Social Choice, «Jeux Simples», and Logic

Tijmen R. Daniëls

tijmen.daniels@uva.nl

Universiteit van Amsterdam
Amsterdam School of Economics

and

Institute for Logic, Language and Computation

December 6, 2006

Introduction

Social Choice Theory is concerned with questions on how a group of agents can decide as a collective, in a way that reflects the individual opinions of those involved.

The history of the subject can be traced back to 18th century enlightenment thinkers (Bentham, Borda, Condorcet).

The subject's popularity amongst economists is due to Arrow's Theorem. Recent work by logician's has been motivated by Judgement Aggregation.

Motivation

1. Modal definability results are interesting for social choice theorists that care about complexity.

2. Studying the behaviour of aggregation functions results in a nice application of neighbourhood semantics (vs Kripke semantics).

Basic Setup

- Basic Language: $\mathscr{L}_c$ (=classical propositional logic)
- Formulae are constructed from a finite set of sentence letters $\mathsf{Q} = \{q_1, q_2, \ldots, q_h\}$, and the logical connectives $\wedge, \neg$.
- $\models$ is standard (semantic) entailment relationship
- Given $Q \subseteq \mathsf{Q}$, φ_Q is the formula:

$$\varphi_Q := \bigwedge_{q_i \in Q} q_i \wedge \bigwedge_{q_i \in (\mathsf{Q} - Q)} \neg q_i$$

Basic Setup

- Basic Language: $\mathscr{L}_c$ (=classical propositional logic)
- Formulae are constructed from a finite set of sentence letters $\mathsf{Q} = \{q_1, q_2, \ldots, q_h\}$, and the logical connectives $\wedge, \neg$.
- $\models$ is standard (semantic) entailment relationship
- Given $Q \subseteq \mathsf{Q}$, φ_Q is the formula:

$$\varphi_Q := \bigwedge_{q_i \in Q} q_i \wedge \bigwedge_{q_i \in (\mathsf{Q} - Q)} \neg q_i$$

Basic Setup

- Basic Language: $\mathscr{L}_c$ (=classical propositional logic)
- Formulae are constructed from a finite set of sentence letters $\mathsf{Q} = \{q_1, q_2, \ldots, q_h\}$, and the logical connectives $\wedge, \neg$.
- $\models$ is standard (semantic) entailment relationship
- Given $Q \subseteq \mathsf{Q}$, φ_Q is the formula:

$$\varphi_Q := \bigwedge_{q_i \in Q} q_i \wedge \bigwedge_{q_i \in (\mathsf{Q} - Q)} \neg q_i$$

Basic Setup

- Basic Language: $\mathscr{L}_c$ (=classical propositional logic)
- Formulae are constructed from a finite set of sentence letters $\mathsf{Q} = \{q_1, q_2, \ldots, q_h\}$, and the logical connectives $\wedge, \neg$.
- $\models$ is standard (semantic) entailment relationship
- Given $Q \subseteq \mathsf{Q}$, φ_Q is the formula:

$$\varphi_Q := \bigwedge_{q_i \in Q} q_i \ \wedge \bigwedge_{q_i \in (\mathsf{Q} - Q)} \neg q_i$$

Choices of agents...

- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \rightarrow \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

Choices of agents...

- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \to \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

Choices of agents...

- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \rightarrow \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

Choices of agents...

- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \to \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

Choices of agents. . .

- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \to \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

Choices of agents…

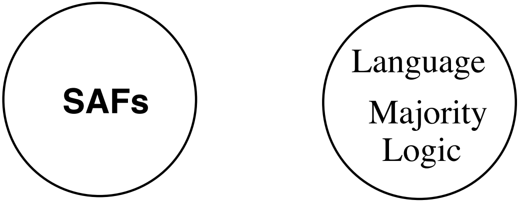
- N is a set of agents
- A CHOICE FUNCTION is a function $\pi : N \to \mathscr{P}(\mathsf{Q})$; intuitively $\pi(i)$ provides the information on the choices of agent i
- Π is the set of all such functions.
- If $\varphi_{\pi(i)} \models \psi$ then we say that "agent i accepts ψ"
- The set of all agents that accept $q_j \in \mathsf{Q}$, that is $\{i \in N \mid q_j \in \pi(i)\}$, is denoted by $[\![q_j]\!]_\pi$
- More generally, for $\psi \in \mathscr{L}_c$, $[\![\psi]\!]_\pi := \{i \in N \mid \varphi_{\pi(i)} \models \psi\}$.

