Bicriteria Models for Fair Resource Allocation

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Resource Allocation Problem

- $I = \{1, 2, \ldots, m\}$ – set of services (agents)
- Q – feasible set of allocation decisions
- $\mathbf{x} \in Q$ – allocation pattern
- $y_i = f_i(\mathbf{x})$ – result (effect) of allocation $\mathbf{x}$ for service i

- Networking, e.g. bandwidth allocation, network dimensioning
- Results maximization (overall efficiency)
- Fairness

Fair (Equitable) Optimization

- Multicriteria Problem

$$\max\{(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q\}$$

strict monotonicity: $\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \Rightarrow \quad$ Pareto-optimization

- Equally important homogeneous outcomes:
$\mathbf{y} = (y_1, y_2, \ldots, y_m)$ – distribution of m individual outcomes:
Fairness 1: impartiality (anonymity)

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m)$$

- Fairness 2: equitability – strict preference of equitable transfer

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y}$$

Ordered Outcomes

- Ordered outcomes: $\langle . \rangle$ nonincreasing ordering
  $\langle \mathbf{a} \rangle = (a_{\langle 1 \rangle}, a_{\langle 2 \rangle}, \ldots, a_{\langle m \rangle})$ result of nonincreasing ordering of $\mathbf{a}$:
  $a_{\langle 1 \rangle} \le a_{\langle 2 \rangle} \le \ldots \le a_{\langle m \rangle}$ and $a_{\langle i \rangle} = a_{\tau(i)}$ for some permutation τ.

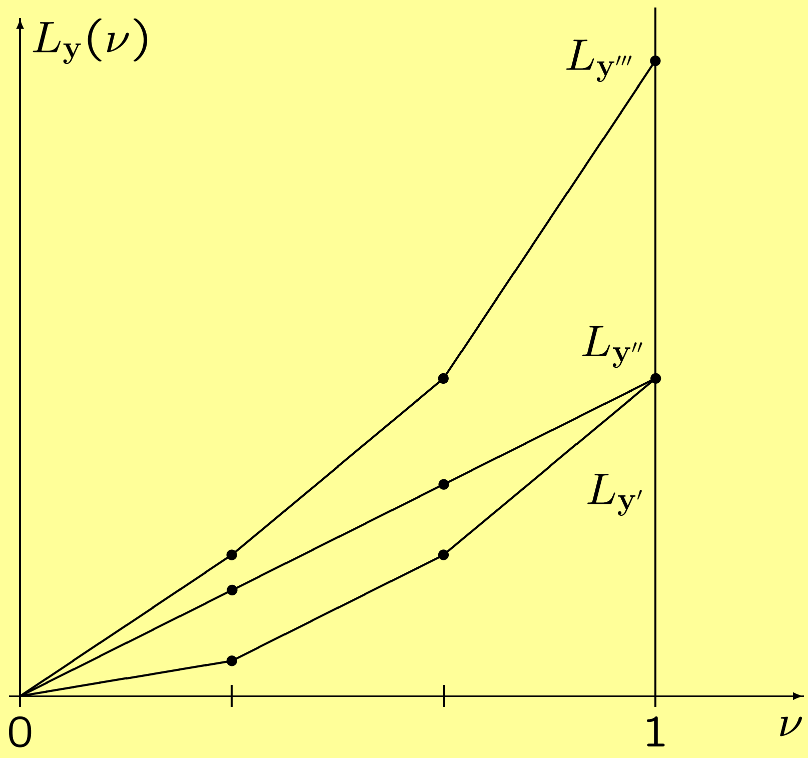
- Cumulated ordered outcomes: $\bar{\theta}_i(\mathbf{y}) = \dfrac{1}{m} \sum_{i=1}^{k} y_{\langle i \rangle}$
  Absolute Lorenz curve $L_{\mathbf{y}}(\frac{i}{m}) = \frac{1}{m}\bar{\theta}_i(\mathbf{y})$

