Equal Representation in Two-tier Voting Systems

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Introduction

- History and efficiency considerations often call for *two-tier electoral systems*:
  1. People's preferences are aggregated in *constituencies*
  2. Constituencies' preferences are aggregated in an *electoral college*
- Question:
  How should constituencies' voting weights in the college be chosen s.t. a priori all individuals have identical influence?
- Allocating weights proportional to population sizes seems straightforward, but:
- In general, voting *power* is not linear in voting *weight,* e.g. EU Council of Ministers 1958.
- Power measures as the Penrose-Banzhaf- or the Shapley-Shubik-Index are designed to capture the non-trivial relationship between weight and power.

Penrose's square root rule

- *Penrose's square root rule (1946)*:
  Choose weights s.t. constituencies' Penrose-Banzhaf index is proportional to square root of population
- For most practical reasons (especially, if the number of constituencies is "large"), a simpler rule suffices:

  weight = sqrt(Population)

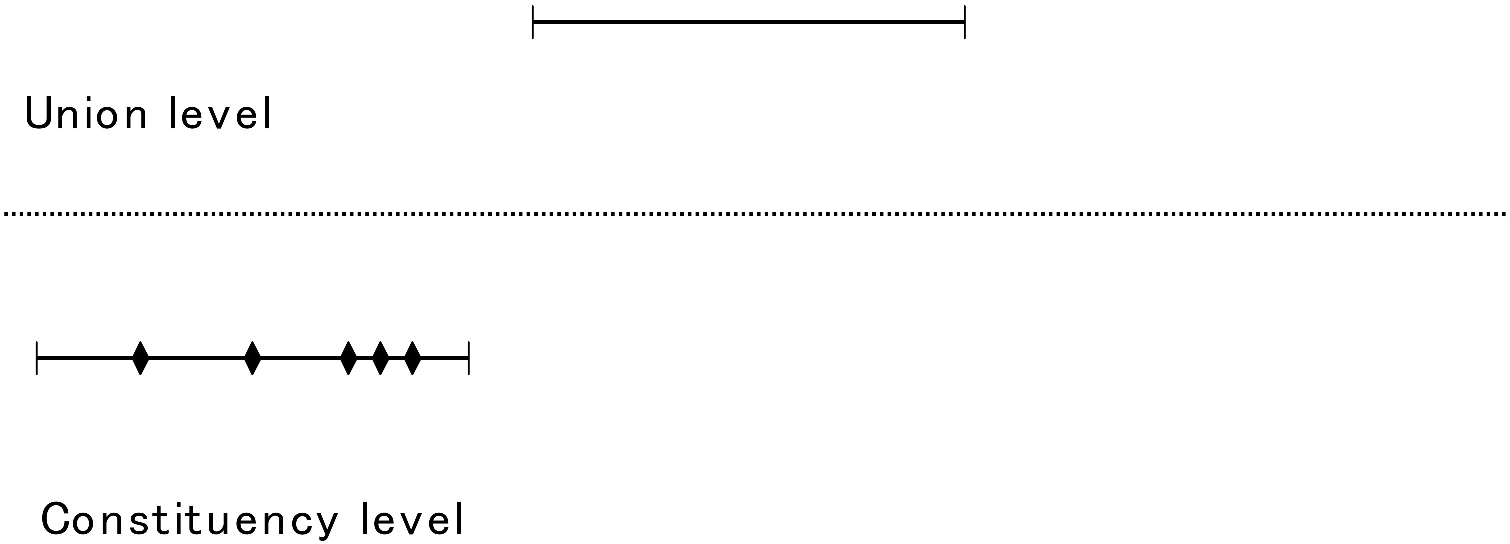
- The rule requires decisions $x \in \{0,1\}$ and (in expectation) equi-probable independent 0 or 1-votes

- What if the world is *not* dichotomous but, e.g., $x \in [0,1]$?

Outline

- Model
- Analytical problems
- Monte Carlo simulation
- Results
- Concluding remarks