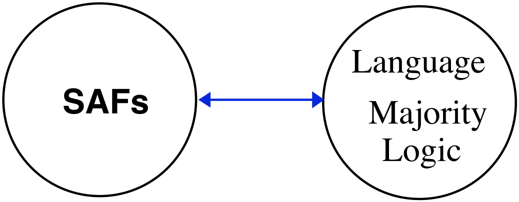
Social Aggregation Functions

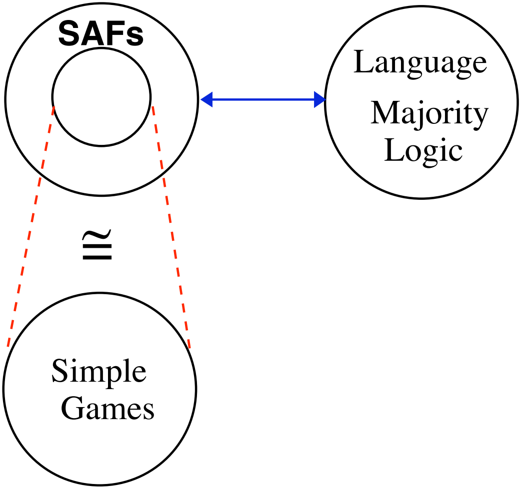
A SOCIAL AGGREGATION FUNCTION (SAF) is a (possibly partial) function $F : \Pi \to \mathscr{P}(\mathscr{L}_\mathsf{c})$;

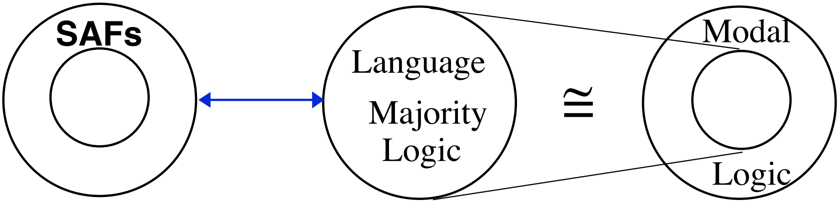
Look at it as a decision procedure.

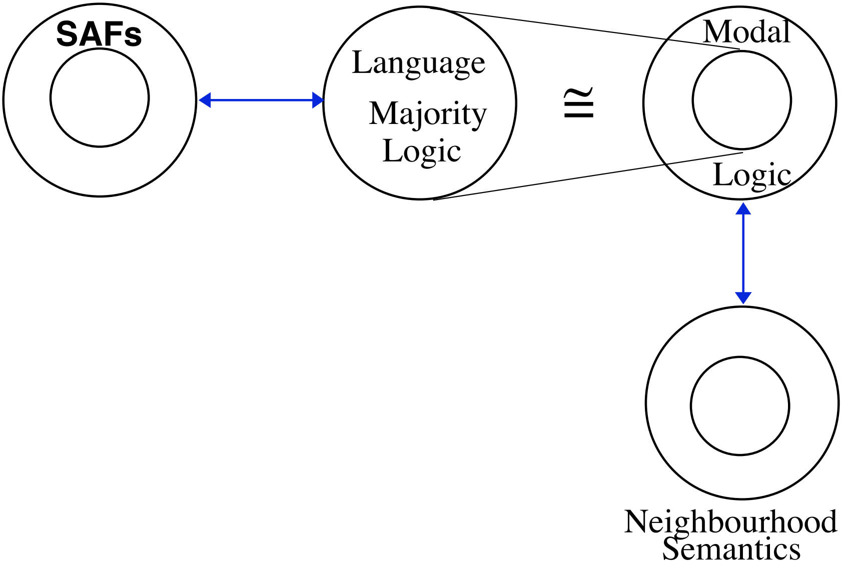
$F(\pi)$ denotes the socially accepted sentences of $\mathscr{L}_c$ given π.

