Approximability of Dodgson's Rule

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What do we know about Dodgson?

Dodgson's is one of the most interesting rules but computationally demanding. Not all news are bad.

- Dodgson's Score is NP-complete (BTT, 1989),
- Dodgson's Winner is NP-hard (BTT, 1989),
- Dodgson's Winner is complete for parallel access to NP (HHR, 1997),
- Dodgson's Winner is FPT parameterized by the number of alternatives m (M-D, 2006) and by the Dodgson's score (Fellows, 2006).
- For fixed m, given a voting situation (succinct input) Dodgson's Winner can be computed in

$$\mathcal{O}(\ln n)$$

operations, where n is the number of voters (M-D, 2006) and in time

$$\mathcal{O}(\ln^2 n \cdot \ln n \cdot \ln \ln n).$$

Why do we want to approximate Dodgson?

Sometimes a small percent of mistakes does not matter or we just want to have a lower bound on the Dodgson's score. The following approximations are known:

- Tideman rule (Tideman, 1987),
- Dodgson Quick rule (McCabe-Dansted, 2006).

In this paper we investigate under which circumstances and how fast these rules converge to Dodgson when $n \to \infty$.

Basic Concepts

- A is a set of m alternatives, $\mathcal{N}$ is a set of n voters (agents).
- $\mathcal{L}(A)$ is the set of all linear orders on A.

$$|\mathcal{L}(A)| = m!$$

- $\mathcal{L}^n(A)$ is the set of all profiles on A (ordered n-tuples of linear orders)

$$|\mathcal{L}^n(A)| = (m!)^n.$$

- $\mathcal{S}^n(A)$ is the set of all voting situations on A (unordered n-tuples of linear orders)

$$|\mathcal{S}^n(A)| = \binom{n+m!-1}{n}.$$

- A voting situation from $\mathcal{S}^n(A)$ can be given by

$$(n_1, n_2, \ldots, n_{m!}),$$

where $n_1 + n_2 + \ldots + n_{m!} = n$.

Social Choice Rules

A family of mappings $F = \{F_n\}$, $n \in \mathbb{N}$,

$$F_n \colon \mathcal{L}(A)^n \to 2^A,$$

is called a social choice rule (SCR).

Having the canonical mapping

$$\mathcal{L}^n(A) \to \mathcal{S}^n(A),$$

in mind, sometimes a SCR is defined as a family of mappings

$$F_n \colon \mathcal{S}(A)^n \to 2^A.$$

(succinct input). This way we end up with anonymous rules only.

One voting situation may represent several profiles.

Main Probability Assumptions

- The IC (Impartial Culture):

  assumes $\mathcal{L}^n(A)$ to be a discrete probability space with the uniform distribution, i.e. all profiles are equally likely

  Under the IC all voters are independent.

- The IAC (Impartial Anonymous Culture):

  assumes $\mathcal{S}^n(A)$ to be a discrete probability space with the uniform distribution, i.e. all voting situations are equally likely

  The IAC implicitly assumes some dependency between voters. This distribution is slightly contagious.

Advantages

Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be a profile. By aP_ib, where $a, b \in A$, we denote that the i^{th} agent prefers a to b. We define

$$n_{xy} = \#\{i \mid xP_iy\}.$$

Many of the rules to determine the winner use scores made up from the numbers

$$\mathsf{adv}(a, b) = \max(0, n_{ab} - n_{ba})$$

which are called advantages, e.g. the Tideman score is defined as follows:

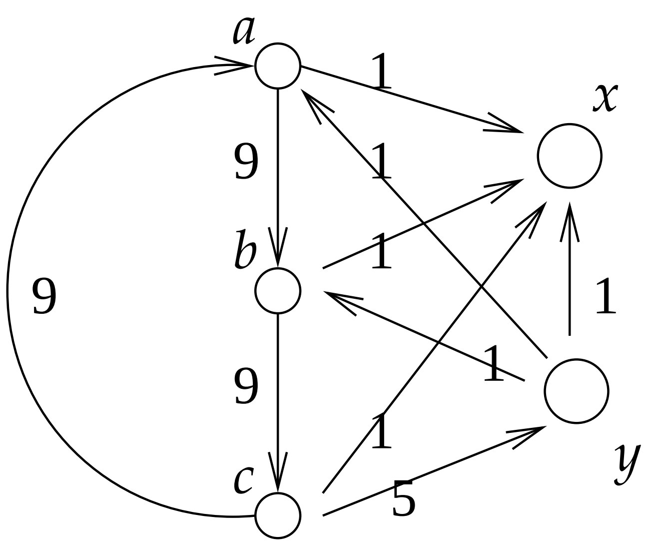
$$Sc_{\mathbf{t}}(a) = \sum_{b \neq a} \mathsf{adv}(b, a).$$

We also define the DQ-score

$$Sc_{\mathbf{d}}(a) = \sum_{b \neq a} \left\lceil \frac{\mathsf{adv}(b, a)}{2} \right\rceil.$$

Tideman and DQ are different

For $m = 5$ let us consider a profile with the following advantages. It exists by Debord's theorem.



