

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Suppose Ann and Bob are dividing three goods: A, B, and C.

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Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$.

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yielding $p = 100/111 = 0.9009$

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Step 3. Equitability adjustment:

yielding $p = 100/111 = 0.9009$

Item	Ann	Bob
A	0	4
B	58.559	4.559
C	0	50
Total	58.559	58.559

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A **valuation** of these goods is a vector of natural numbers $\langle a_1, \ldots, a_n \rangle$ whose sum is 100.

Let $\alpha, \alpha', \alpha'', \ldots$ denote possible valuations for Ann and $\beta, \beta', \beta'', \ldots$ denote possible valuations for Bob.

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An **allocation** is a vector of n real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows.

For each $i = 1, \ldots, n$, s_i is the proportion of G_i given to Ann.

Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, "Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob."

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$V_A(\alpha, \sigma) = \sum_{i=1}^{n} a_i s_i$ is the total number of points that Ann receives.

$V_B(\beta, \sigma) = \sum_{i=1}^{n} b_i(1 - s_i)$ is the total number of points that Bob receives.

Thus AW can be viewed as a function from pairs of valuations to allocations: $\mathsf{AW}(\alpha, \beta) = \sigma$ if σ is the allocation produced by the AW algorithm.

Fairness

- **Proportional** if both Ann and Bob receive at least 50% of their valuation: $\sum_{i=1}^{n} s_i a_i \geq 50$ and $\sum_{i=1}^{n} (1 - s_i) b_i \geq 50$

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- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation:
$\sum_{i=1}^{n} s_1 a_i \geq \sum_{i=1}^{n}(1 - s_i)a_i$ and $\sum_{i=1}^{n}(1 - s_i)b_i \geq \sum_{i=1}^{n} s_i b_i$

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- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party: for each allocation $\sigma' = \langle s'_1, \ldots, s'_n \rangle$ if $\sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i$, then $\sum_{i=1}^{n}(1-s'_i)b_i < \sum_{i=1}^{n}(1-s_i)b_i$. (Similarly for Bob)

Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.

- AW only produces equitable allocations (equitability is essentially built in to the procedure).

- AW produces allocations σ that in which at most one good is split.

***Adjusted Winner* is Fair**

Theorem (Brams and Taylor) *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)*

Some Questions

- Can we make use of geometric intuitions?
- Is AW a “continuous" function?
- It seems that the more the agents’ utilities differ, the more points AW gives to each agent.
- The agents’ utility functions are assumed to be linear, what about non-linear utility functions?
- Can an agent benefit by making use of information about the other agent’s valuation?

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Item	Ann	Bob
G_1	$50 + \varepsilon/2$	$50 - \varepsilon/2$
G_2	$50 - \varepsilon/2$	$50 + \varepsilon/2$

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- The agents’ utility functions are assumed to be linear, what about non-linear utility functions? The nonlinear situation may be interesting.
- Can an agent benefit by making use of information about the other agent’s valuation? Yes, but in most cases it is not a “safe” strategy.

Conclusion and Future Work

- *AW* is an *algorithm* to "fairly" divide n goods among two people. We have studied a number of general properties about the corresponding function. (*Why does such an algorithm exist?*)

- A more detailed analysis of strategizing in *AW* (safe strategizing requires *perfect* knowledge: expected utility calculations).

- Can we make the discussion on nonlinear utilities *practical*?

Thank you.