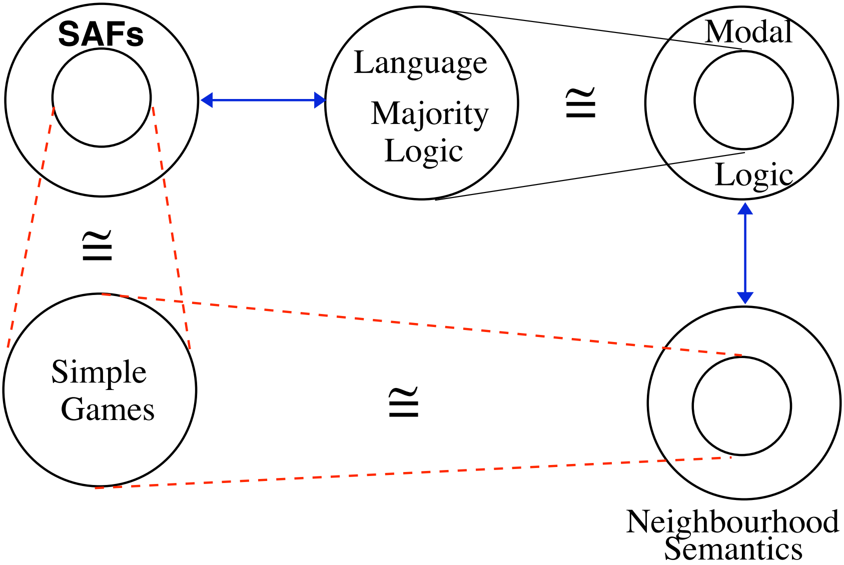

SAFs
Language
Majority
Logic


SAFs
Language
Majority
Logic


SAFs
Language
Majority
Logic
≅
Simple
Games


SAFs
Language
Majority
Logic
≅
Modal
Logic


SAFs
Language
Majority
Logic
≅
Modal
Logic
Neighbourhood
Semantics


SAFs
Language
Majority
Logic
≅
Modal
Logic
≅
Simple
Games
≅
Neighbourhood
Semantics

Semantics for Majority Logic

The language $\mathscr{L}_\Box$ is generated by:

$$\psi ::= \Box\alpha \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid \bot \qquad \text{with each } \alpha \in \mathscr{L}_c.$$

Let F be a SAF, and π a choice function in the domain of F. The pair (F, π) is called a MODEL. Let $\psi, \psi_1, \psi_2 \in \mathscr{L}_\Box$ and $\Psi \subseteq \mathscr{L}_\Box$. We write:

$$\begin{array}{rll} & (F,\pi) \Vdash \Box\varphi & \text{iff } \alpha \in \mathscr{L}_c \text{ and } \alpha \in F(\pi); \\ & (F,\pi) \Vdash \psi_1 \wedge \psi_2 & \text{iff } (F,\pi) \Vdash \psi_1 \text{ and } (F,\pi) \Vdash \psi_2; \\ & (F,\pi) \Vdash \neg\psi & \text{iff } (F,\pi) \nVdash \psi; \\ & (F,\pi) \Vdash \bot & \text{never}, \\ \text{and:} & F \Vdash \psi & \text{iff for all } \pi \in \mathrm{dom}(F), (F,\pi) \Vdash \psi, \\ \text{and finally:} & F \Vdash \Psi & \text{iff for all } \psi \in \Psi, F \Vdash \psi. \end{array}$$

Simple Games

Dedekind, Von Neumann and Morgenstern, Monjardet, Taylor and Zwicker (book), Mihara…

Let W be a collection of subsets of N, closed under supersets.

$$A \in W, A \subseteq B \subseteq N \text{ implies } B \in W.$$

The pair (N, W) is called a SIMPLE GAME.

Simple games generalise the notion of 'MAJORITY'.

A simple game is called FINITE if N is a finite set.

Examples of simple games. (1) The simple majority game.
(2) The collections of subsets of N that contain some fixed player i ("Ultrafilters").

A correspondence...

