MINIMAL MANIPULABILITY OF SOCIAL CHOICE FUNCTIONS

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This presentation is based on the following four papers:

1. Minimal manipulability: anonymity and surjectivity (2004)
2. Minimal manipulability: anonymity and unanimity (SCW, forthcoming)
3. Anonymous voting and minimal manipulability (JET, forthcoming)
4. Minimal manipulability: unanimity and nondictatorship (JME, forthcoming)

Related works:

Kelly (1988, 1989, 1993), Fristrup and Keiding (1998), Aleskerov and Kurbanov (1999), Slinko (2002)

Outline of the talk:

1. Basic model
2. Nondictatorial social choice functions
3. Nondictatorial and unanimous social choice functions
4. Nondictatorial and anonymous social choice functions
5. Nondictatorial unanimous anonymous social choice functions
6. Nondictatorial tops-only and anonymous social choice functions
7. Other related work and concluding remarks

1. Basic model

A, $|A| =: m \geq 3$: finite set of *alternatives*.

N, $|N| =: n \geq 2$: finite set of *agents*.

P : Set of all linear orders on A, (strict) *preferences*.

P^N : set of all preference *profiles*.

A *social choice function* is a surjective map $f : P^N \to A$.

The social choice function f is *dictatorial* if there is an agent $d \in N$ such that $f(p)$ is the top of $p(d)$ for every $p \in P^N$.

A profile $p \in P^N$ is *manipulable*, given f, if there is an $i \in N$ and a $t \in P$ such that $f(p_{-i}, t) \neq f(p)$ and $f(p_{-i}, t)\, p(i)\, f(p)$.

$M_f := \{p \in P^N \mid f \text{ manipulable at } p\}$: Set of profiles at which f is manipulable.

$m_f := |M_f|$.

Theorem (Gibbard-Sattertwaite) Let f be nondictatorial. Then $m_f > 0$.

Main question What is the minimum of m_f for nondictatorial f in a certain class, and which f attain the minimum?

2. Nondictatorial social choice functions

(1) $n = 2$, $m = 3$

Kelly (1988,1989): $m_f \geq 2$, and the lower bound 2 is reached by "nice" social choice functions, e.g., anonymous and Pareto optimal SCF:

$$
\begin{array}{c|cccccc}
 & xyz & xzy & yxz & yzx & zxy & zyx \\
\hline
xyz & x & x & y & y & x & y \\
xzy & x & x & y & y & \mathbf{z} & z \\
yxz & y & y & y & y & y & y \\
yzx & y & y & y & y & y & y \\
zxy & x & \mathbf{z} & y & y & z & z \\
zyx & y & z & y & y & z & z
\end{array}
$$

How many SCFs reach the lower bound? Answer: 135 (found by computer). (In total there are 3^{36} different SCFs!)

(2) $n = 2$, $m \geq 4$

Fristrup and Keiding (1998): $m_f \geq (n-1)\left(\frac{m!}{2} - 1\right) + 1$, and the lower bound is reached, among others, by the "almost dictatorial social choice functions".

The social choice function $f : P^N \longrightarrow A$ is *almost dictatorial* if there is an agent $d \in N$ and a profile $\tilde{p} \in P^N$ and an alternative $x \in A$ such that

(i) in $\tilde{p}$ all agents $i \neq d$ prefer x to the top alternative of d, i.e., the highest alternative, say y, in the preference $\tilde{p}(d)$;

(ii) f assigns x to $\tilde{p}$;

(iii) d is a dictator at every other profile $p \neq \tilde{p}$, i.e., $f(p)$ is the top alternative in the preference $p(d)$.

Observe that agent d can manipulate at $\tilde{p}$; each agent $i \neq d$ can manipulate at each profile $(\tilde{p}_{-i}, t)$ with $t \neq \tilde{p}(i)$ and $x\,t\,y$. Results, indeed, in $m_f \geq (n-1)\left(\frac{m!}{2} - 1\right) + 1$ manipulable profiles.

Also observe that for $n = 2$ and $m = 3$ almost dictatorial social choice functions have 3 manipulable profiles, and are thus not minimally manipulable.

(3) $n \geq 3$, $m = 3$

In this case, $m_f \geq n$, and the lower bound is reached by exactly 6 social choice functions m^t, namely one for each preference $t \in P$.

For $t = xyz$, m^t is defined as follows:

(i) If each agent has the preference $t = xyz$, then m^t assigns x;

(ii) otherwise, if each agent prefers y over z, then m^t assigns y;

(iii) in all other cases, m^t assigns z.