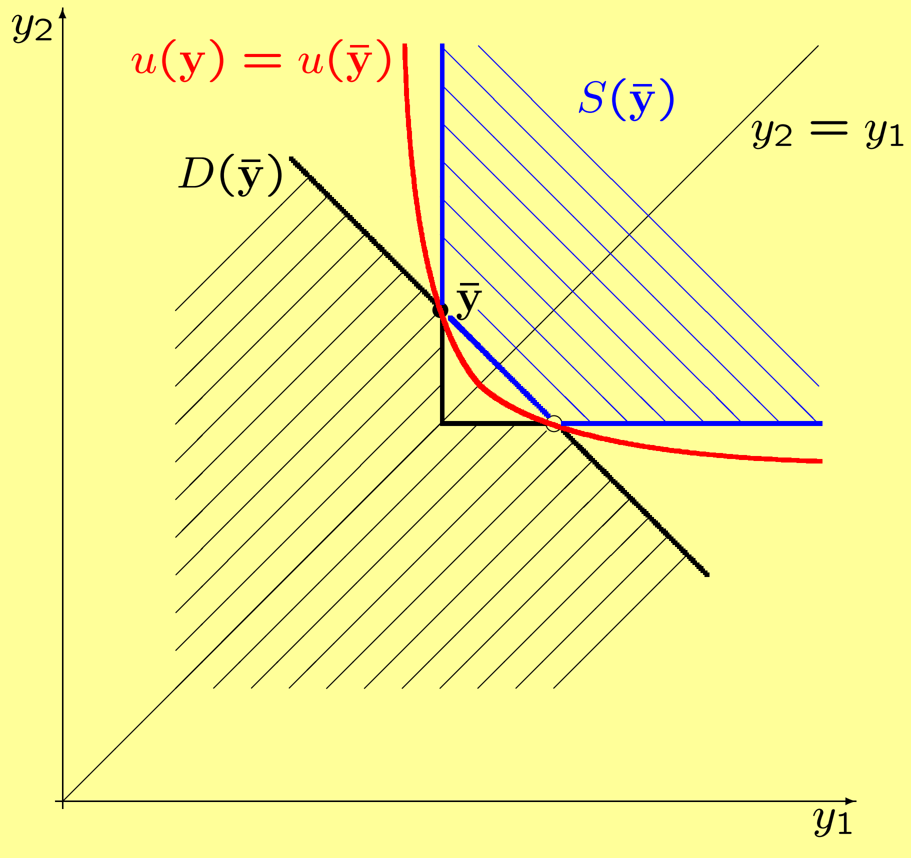
- Fair (equitable) dominance $\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Leftrightarrow \quad \bar{\theta}_i(\mathbf{y}') \ge \bar{\theta}_i(\mathbf{y}'')\ \forall i$

- Max-Min Fairness

$$\text{lexmax}\ \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$$

Fair (Equitable) Dominance





Bicriteria Mean-Equity Models

$$\max \{(\mu(\mathbf{f}(\mathbf{x})), -\varrho(\mathbf{f}(\mathbf{x}))) \ : \ \mathbf{x} \in Q\}$$

- Mean $\mu(\mathbf{y}) = \dfrac{1}{m}\sum_{i=1}^{m} y_i$,

  Inequality measure $\varrho(\mathbf{y})$: standard deviation, MAD, Gini's mean difference, semideviations, etc

- Modeling efficiency/equity trade-off

- Inequality measure minimization may result in inefficient decisions.

  E.g., allocating no resource to any service will provide perfectly equal (zero) effect to all the services (real communism).