Let $\pi, \pi' \in \Pi$, $\varphi, \psi \in \mathscr{L}_c$ be arbitrary. A SAF is said to satisfy:

UNIVERSAL DOMAIN (UD) iff the domain of F is Π;

MONOTONICITY (M) iff whenever $[\![\varphi]\!]_\pi \subseteq [\![\varphi]\!]_{\pi'}$ then $\varphi \in F(\pi) \implies \varphi \in F(\pi')$;

NEUTRALITY (N) (or 'systematicity') iff whenever $[\![\varphi]\!]_\pi = [\![\psi]\!]_{\pi'}$ then $\varphi \in F(\pi) \iff \psi \in F(\pi')$.

Theorem

The neutral, monotonic and universal domain SAFs stand in 1-1 correspondence with simple games.

For such SAFs we can restate our semantics:

$$(F_{(N,W)}, \pi) \Vdash \Box\alpha \quad \text{iff} \quad [\![\alpha]\!]_\pi \in W$$

A correspondence...

Let $\pi, \pi' \in \Pi$, $\varphi, \psi \in \mathscr{L}_c$ be arbitrary. A SAF is said to satisfy:
UNIVERSAL DOMAIN (UD) iff the domain of F is Π;
MONOTONICITY (M) iff whenever $[\![\varphi]\!]_\pi \subseteq [\![\varphi]\!]_{\pi'}$ then $\varphi \in F(\pi) \implies \varphi \in F(\pi')$;
NEUTRALITY (N) (or 'systematicity') iff whenever $[\![\varphi]\!]_\pi = [\![\psi]\!]_{\pi'}$ then $\varphi \in F(\pi) \iff \psi \in F(\pi')$.

Theorem

The neutral, monotonic and universal domain SAFs stand in 1-1 correspondence with simple games.

For such SAFs we can restate our semantics:

$$(F_{(N,W)}, \pi) \Vdash \Box\alpha \quad \text{iff} \quad [\![\alpha]\!]_\pi \in W$$

A correspondence...

Let $\pi, \pi' \in \Pi$, $\varphi, \psi \in \mathscr{L}_c$ be arbitrary. A SAF is said to satisfy:

UNIVERSAL DOMAIN (UD) iff the domain of F is Π;

MONOTONICITY (M) iff whenever $[\![\varphi]\!]_\pi \subseteq [\![\varphi]\!]_{\pi'}$ then $\varphi \in F(\pi) \implies \varphi \in F(\pi')$;

NEUTRALITY (N) (or 'systematicity') iff whenever $[\![\varphi]\!]_\pi = [\![\psi]\!]_{\pi'}$ then $\varphi \in F(\pi) \iff \psi \in F(\pi')$.

Theorem

The neutral, monotonic and universal domain SAFs stand in 1-1 correspondence with simple games.

For such SAFs we can restate our semantics:

$$(F_{(N,W)}, \pi) \Vdash \Box\alpha \quad \text{iff} \quad [\![\alpha]\!]_\pi \in W$$

More on simple games

A simple game is called PROPER if it satisfies:

$$A \in W \text{ implies } N - A \notin W.$$

A simple game is called STRONG if it satisfies:

$$A \notin W \text{ implies } N - A \in W.$$

A player $i \in N$ is called a DUMMY PLAYER of (N, W) if:

$$\text{for all } X \in \mathscr{P}(N),\ X \in W \iff X \cup \{i\}. \in W$$

Generalising this notion to sets, a set $A \subseteq N$ is called a SET OF DUMMY PLAYERS if:

$$\text{for all } X \in \mathscr{P}(N), \text{ and any } B \subseteq A,\ X \in W \iff X \cup B \in W.$$

Given $\Omega = (N, W)$, denote the set of its dummy players by $\mathscr{D}(\Omega)$.

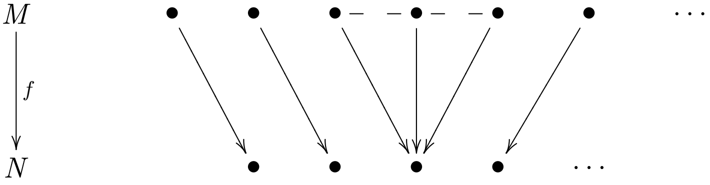
The Rudin Keisler Ordering

If N and M are two sets of agents, and $\Omega = (N, W)$ is a simple game and f is a map from N to M, $f_*(W)$ is the subset of $\mathscr{P}(M)$ given by:

$$A \in f_*(W) \iff f^{-1}[A] \in W,$$

where $f^{-1}[A]$ is the preimage of A (that is: $\{i \in N \mid f(i) \in A\}$).