Observe that for each i, the unique profile at which i can manipulate is the profile p with $p(i) = xzy$ and $p(j) = xyz$ for all $j \neq i$. Hence, indeed, $|M_{m^t} = n|$.

Observe also that these social choice functions are anonymous. They are not Pareto optimal nor even unanimous.

A social choice function $f : P^N \to A$ is *unanimous* if for every profile p with $p(i) = p(j)$ for all $i, j \in N$, we have $f(p)$ is the top alternative of this common preference.

(4) $n \geq 3$, $m \geq 4$: Still open.

3. Nondictatorial and unanimous social choice functions

(1) $n = 2$, $m = 3$

Lower bound $m_f = 2$ (Kelly, 1988).

(2) $n = 2$, $m \geq 4$

Lower bound $m_f = (n-1)\left(\frac{m!}{2} - 1\right) + 1$ (Fristrup and Keiding, 1998): note that almost dictatorial social choice functions are unanimous.

(3) $n \geq 3$, $m \geq 3$

Lower bound $m_f = (n-1)\left(\frac{m!}{2} - 1\right) + 1$, attained by the almost dictatorial social choice functions and no others.

4. Nondictatorial and anonymous social choice functions

(1) $n = 2$, $m = 3$

Lower bound $m_f = 2$ (Kelly, 1988).

(2) $n \geq 3$, $m = 3$

Lower bound $m_f = n$, and this lower bound is reached by exactly 6 social choice functions m^t: recall that these are anonymous.

(3) $m \geq 4$ and $n \geq m + 1$ if $m = 4$, $n \geq m + 2$ if $m > 4$

Lower bound $m_f = n\left(\frac{m!}{3} - 1\right)(m - 2)$, reached uniquely by extensions of the m^t solutions.

This lower bound is higher than for the almost dictatorial solutions (which are not anonymous). This is in contrast to the case $m = 3$.

(4) All other cases

Still open.

5. Nondictatorial unanimous anonymous social choice functions

(1) $m = 3$

Lower bound $m_f = 2^n - 2$, reached by several different classes of social choice functions.

Requiring, additionally, Pareto optimality does not increase the lower bound. A typical social choice function in this class looks as follows:

(i) Choose a fixed alternative z if this is not Pareto dominated.

(ii) Otherwise, choose some other, Pareto optimal, outcome (e.g., according to some unanimous social choice function).

We conjecture that these are actually all minimally manipulable SCFs under the additional requirement of Pareto optimality.

(2) $m \geq 4$

Still open.

6. Nondictatorial tops-only and anonymous social choice functions

A social choice function f is *tops-only* if $f(p) = f(\tilde{p})$ whenever the top alternatives of $p(i)$ and $\tilde{p}(i)$ are equal for every $i \in N$.

(1) $n > m \geq 3$

Lower bound $m_f = \frac{1}{2} n\,(m-1)\,(m-2)\,\left((m-1)!\right)^n$.

The lower bound is reached uniquely by "unanimity rules with status quo".

A social choice function f is a *unanimity rule with status quo* of there is an alternative x (the *status quo*) that is always assigned unless every agent in a given profile has the same top alternative, say $y \neq x$: in that case y is assigned.

The number of manipulable profiles of such an f is equal to

$$n\,(m-1)\,\left(\frac{m!}{2} - (m-1)!\right)\left((m-1)!\right)^{n-1}$$

which is equal to the lower bound above.

(2) $n \leq m$

We have some results, namely for $n = 2$ and with Pareto optimality. Other cases still open.

7. Other related work and concluding remarks

(1) Other related work

- In Kelly (1993), a computer draws social choice functions uniformly from all social choice functions satisfying conditions like anonymity, neutrality and Pareto optimality, or combinations thereof. It turns out that imposing anonymity in the case of three alternatives causes a shift towards social choice functions with less manipulable profiles, compared to the sample obtained without any constraints.
- Aleskerov and Kurbanov (1999) contains simulation and enumeration results on 26 different social choice functions for different indices of manipulability.
- Slinko (2002) counts the number of instable profiles of classical social choice functions, which is an upper bound for the number of manipulable profiles of these social choice functions.

(2) Concluding remarks

- Most minimally manipulable social choice functions obtained in these works serve as benchmarks rather than as attractive social choice functions – although some of them can be observed in pratice, e.g., unanimity rules with status quo (EU).
- Most results were first obtained by running computer simulation and optimization programs, and only proved formally later.
- Future work in our view should focus on imposing additional restrictions next to minimal manipulability, but the question is whether this is analytically tractable.