Weak Monotonicity and Bayes-Nash Incentive Compatibility

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Introduction

Setting

- Set of agents: $N = \{1, \dots, n\}$
- Multi-dimensional type of agent i: $t^i \in T^i$ with $T^i \subseteq \mathbb{R}^k$
- Set of outcomes Γ
- Valuations $v(\alpha | t^i, t^{-i})$
- Types independently distributed
- T set of type profiles $t = (t^1, \dots, t^n)$
- Allocation rule: $f : T \mapsto \Gamma$

Introduction

Goal

- Characterize allocation rules for which there is a $P : T \mapsto \mathbb{R}^n$ such that (f, P) is Bayes-Nash incentive compatible.
- Can we extend weak monotonicity characterization for dominant strategy i.c. (Bikhchandani, Chatterji, Sen, Lavi, Mu'alem, Nisan, Sen (2006), Gui, Müller, Vohra (2004), Saks, Yu (2005) to Bayes-Nash i.c.?

Notation

Bayes-Nash Incentive Compatibility

- f is Bayes-Nash i.c. if $\exists P$ s.t. $\forall i \in N, \forall r^i, \tilde{r}^i \in T^i$

$$\begin{aligned} & E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid r^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid r^i, t^{-i}\right)\right] \\ \geq \quad & E_{-i}\left[P_i\left(r^i, t^{-i}\right) - P_i\left(\tilde{r}^i, t^{-i}\right)\right] \end{aligned}$$

- Implies weak monotonicity: $\forall i \in N, \forall r^i, \tilde{r}^i \in T^i$

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Network approach

Network

- $\forall i \in N$ complete directed graph T_f^i
- Node associated with each type
- Length of edge from $\tilde{r}^i$ to r^i (cost of manipulation):

$$l^i\left(\tilde{r}^i, r^i\right) =$$
$$E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid r^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid r^i, t^{-i}\right)\right]$$

- weak-monotonicity becomes no-negative 2-cycle:

$$l^i\left(\tilde{r}^i, r^i\right) + l^i\left(r^i, \tilde{r}^i\right) \geq 0$$

Network approach

Theorem

An allocation rule f is Bayes-Nash incentive compatible if and only if $\forall i \in N$, T_f^i has no negative cycle.

Proof

Similar to Rochet (1987), and infinite graph equivalent of "shortest path lengths exist if and only if no negative cycle".

Question

No negative 2-cycle (i.e., weak-monotonicity) if and only if no negative cycle?

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One-dimensional types

$T^i \subseteq \mathbb{R}$

- **Definition:** The costs of manipulation are decomposition monotone if $\forall \underline{r}^i, \bar{r}^i \in T^i$ and $\forall \alpha \in (0,1)$ we have

$$l^i\left(\underline{r}^i, \bar{r}^i\right) \geq l^i\left(\underline{r}^i, (1-\alpha)\underline{r}^i + \alpha\bar{r}^i\right) + l^i\left((1-\alpha)\underline{r}^i + \alpha\bar{r}^i, \bar{r}^i\right).$$

- **Theorem** If costs of manipulation are decomposition monotone, T^i convex, then f is Bayes-Nash i.c. if and only if for all $i \in N$, T^i_f has no negative 2-cycle. (Example: Myerson (1981) *"Optimal Auction Design"*)

Multi-Dimensional Types

Additional Assumption

- $T^i \subseteq \mathbb{R}^k$ convex
- Valuations linear w.r.t. own type: $\forall \gamma \in \Gamma$

$$v^i\left(\gamma \mid t^i, t^{-i}\right) = \alpha^i\left(\gamma \mid t^{-i}\right) + \beta^i\left(\gamma \mid t^{-i}\right) t^i$$

- $\alpha^i : \Gamma \times T^{-i} \mapsto \mathbb{R}$, $\beta^i : \Gamma \times T^{-i} \mapsto \mathbb{R}^k$
- Expected valuation: $E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right)\right]$

$$= \quad E_{-i}\left[\alpha^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] + E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] t^i$$

Lemma

If v^i is linear in the own type and f satisfies weak monotonicity then the costs of manipulation are decomposition monotone.

Multi-Dimensional Types

Potential function and Path independence

- $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ is vector field $T^i \mapsto \mathbb{R}^k$
- A vector field ψ: $T^i \mapsto \mathbb{R}^k$ has a potential function φ: $T^i \mapsto \mathbb{R}$ if for any smooth path A joining $\underline{t}^i, \bar{t}^i \in T^i$

$$\int_A \psi = \varphi\left(\bar{t}^i\right) - \varphi\left(\underline{t}^i\right).$$

- Equivalent: ψ is path-independent, that is for any closed path B

$$\int_B \psi = 0.$$

Multi-Dimensional Types

Theorem

Suppose that $\forall i \in N$, T^i is convex and that agents have valuation functions that are linear w.r.t. their own true types then: f is Bayes-Nash incentive compatible if and only if for all $i \in N$
(1) T^i_f has no negative 2-cycle and
(2) $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ is path independent.

Multi-Dimensional Types

Proof sketch

- Necessity of (2):
  No-negative cycle $\Rightarrow E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ cyclically monotone (Rockafellar 1966) $\Rightarrow$ is a selection of the sub-differential of a convex function (Rockafellar 1970) $\Rightarrow$ path-independence (Krishna & Maenner 2001).
- Sufficiency:
  Take a negative cycle. Decomposition monotonicity allows to bound edge lengths $l(s, r)$ from below by integrals. Path-independence shows that the resulting integral is equal to 0.
- **Remark:** Neither of (1) or (2) implies the other, in particular this means that only (1) is not sufficient for B.N.I.C.

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