The game $(M, f_*(W))$ is obtained by considering the players of Ω identified by f to vote as a bloc.



$f : M \to N$ identifies the three indicated points.

RK projection and isomorphism

- **Definition.** We say that $\Omega = (N, W)$ is RK-BELOW $\Omega' = (N', W')$, iff there exists a map f such that $W = f_*(W')$; in this case we write $\Omega \leq_{\mathrm{RK}} \Omega'$.
- Games Ω and Ω' are called ISOMORPHIC if $\Omega \leq_{\mathrm{RK}} \Omega' \leq_{\mathrm{RK}} \Omega$.
- If $\Omega \subseteq \mathscr{P}(N \cup A)$ is obtained from $\Omega' \subseteq \mathscr{P}(N)$ by adding a set of dummy players A, then Ω is isomorphic to Ω'.
- If the projection function $f : N \to M$ isn't surjective, then the set $M - \mathrm{ran}(f)$ will consist of dummy players.

RK projection and isomorphism

- **Definition.** We say that $\Omega = (N, W)$ is RK-BELOW $\Omega' = (N', W')$, iff there exists a map f such that $W = f_*(W')$; in this case we write $\Omega \leq_{\text{RK}} \Omega'$.
- Games Ω and Ω' are called ISOMORPHIC if $\Omega \leq_{\text{RK}} \Omega' \leq_{\text{RK}} \Omega$.
- If $\Omega \subseteq \mathscr{P}(N \cup A)$ is obtained from $\Omega' \subseteq \mathscr{P}(N)$ by adding a set of dummy players A, then Ω is isomorphic to Ω'.
- If the projection function $f : N \to M$ isn't surjective, then the set $M - \text{ran}(f)$ will consist of dummy players.

RK projection and isomorphism

- **Definition.** We say that $\Omega = (N, W)$ is RK-BELOW $\Omega' = (N', W')$, iff there exists a map f such that $W = f_*(W')$; in this case we write $\Omega \leq_{\mathrm{RK}} \Omega'$.
- Games Ω and Ω' are called ISOMORPHIC if $\Omega \leq_{\mathrm{RK}} \Omega' \leq_{\mathrm{RK}} \Omega$.
- If $\Omega \subseteq \mathscr{P}(N \cup A)$ is obtained from $\Omega' \subseteq \mathscr{P}(N)$ by adding a set of dummy players A, then Ω is isomorphic to Ω'.
- If the projection function $f : N \to M$ isn't surjective, then the set $M - \mathrm{ran}(f)$ will consist of dummy players.

RK projection and isomorphism

- **Definition.** We say that $\Omega = (N, W)$ is RK-BELOW $\Omega' = (N', W')$, iff there exists a map f such that $W = f_*(W')$; in this case we write $\Omega \leq_{\text{RK}} \Omega'$.
- Games Ω and Ω' are called ISOMORPHIC if $\Omega \leq_{\text{RK}} \Omega' \leq_{\text{RK}} \Omega$.
- If $\Omega \subseteq \mathscr{P}(N \cup A)$ is obtained from $\Omega' \subseteq \mathscr{P}(N)$ by adding a set of dummy players A, then Ω is isomorphic to Ω'.
- If the projection function $f : N \to M$ isn't surjective, then the set $M - \text{ran}(f)$ will consist of dummy players.

Time to return to logic. . .

- **observation.** $\mathscr{L}_{\Box}$ is a fragment of modal logic, $\mathscr{L}_{\Box\Box}$, which makes use of the grammar:

$$\psi ::= q \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Box\psi \mid \bot \qquad \text{with each } q \in \mathsf{Q}$$

- For majority logic, distributivity of $\Box$ fails. One semantics for modal logic that deals with this adequately is provided by NEIGHBOURHOOD STRUCTURES.
- **definition.** A (monotonic) NEIGHBOURHOOD FRAME (N.F.) is a pair (S, ν), S is a nonempty set of states, $\nu : S \to \mathscr{P}(\mathscr{P}(S))$ is the neighbourhood function; for each $s \in S$, $\nu(s)$ is closed under supersets. A NEIGHBOURHOOD MODEL (N.M.), $\mathfrak{M} = (S, \nu, V)$, is a n.f. paired with a valuation $V : W \to \mathscr{P}(\mathsf{Q})$.
- **observation.** If the neighbourhood function is a constant function, then a neighbourhood frame is just a simple game.
- **observation.** The sematics for Neutral Monotonic UD SAFs sits 'inside' neighbourhood semantics.

