The Probability of Sen's Liberal Paradox

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A. Introduction

1. Research Question

- How likely is Sen's Liberal Paradox in a relatively unrestricted domain?
- **More Precisely: How often will Acyclicity, weak Pareto, and Minimal Liberalism conflict in a relatively unrestricted domain?**

B. Sen's Theorem

1. Theorem: there is no social decision function that can simultaneously satisfy unrestricted domain, weak Pareto, and minimal liberalism.
2. Definitions
   a. Unrestricted Domain (**U***): Every logically possible set of individual orderings is included in the domain of the social decision function.
   b. Weak Pareto (**P***): If every individual prefers alternative x to alternative y, then society must prefer x to y.

B. Sen's Theorem

c. Minimal Liberalism (**L***): There are at least two individuals such that for each of them there is at least one pair of alternatives over which he/she is decisive.

An individual is decisive over {x, y}, if his/her preference on {x, y} implies the social preferences on {x, y}.

d. Acyclicity (**A***): A social ordering is acyclical over

X $\leftrightarrow$ for all $x_1 \ldots x_j$ in X:

$[\{x_1 P x_2 \;\&\; x_2 P x_3 \;\&\; \ldots \&\; x_{j-1} P x_j\} \rightarrow x_1 R x_j]$.

B. Sen's Theorem

3. Sen (1979) shows that if a preference relation is reflexive and complete, then acyclicity is necessary and sufficient for a finite choice set. In other words, **A*** is necessary and sufficient for a social decision function.
4. The proof is shown by proving that there exists a set of preferences which cause a contradiction between **A***, **P***, and **L***.

C. The Probability of the Paradox

1. Distinction:
   a. Analytical representations of an unrestricted domain.

      vs.

   b. How often the paradox occurs in an empirical setting.

      The former attempts to get at the probability in an unrestricted domain. The latter implicitly restricts the domain.
2. Gehrlein (1983) claims that his formula for the probability of a transitive order requires 24 summation signs.

   Hence, the probabilities must be approximated or simulated.
3. We simulate using two types of spatial voting models.

D. Euclidean Preferences, Undefined Dimensions

1. Assumptions:
   a. Multi-dimensional spatial voting model, bounded on the unit square (2 dimensions).
   b. N individuals.
   c. A alternatives.
   d. Euclidean preferences.
      (i.e. an individual prefers alternatives closer to his/her ideal point more than alternatives farther away).
   e. Each individual is decisive over *da* pairs of alternatives. Pairs are randomly assigned, such that an individual can be assigned any pair but no two individuals will be assigned decisive rights over the same pair.
   f. Strict transitivity of social relation is evaluated in place of acyclicity (for computational reasons).
      Suggestions for how to evaluate acyclicity are welcome.

D. Euclidean Preferences, Undefined Dimensions

2. Trial
   a. Draw N ideal points from a uniform distribution.
   b. Draw A alternatives from a uniform distribution.
   c. Determine individual preferences based on distance.
   d. Pareto: if everyone prefers some alternative x to another alternative y, record xPy in the social ranking.
   e. Minimal Liberalism: Randomly draw a cell address from an AxA upper triangular matrix of social preferences (without replacement) and assign this address to individual i. Then record individual i's preference for this pair as the social preference.

      Repeat for N individuals and then for *da* pairs per individual.
   f. Transitivity: test for the strict transitivity of the social ranking and update preferences required by transitivity.

      If a contradiction is found, the contradiction is noted and the trial terminates. If a contradiction is not found, the routine is repeated until all social preferences have been updated.

D. Euclidean Preferences, Undefined Dimensions

3. We repeat the process a large number of times and count the relative frequency of a contradiction.
    a. For 500,000 trials we are 95% confident that the true probability is within 0.0015 of the figures reported.

D. Euclidean Preferences, Undefined Dimensions

Table 1: Probability of a Contradiction in 2 Dimensions

	da		
N	1	10	20
3	0.015	0.676	0.947
41	0.274	1.000	1.000
60	0.703	1.000	1.000

Note: $A = 50$; hence, $\binom{A}{2} = 1,225$. Rounded figures indicate the probability of a contradiction between A^*, P^*, and L^*. Trials $= 500,000$.

Note: the prob. of a contradiction increases as N, or *da,* increases.

As *da* increases, a greater proportion of the 1,225 possible pairs of alternatives are determined by decisiveness.

As N increases, the probability of a Pareto preferred alternative decreases. The conflict is primarily between L* and A* for N ≥ 41.

D. Euclidean Preferences, Undefined Dimensions

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Less someone think the results can be fully explained by the assignment of too many decisive pairs…

If N=41, *da* = 10, and A = 200, the probability of a contradiction still rounds to 1 (at 10^{-5}).

D. Euclidean Preferences, Undefined Dimensions

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If N=41, *da* = 10, and A = 200, the probability of a contradiction still rounds to 1 (at 10^{-5}).

Note: for A=200, the decisive pairs are only 2% of the 19,900 possible pairs.