Model

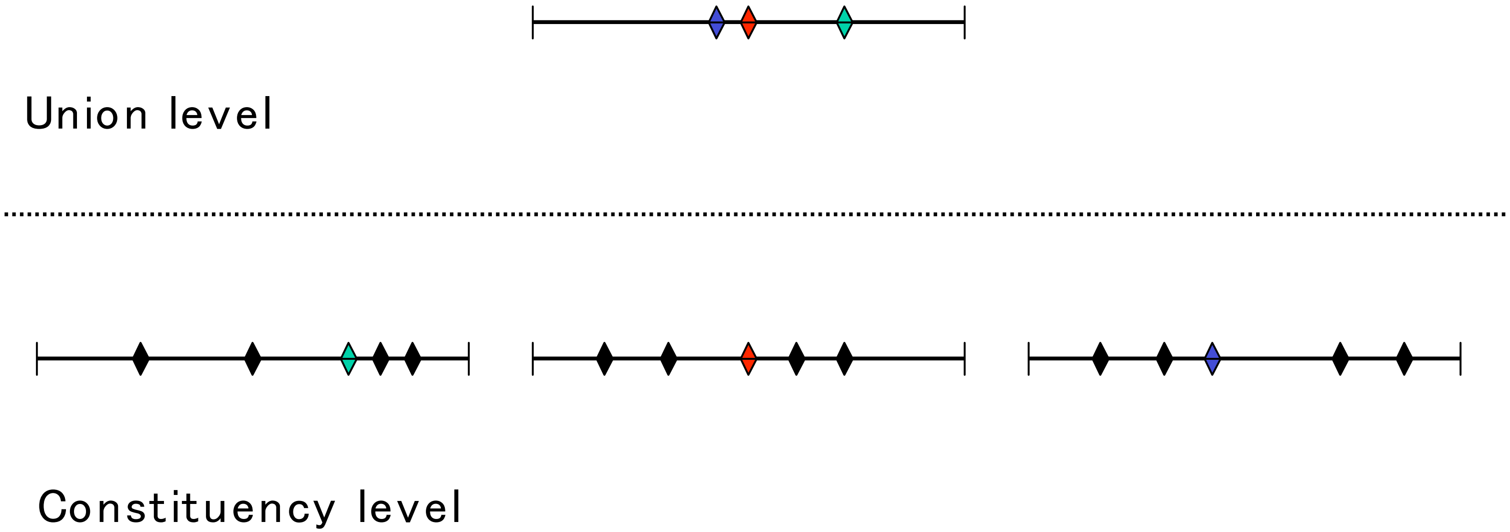


Model

Union level

Constituency level

Model



Model

- Voters are partitioned into *m* constituencies and have single-peaked preferences with a priori uniformly distributed ideal points $\lambda \in X \equiv [0,1]$
- Constituency *j*'s representative is chosen to match the *median voter* in his constituency
- Each constituency *j* has weight w_j in the electoral college; a 50%-quota *q* is used
- *Pivotal constituency* (*P*) is defined by $P \equiv \min\{r : \sum_{k=1}^{r} w_{(k)} > q\}$ [permutation (·) orders constituencies from left to right]
- (*P*) gets its will, i.e. $x^* = \lambda_{(P)}$

➢ *Problem of equal representation:*
Given population sizes $n_1, \ldots, n_m$, find weights $w_1, \ldots, w_n$ s.t. each voter has equal chance of determining x^*

First analysis

- Each voter in constituency j has chance $1/n_j$ to be its median
  $\Rightarrow \Pr(\lambda_j = \lambda_{P:m}) \overset{!}{=} c \cdot n_j$ for all j (with $c>0$)
- Assuming *i.i.d. voters*, different n_j imply different a priori distributions of medians
- With density f and c.d.f. F for individual voters' ideal points, representatives' ideal points are asymptotically normal with

$$\mu_j = F^{-1}(0.5), \qquad \sigma_j = [2f(\mu_j) \cdot \mathrm{sqrt}(n_j)]^{-1}$$

$\Rightarrow$ Larger constituencies are a priori more central in the electoral college and more likely to be pivotal under a 50%-quota.

Analytical problems

- Already for *unweighted* voting, i.e. $P \equiv (m+1)/2$, we run into trouble:

$$\begin{aligned}\Pr(j = (p)) &= \Pr\Big(\text{exactly } p-1 \text{ of the } \lambda_k, k \neq j, \text{ satisfy } \lambda_k < \lambda_j\Big) \\ &= \int \sum_{\substack{S \subset N \setminus j, \\ |S| = p-1}} \prod_{k \in S} F_k(x) \cdot \prod_{k \in N \setminus j \setminus S} (1 - F_k(x)) \cdot f_j(x)\, dx\end{aligned}$$

- Asymptotic approximation with only n_1 varying and $n_2 = \ldots = n_m$ seems possible, but for general $n_1, \ldots, n_m$?

Monte Carlo simulation

- Probability $\pi_j := \Pr(j = (P))$ is the expected value of random variable $H_j(\lambda_1, \ldots, \lambda_m)$ which is 1 if $j = (P)$ and 0 otherwise
- H_j's expected value can be approximated by the *empirical average* of many independent draws of H_j
- Weight vectors are constructed from given population sizes by