Inequality Measures vs. Underachievements

- $M_\varrho(\mathbf{y}) = \mu(\mathbf{y}) - \varrho(\mathbf{y})$ underachievement measure (maximized)

- $\varrho(\mathbf{y})$ is mean-complementary fairly consistent $\Leftrightarrow$

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \varrho(\mathbf{y}'')$$

- $\varrho(\mathbf{y})$ is fairly α-consistent $\Leftrightarrow$ $\alpha\varrho(X)$ is mean-complementary fairly consistent

$$\mathbf{y}' \succeq_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \alpha\varrho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\varrho(\mathbf{y}'')$$

- strong consistency

$$\mathbf{y}' \succ_e \mathbf{y}'' \quad \Rightarrow \quad \mu(\mathbf{y}') - \varrho(\mathbf{y}') > \mu(\mathbf{y}'') - \varrho(\mathbf{y}'')$$

Fair Consistency

- If the inequality measure $\varrho(\mathbf{y})$ is fairly α-consistent, then except for allocation patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of the parametric problem

$$\max\{\mu(\mathbf{f}(\mathbf{x})),\ \mu(\mathbf{f}(\mathbf{x})) - \alpha\varrho(\mathbf{f}(\mathbf{x})) \ :\ \mathbf{x} \in Q\}$$

  is a fairly efficient solution.

- If the inequality measure $\varrho(\mathbf{y})$ is fairly α-consistent, then except for allocation patterns with identical values of $\mu(\mathbf{y})$ and $\varrho(\mathbf{y})$, every optimal solution of the parametric problem

$$\max\{\mu(\mathbf{f}(\mathbf{x})) - \lambda\varrho(\mathbf{f}(\mathbf{x})) \ :\ \mathbf{x} \in Q\}$$

  with $0 < \lambda < \alpha$ is a fairly efficient solution.

- In the case of strong α-consistency, every efficient/optimal solution is, unconditionally, fairly efficient.

Consistency Conditions

- Let $\varrho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure.
  If the measure is bounded by the maximum downside semideviation (is Δ-bounded)

  $$\alpha\varrho(\mathbf{y}) \leq \Delta(\mathbf{y}) = \max_{i=1,\ldots,m} (\mu(\mathbf{y}) - y_i) \quad \forall \mathbf{y},$$

  then $\varrho(\mathbf{y})$ is fairly α-consistent.

- If $\varrho(\mathbf{y})$ is also strictly Δ-bounded on unequal outcomes and strictly convex on identically distributed outcomes, then the consistency is strong.

Fair Consistency Results

Measure		α–consistency	
Mean absolute semideviation	$\bar{\delta}(\mathbf{y})$	1	
Mean absolute deviation	$\delta(\mathbf{y})$	0.5	
Maximum semideviation	$\Delta(\mathbf{y})$	1	
Maximum absolute deviation	$R(\mathbf{y})$	$1/(m-1)$	
Mean absolute difference	$\Gamma(\mathbf{y})$	1	strong
Maximum absolute difference	$d(\mathbf{y})$	$1/m$	
Standard semideviation	$\bar{\sigma}(\mathbf{y})$	1	
Standard deviation	$\sigma(\mathbf{y})$	$1/\sqrt{m-1}$	strong

- Convex combination of measures preserves mean-complementary fair consistency.

- One strongly consistent measure is enough for strong consistency of the entire combination.

Conclusions

- In order to comply with the maximization of outcomes as well as with an fair treatment of agents, the concept of equitable efficiency must be used for the multiple criteria model. Simplified mean-equity approaches, in general, may lead to inferior conclusions.

- Several inequality measures can be combined with the mean itself into fairly consistent underachievement measures. We have shown that properties of convexity and positive homogeneity together with boundedness by the maximum semideviation are sufficient for a typical absolute inequality measure to guarantee the corresponding fair consistency.

- Many of the inequality measures, we analyzed, can be implemented with auxiliary linear programming constraints.