Scores	a	b	c	x	y
Tideman	10	10	9	4	5
DQ	6	6	5	4	3

Here x is the sole Tideman winner, but y is the sole DQ-winner.

The IC results

For a profile $\mathcal{P} \in \mathcal{L}^n(A)$ let $W_D(\mathcal{P})$, $W_T(\mathcal{P})$, $W_{DQ}(\mathcal{P})$ be the set of Dodgson winners, Tideman winners and DQ-winners, respectively.

Theorem 1 (M-DPS, 2006) *When $m \geq 5$ is fixed and $n \longrightarrow \infty$*

$$\mathit{Prob}\,(W_T(\mathcal{P}) \neq W_D(\mathcal{P})) = \Theta\left(n^{-\frac{m!}{4}}\right)$$

for some $k > 0$.

Theorem 2 (M-DPS, 2006) *When m is fixed and $n \longrightarrow \infty$*

$$\mathit{Prob}\left(W_{DQ}(\mathcal{P}) \neq W_D(\mathcal{P})\right) = O(e^{-n}).$$

For odd n the DQ-approximation is a much better one.

Average complexity (IC) partial result

Corollary 1 (M-DPS, 2006) *When* m *is fixed and* $n \longrightarrow \infty$*. Given the uniform distribution on the set of profiles* $\mathcal{L}(A)^n$*, there exists an algorithm that given a succinct input of a profile* $\mathcal{P}$ *computes the Dodgson's score of an alternative* a *with expected running time*

$$\mathcal{O}(\ln n),$$

i.e. logarithmic with respect to the number of agents.

Without fixing m the average case complexity of Dodgson remains unknown.

The IAC results

For a profile $\mathcal{P} \in \mathcal{L}^n(A)$ let $W_D(\mathcal{P})$, $W_T(\mathcal{P})$, $W_{DQ}(\mathcal{P})$ be the set of Dodgson winners, Tideman winners and DQ-winners, respectively.

Theorem 3 (M-D, 2006) *When $m \geq 4$ is fixed and $n \longrightarrow \infty$*

$$\textit{Prob}\,(W_T(\mathcal{P}) \neq W_D(\mathcal{P})) \rightarrow c_m > 0.$$

$$\textit{Prob}\,\big(W_{DQ}(\mathcal{P}) \neq W_D(\mathcal{P})\big) \rightarrow c_m > 0.$$

The constant c_m is miniscule for small m and is the same in both equations since

Theorem 4 (M-D, 2006) *When $n \longrightarrow \infty$ and $m = o(n)$, then*

$$\textit{Prob}\,\big(W_{DQ}(\mathcal{P}) \neq W_T(\mathcal{P})\big) = \mathcal{O}(n^{-1}).$$

Experimental results

Number of occurrences per 1,000 Elections with 5 alternatives that the Dodgson Winner was not chosen

The IC results

Voters	3	5	7	9	15	17	25	85	257	1025
DQ	1.5	1.9	1.35	0.55	0.05	0.1	0	0	0	0
Tideman	1.5	2.3	2.7	3.95	6.05	6.85	7.95	8.2	5.9	2.95
Simpson	57.6	65.7	62.2	57.8	48.3	46.6	41.9	30.2	23.4	21.6

The IAC results

Voters	3	5	7	9	15	17	25	85	257	1025
DQ	1.30	2.11	1.55	0.91	0.20	0.13	0.02	0.00	0.00	0.00
Tideman	1.30	2.28	3.12	4.16	6.41	6.86	7.99	6.20	3.25	0.91
Simpson	55.9	63.3	60.7	56.3	46.5	43.9	38.2	25.3	20.5	17.9

Publications

- McCabe-Dansted, J. (2006) Approximability and Computational Feasibility of Dodgson's Rule. Master's thesis. The University of Auckland, 2006.

  http://dansted.org/thesis06/DanstedThesis06.pdf

- McCabe-Dansted, J., Pritchard, G., and Slinko, A. Approximability of Dodgson's rule, Report Series N.551. Department of Mathematics. The University of Auckland, June, 2006.

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- Oral communications by M. Fellows and J. McCabe-Dansted