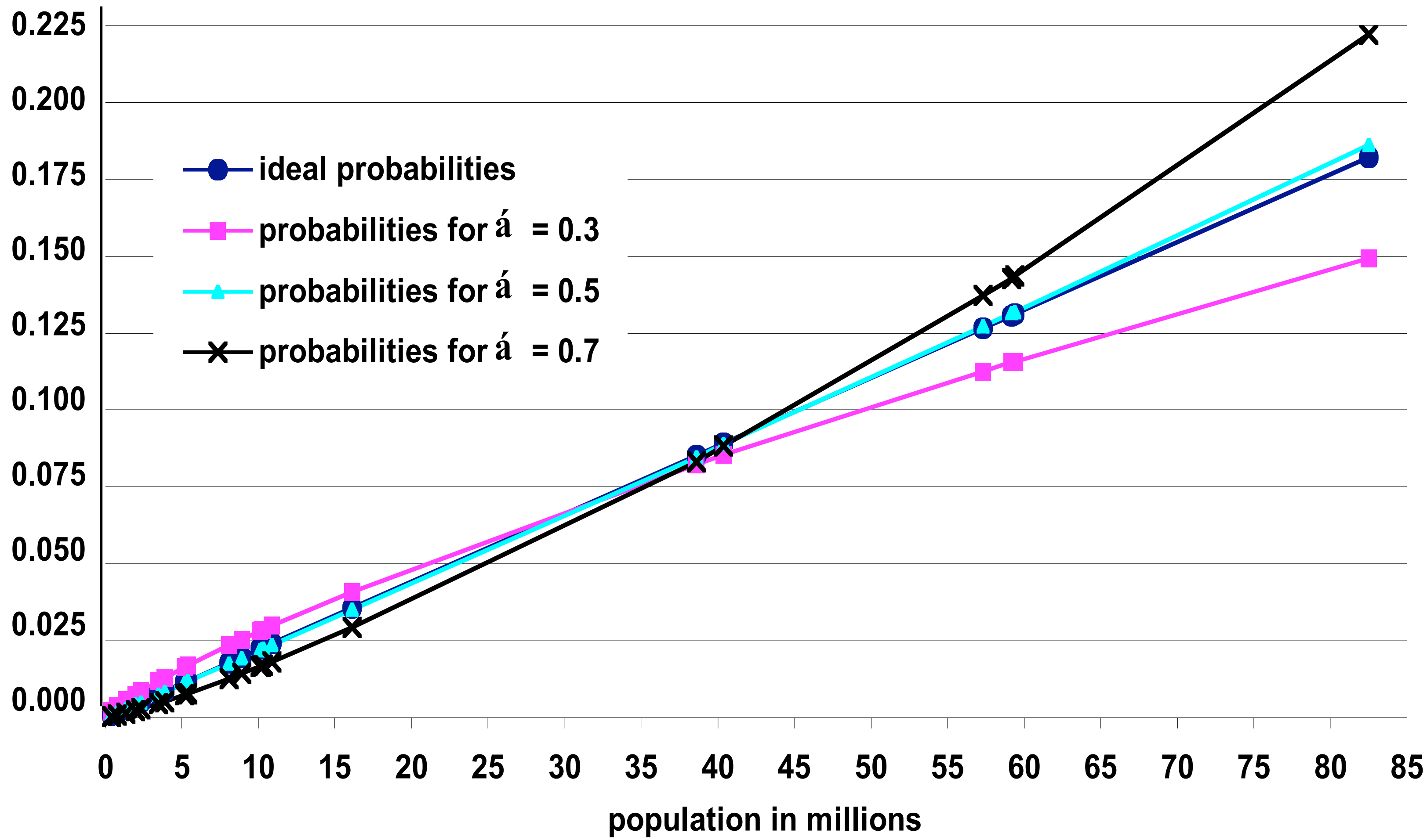
$$w_j = n_j^{\alpha}$$

- For fixed weights $(w_1, \ldots, w_m)$ and populations $(n_1, \ldots, n_m)$, we draw $\lambda_1, \ldots, \lambda_m$ from the beta distributions corresponding to i.i.d. $U[0,1]$ voters in all constituencies and average $H_1, \ldots, H_m$ over 10 million draws

➢ We search for the α which yields smallest cumulative (individual) quadratic deviation of π_j from the ideal egalitarian probability $\pi_j^* = n_j / \sum n_k$ $(j = 1, \ldots, m)$