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Table 2: Probability of a Contradiction in 20 Dimensions

	da		
N	1	10	20
3	0.001	0.637	0.971
41	0.236	1.000	1.000
60	0.684	1.000	1.000

Note: $A = 50$; hence, $\binom{A}{2} = 1,225$. Rounded figures indicate the probability of a contradiction between $A*$, $P*$, and L^*. Trials $= 500,000$.

E. Uniform Preferences, Decisive Dimensions

1. Motivation.
   a. Two Individuals (Billie and Muddy) with decisive rights over 1 attribute each (whether they sleep on their belly or their back, everything else equal).
   b. Four possible alternatives (or states):

      s_1 = {Muddy belly, Billie belly}

      s_2 = {Muddy belly, Billie back}

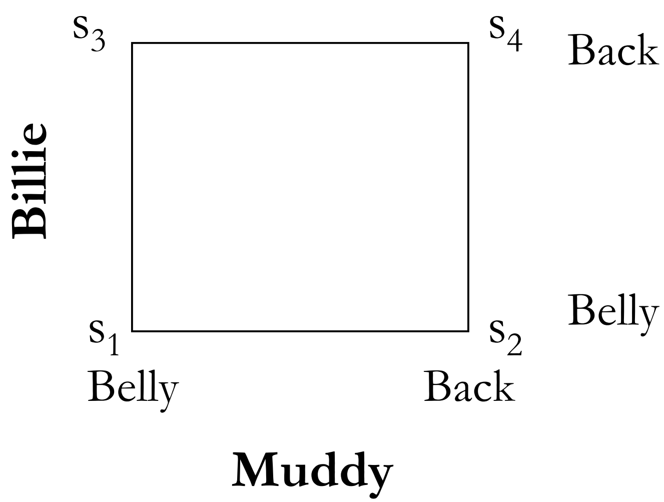
      s_3 = {Muddy back, Billie belly}

      s_4 = {Muddy back, Billie back}
   c. Billie is decisive over: $\{s_1, s_2\}$; $\{s_3, s_4\}$

      Muddy is decisive over: $\{s_1, s_3\}$; $\{s_2, s_4\}$

E. Uniform Preferences, Decisive Dimensions

d. Represented as dichotomous alternatives on a unit square.



e. In the spirit of Sen (1970), an individual will be decisive over a pair of alternatives $\{s_j, s_k\}$ if and only if s_j and s_k are identical on every other dimension, except his/her decisive dimension.

E. Uniform Preferences, Decisive Dimensions

2. Preliminaries
   a. Assign each individual a decisive dimension.
   b. Allow 2^d alternatives for d dimensions.
3. Trial.
   a. Randomly assign each individual a most preferred alternative, second most preferred, etc. such that each of the A! possible preference rankings are equally likely.
   b. Determine Pareto preferences, as before.
   c. Allow individual i to be decisive over $\{s_j, s_k\}$ if and only if s_j and s_k are identical on every other dimension, except i's decisive dimension.
   d. Determine the transitivity of the social relation, as before.

E. Uniform Preferences, Decisive Dimensions

Table 3: Probability of a Contradiction, Dichotomous Choice and Decisive Dimensions

	$Dim = 2$	$Dim = 3$	$Dim = 4$	$Dim = 5$
N	$A = 4$	$A = 8$	$A = 16$	$A = 32$
2	0.215	0.398	0.668	0.916
3		0.667	0.895	0.992
4			0.977	1.000

Note: Each individual is assigned 1 decisive dimension. Hence, the number of dimensions, d, must be at least as great as N. Furthermore, $A = 2^d$. Rounded figures indicate the probability of a contradiction between $A*$, $P*$, and L^*. Trials $= 500,000$.

Note: On diagonal figures reflect cases where the number of dimensions equal the number of individuals.
Off diagonal figures reflect cases where there are additional dimensions representing attributes that no individual would be decisive over (such as Bush being president or not).

E. Uniform Preferences, Decisive Dimensions

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Note: Each individual is assigned 1 decisive dimension. Hence, the number of dimensions, d, must be at least as great as N. Furthermore, $A = 2^d$. Rounded figures indicate the probability of a contradiction between $A*$, $P*$, and L^*. Trials $= 500,000$.

Results: The probability of a contradiction increases as N increases.
More surprisingly, the probability of a contradiction increases as the number of unassigned dimensions increases.

This is surprising because, as the number of dimensions increases, the proportion of alternative pairs determined by decisiveness decreases. Hence, the paradox appears pervasive.

F. Conclusion

1. Sen's theorem shows that conflicts between the Pareto criterion, minimum liberalism, and acyclicity can always occur.
2. We find that not only do these conflicts exist, our preliminary analysis suggest that they are fairly common in the domain of possible preferences.
3. Furthermore, as N increases, the probability of a contradiction increases as well. This implies that the paradox is more likely to occur in populations the size of a small town than in committees with few decisive rights.