Inequality measures

- Maximum absolute deviation $\Delta = \max_{i=1,\ldots,m} |y_i - \frac{1}{m}\sum_{j=1}^{m} y_j|$
- Mean absolute deviation: $\delta = \frac{1}{m}\sum_{i=1}^{m} |y_i - \frac{1}{m}\sum_{j=1}^{m} y_j|$
- Mean absolute difference: $D = \frac{1}{2m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} |y_i - y_j|$
- Variance: $\sigma^2 = \frac{1}{m}\sum_{i=1}^{m}(y_i - \frac{1}{m}\sum_{j=1}^{m} y_j)^2$
- Relative inequality measures: $\varrho(\alpha\mathbf{y}) = \varrho(\mathbf{y})$ dla $\alpha > 0$

Conditional means – computational models

- LP for a given vector $\mathbf{y}$ while nonlinear for variable $\mathbf{y}$:

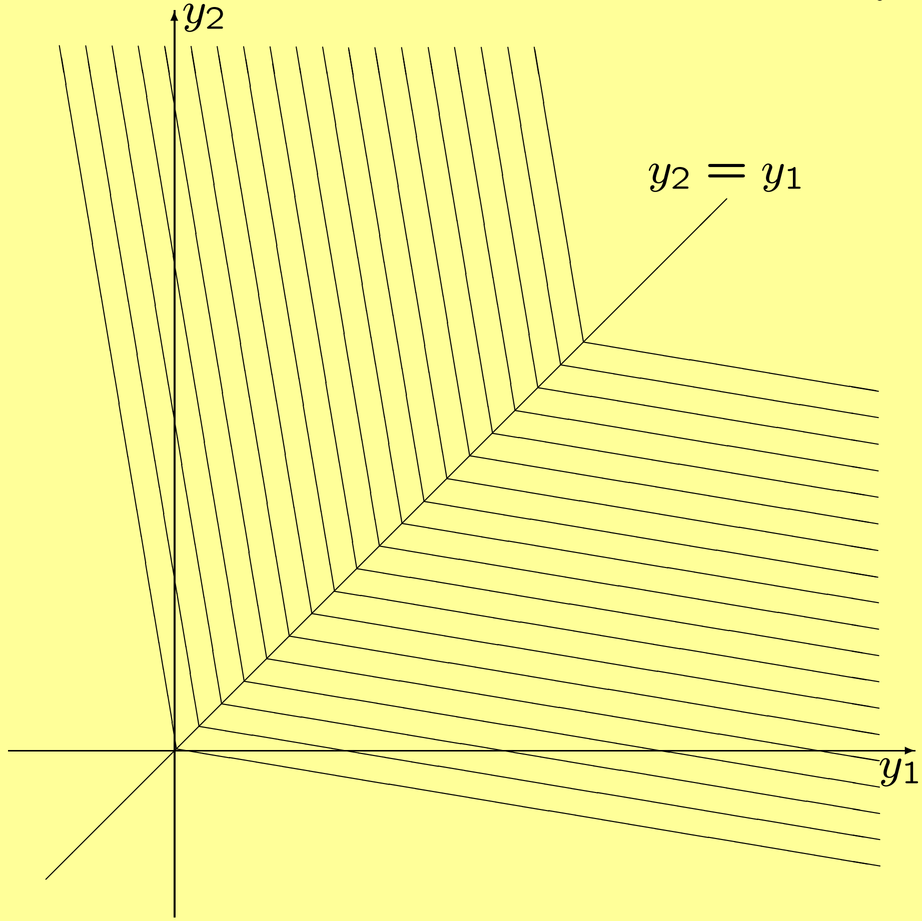
$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad = \quad & \text{min} \ \sum_{i=1}^{m} y_i u_{ki} \\ \text{p.w.} \ & \sum_{i=1}^{m} u_{ki} = k, \ \ 0 \le u_{ki} \le 1 \qquad \forall \ i \end{aligned}$$

- By taking advantage of the LP duality:

$$\begin{aligned} \bar{\theta}_k(\mathbf{y}) \quad = \quad & \text{max} \ \left(k r_k - \sum_{i=1}^{m} d_{ki} \right) \\ \text{p.w.} \ & r_k - y_i \le d_{ki}, \ \ d_{ki} \ge 0 \qquad \forall \ i \end{aligned}$$

LP even for variable $\mathbf{y}$

Ordered Weighted Average OWA: $a_w(\mathbf{y}) = \sum_{i=1}^{m} w_i y_{\langle i \rangle},\ w_i > 0$



Equitable OWA – monotonic weights