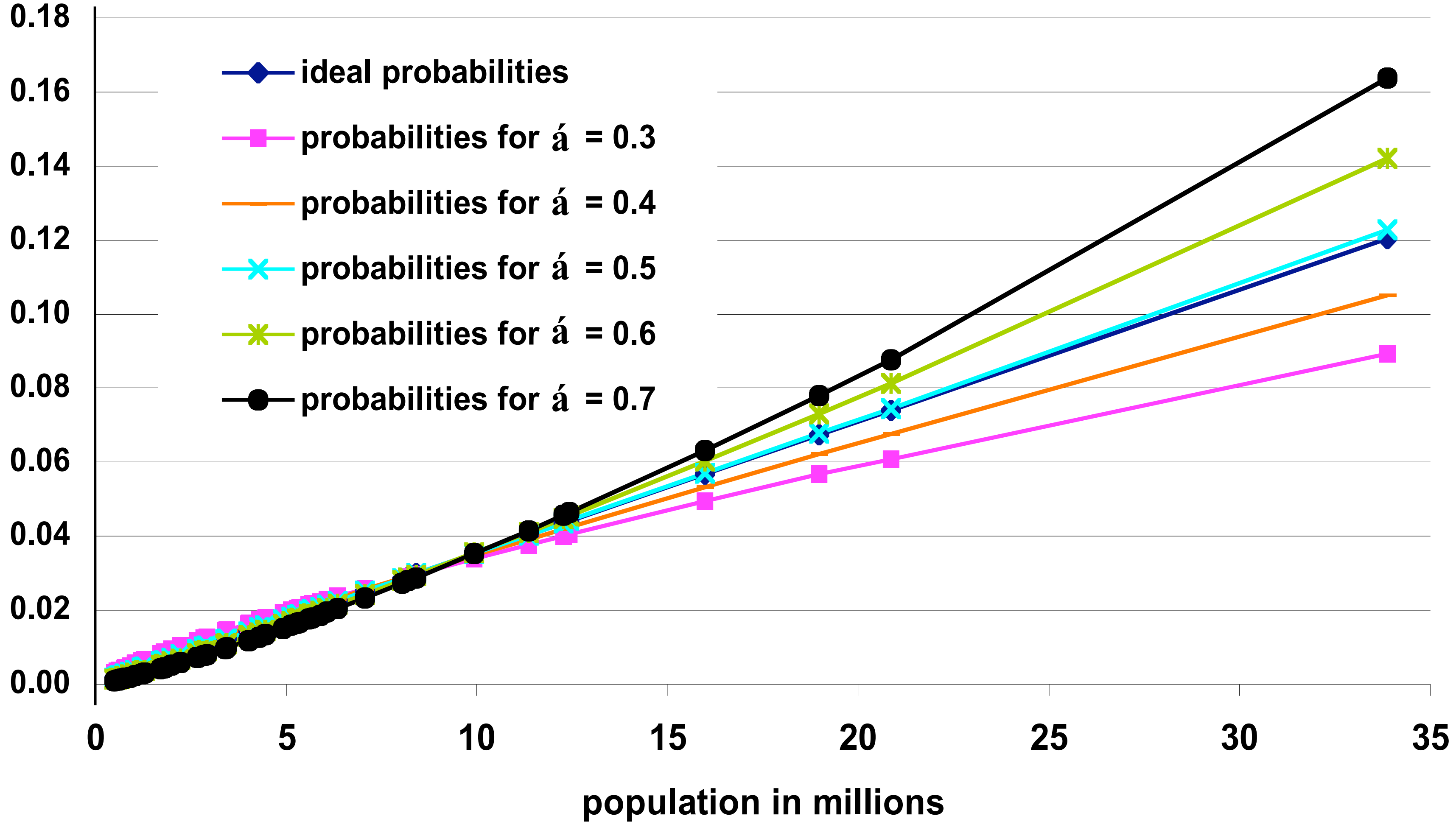
EU Council of Ministers

- Using EU25 population data, α*=0.5 with 50%-quota would give almost equal representation:



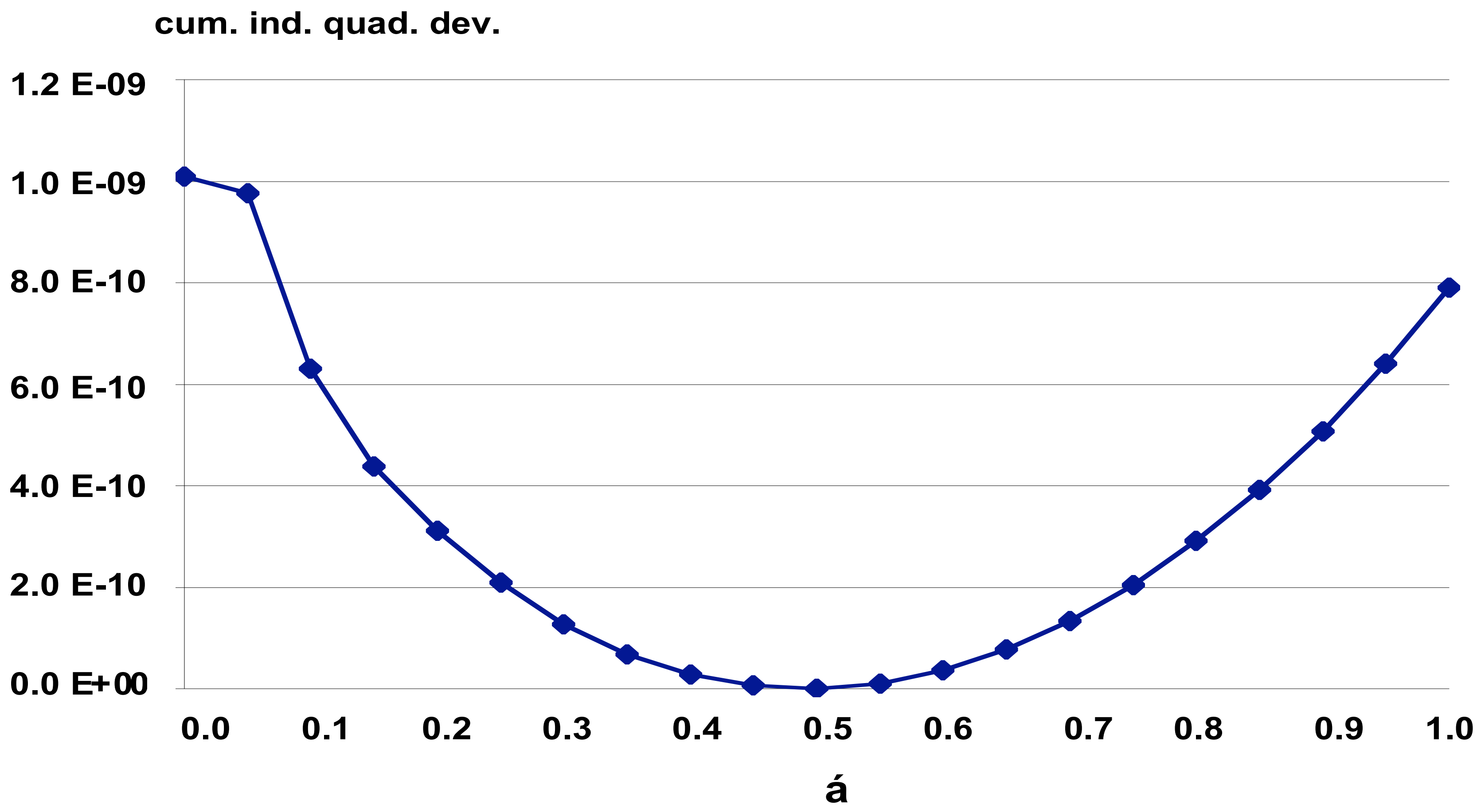
US Electoral College

- Again, α^*=0.5 comes very close to equal representation:



US Electoral College

- Cumulative individual quadratic deviation from equal representation in the US Electoral College:

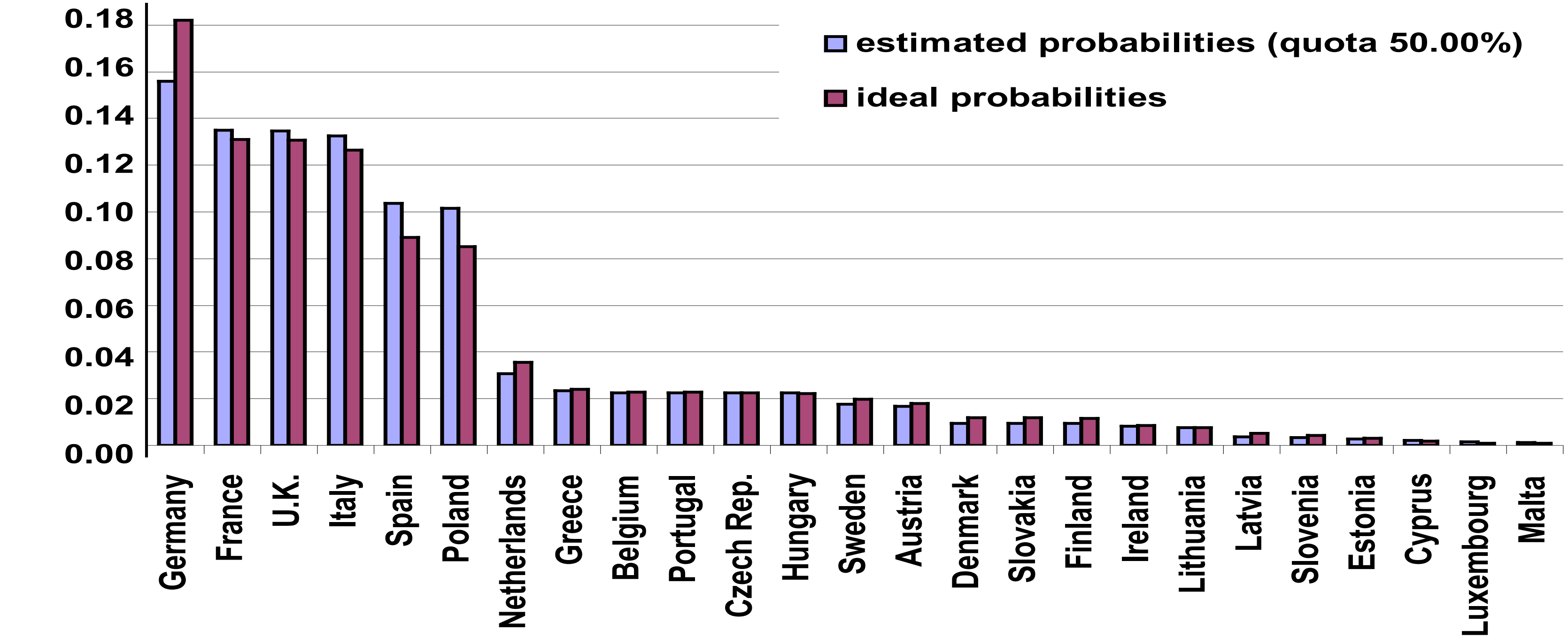


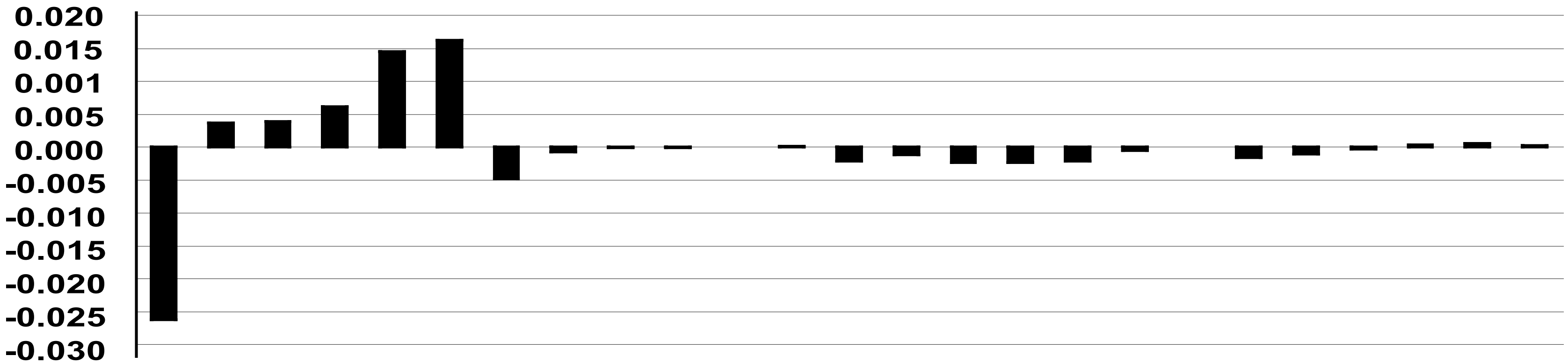
Concluding remarks

- While analytical proof of this looks out of reach, assigning *weights* proportional to *square root of population* provides a quite stable and satisfying answer to our question
- Thus, Penrose's square root rule is much more robust than suggested in the literature;
  unexpectedly, it extends from binary decisions to rich (one-dimensional convex) policy spaces, from simple games to spatial voting
- Future research:
  - A better reference point than voting weight
  - Effects of supermajority rule

EU Council of Ministers

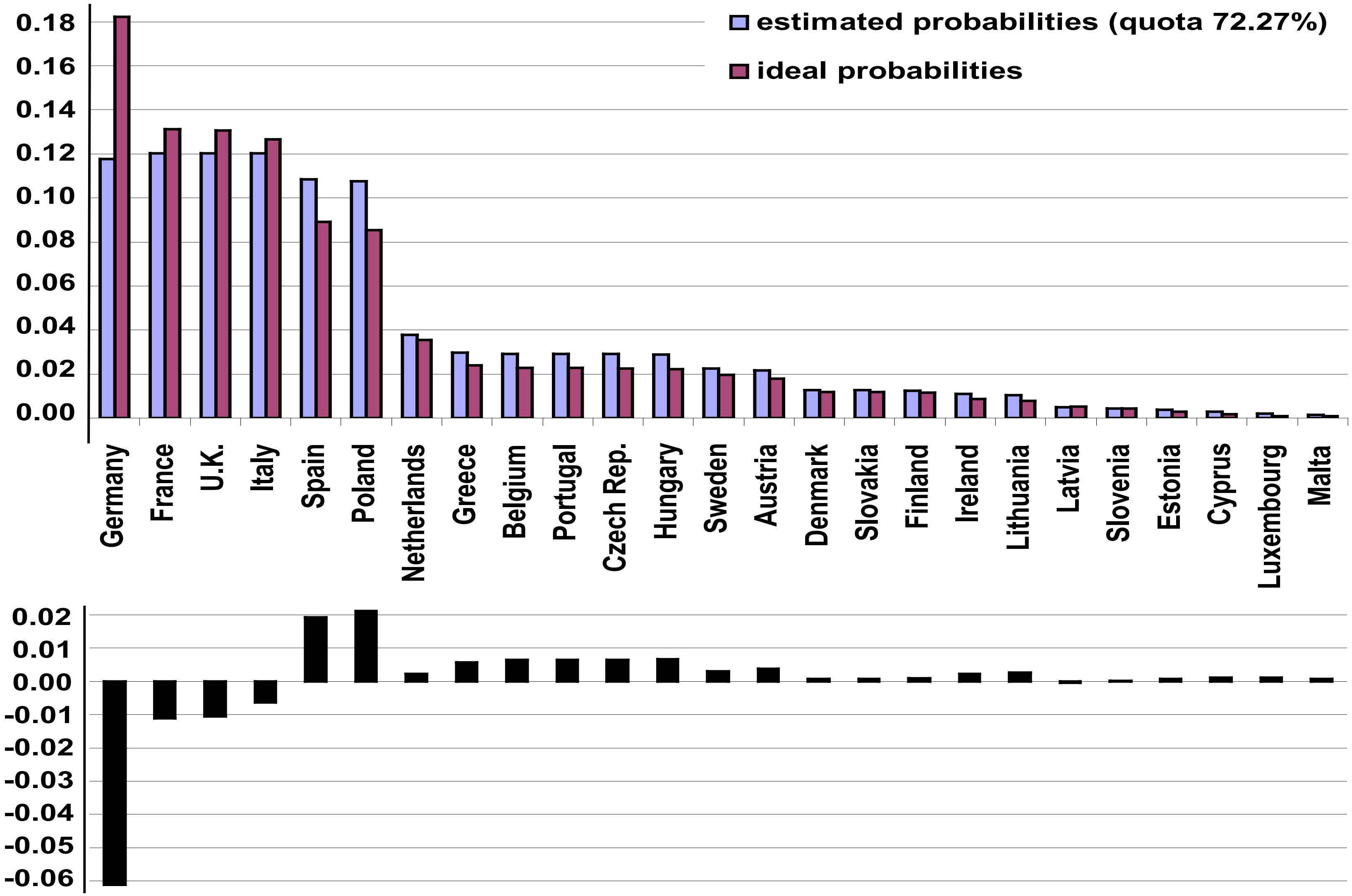
- Nice weights and quota of 50%:





EU Council of Ministers

- Nice weights and quota of 72.2%:



Results: uniformly distributed n_j

- We look, first, at m ranging from 10 to 50 with randomly generated constituency sizes $n_1, \ldots, n_m$ and, second, at two prominent real-world population configurations
- With i.i.d. n_j from $U[0.5 \cdot 10^6, 99.5 \cdot 10^6]$, optimal α is:

# const	(I)	(II)	(III)	(IV)
10	**0.5**	**0.6**	**0.39**	**0.00**
	(1.22×10^{-11})	(1.04×10^{-11})	(2.20×10^{-12})	(2.39×10^{-11})
15	**0.5**	**0.5**	**0.49**	**0.48**
	(1.43×10^{-11})	(1.45×10^{-13})	(2.79×10^{-14})	(8.84×10^{-14})
20	**0.5**	**0.5**	**0.49**	**0.49**
	(4.80×10^{-14})	(8.59×10^{-14})	(5.66×10^{-15})	(6.91×10^{-15})
25	**0.5**	**0.5**	**0.49**	**0.49**
	(9.25×10^{-15})	(1.28×10^{-14})	(5.37×10^{-15})	(7.69×10^{-15})
30	**0.5**	**0.5**	**0.49**	**0.49**
	(1.11×10^{-15})	(5.12×10^{-15})	(7.36×10^{-15})	(2.38×10^{-15})
40	**0.5**	**0.5**	**0.49**	**0.49**
	(3.38×10^{-15})	(5.11×10^{-15})	(3.69×10^{-15})	(7.02×10^{-15})
50	**0.5**	**0.5**	**0.50**	**0.50**
	(3.06×10^{-15})	(4.70×10^{-15})	(3.10×10^{-15})	(3.30×10^{-15})

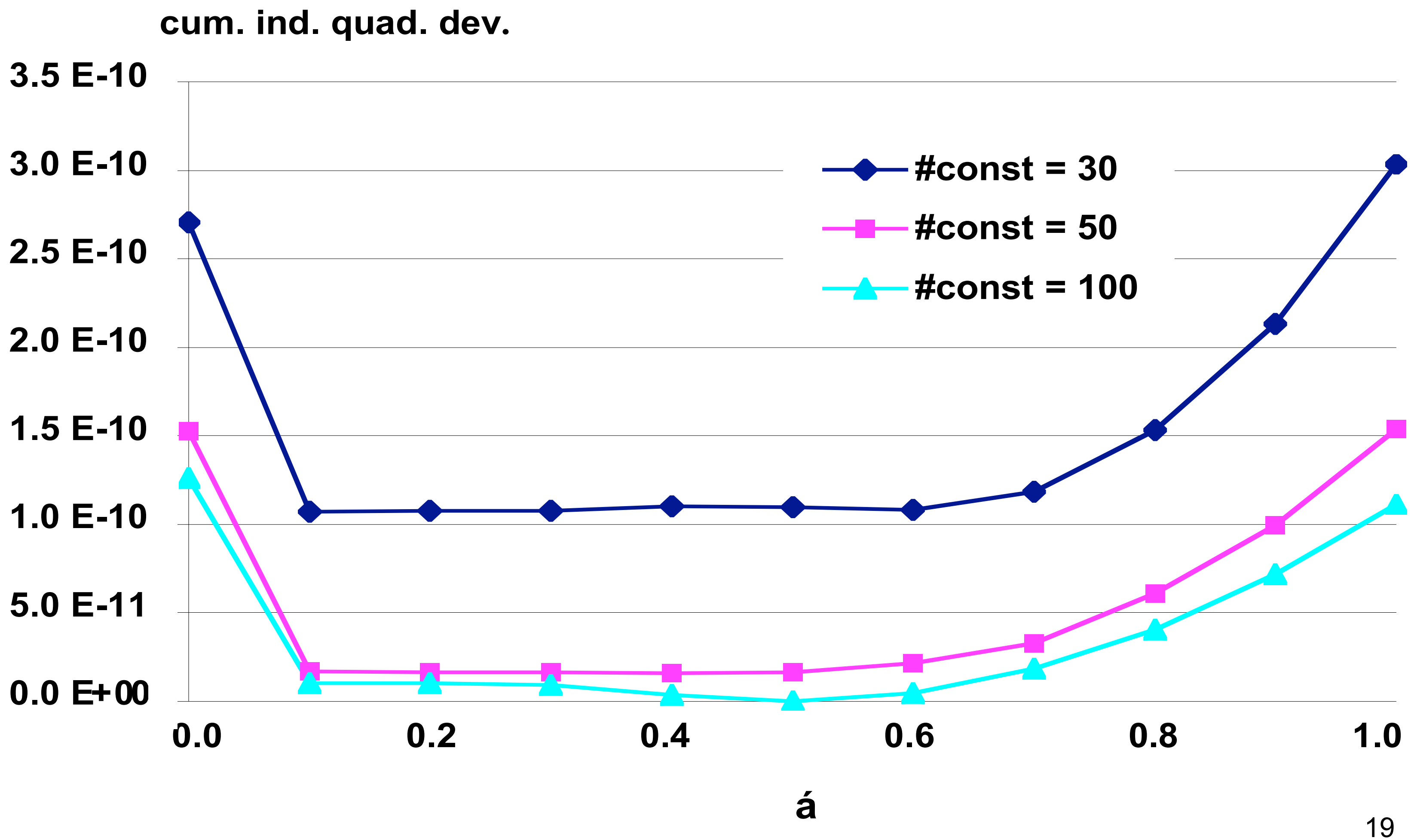
Results: normally distributed n_j

- If constituencies are created for efficiency reasons, sizes possibly are distributed around some 'ideal size'
- With i.i.d. n_j from $N(10^6; 200\,000)$, optimal α is:

# const	(I)	(II)	(III)	(IV)
10	**0.0**	**0.0**	**0.0**	**0.0**
	(1.22×10^{-9})	(1.65×10^{-9})	(9.21×10^{-9})	(1.83×10^{-9})
20	**0.6**	**0.0**	**0.6**	**0.0**
	(2.19×10^{-10})	(2.93×10^{-10})	(2.82×10^{-10})	(3.83×10^{-10})
30	**0.1**	**0.2**	**0.4**	**0.5**
	(1.07×10^{-10})	(1.07×10^{-10})	(6.94×10^{-11})	(6.76×10^{-11})
40	**0.3**	**0.4**	**0.4**	**0.5**
	(1.72×10^{-11})	(2.08×10^{-11})	(2.32×10^{-11})	(2.81×10^{-13})
50	**0.4**	**0.2**	**0.3**	**0.3**
	(1.60×10^{-11})	(7.39×10^{-12})	(3.56×10^{-11})	(4.72×10^{-11})
100	**0.5**	**0.5**	**0.5**	**0.5**
	(1.01×10^{-13})	(2.30×10^{-12})	(1.99×10^{-13})	(3.44×10^{-13})

Results: normally distributed n_j

- For moderately many similar constituencies, weighted voting may allow only quite high (and flat) inequality of representation:



Results: Pareto distributed n_j

- More realistically, with i.i.d. n_j from a Pareto distribution with skewness parameter k, optimal α is:

	Number of constituencies					
κ	10	20	30	40	50	100
1.0	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**
	(1.32×10^{-9})	(6.99×10^{-11})	(1.32×10^{-11})	(1.87×10^{-11})	(1.31×10^{-10})	(3.79×10^{-12})
1.8	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**
	(3.25×10^{-9})	(4.78×10^{-11})	(2.41×10^{-11})	(2.25×10^{-11})	(1.86×10^{-11})	(1.04×10^{-12})
3.4	**0.0**	**0.5**	**0.5**	**0.5**	**0.5**	**0.5**
	(3.72×10^{-9})	(5.64×10^{-11})	(2.41×10^{-11})	(3.27×10^{-12})	(2.67×10^{-12})	(8.88×10^{-13})
5.0	**0.0**	**0.0**	**0.1**	**0.15**	**0.1**	**0.5**
	(1.08×10^{-8})	(3.61×10^{-9})	(1.03×10^{-10})	(2.85×10^{-11})	(1.91×10^{-10})	(7.54×10^{-13})

→ General finding:
As long as $m \geq 15$, $\alpha^*=0.5$ comes close to equal representation;
it does best amongst all considered rules for large m
(and for small m if the electorate's partition is not too equal nor oceanic)