Time to return to logic...

- **observation.** $\mathscr{L}_{\Box}$ is a fragment of modal logic, $\mathscr{L}_{\Box\Box}$, which makes use of the grammar:

$$\psi ::= q \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Box\psi \mid \bot \qquad \text{with each } q \in \mathsf{Q}$$

- For majority logic, distributivity of $\Box$ fails. One semantics for modal logic that deals with this adequately is provided by NEIGHBOURHOOD STRUCTURES.
- **definition.** A (monotonic) NEIGHBOURHOOD FRAME (N.F.) is a pair (S, ν), S is a nonempty set of states, $\nu : S \to \mathscr{P}(\mathscr{P}(S))$ is the neighbourhood function; for each $s \in S$, $\nu(s)$ is closed under supersets. A NEIGHBOURHOOD MODEL (N.M.), $\mathfrak{M} = (S, \nu, V)$, is a n.f. paired with a valuation $V : W \to \mathscr{P}(\mathsf{Q})$.
- **observation.** If the neighbourhood function is a constant function, then a neighbourhood frame is just a simple game.
- **observation.** The sematics for Neutral Monotonic UD SAFs sits 'inside' neighbourhood semantics.

Time to return to logic...

- **observation.** $\mathscr{L}_\Box$ is a fragment of modal logic, $\mathscr{L}_{\Box\Box}$, which makes use of the grammar:

$$\psi ::= q \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Box\psi \mid \bot \qquad \text{with each } q \in \mathsf{Q}$$

- For majority logic, distributivity of $\Box$ fails. One semantics for modal logic that deals with this adequately is provided by NEIGHBOURHOOD STRUCTURES.
- **definition.** A (monotonic) NEIGHBOURHOOD FRAME (N.F.) is a pair (S, ν), S is a nonempty set of states, $\nu : S \to \mathscr{P}(\mathscr{P}(S))$ is the neighbourhood function; for each $s \in S$, $\nu(s)$ is closed under supersets. A NEIGHBOURHOOD MODEL (N.M.), $\mathfrak{M} = (S, \nu, V)$, is a n.f. paired with a valuation $V : W \to \mathscr{P}(\mathsf{Q})$.
- **observation.** If the neighbourhood function is a constant function, then a neighbourhood frame is just a simple game.
- **observation.** The sematics for Neutral Monotonic UD SAFs sits 'inside' neighbourhood semantics.

Time to return to logic. . .

- **observation.** $\mathscr{L}_\square$ is a fragment of modal logic, $\mathscr{L}_{\square\square}$, which makes use of the grammar:

$$\psi ::= q \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \square\psi \mid \bot \qquad \text{with each } q \in \mathsf{Q}$$

- For majority logic, distributivity of $\square$ fails. One semantics for modal logic that deals with this adequately is provided by NEIGHBOURHOOD STRUCTURES.
- **definition.** A (monotonic) NEIGHBOURHOOD FRAME (N.F.) is a pair (S, ν), S is a nonempty set of states, $\nu : S \to \mathscr{P}(\mathscr{P}(S))$ is the neighbourhood function; for each $s \in S$, $\nu(s)$ is closed under supersets. A NEIGHBOURHOOD MODEL (N.M.), $\mathfrak{M} = (S, \nu, V)$, is a n.f. paired with a valuation $V : W \to \mathscr{P}(\mathsf{Q})$.
- **observation.** If the neighbourhood function is a constant function, then a neighbourhood frame is just a simple game.
- **observation.** The sematics for Neutral Monotonic UD SAFs sits 'inside' neighbourhood semantics.

Time to return to logic...

- **observation.** $\mathscr{L}_\Box$ is a fragment of modal logic, $\mathscr{L}_{\Box\Box}$, which makes use of the grammar:

$$\psi ::= q \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \Box\psi \mid \bot \qquad \text{with each } q \in \mathsf{Q}$$

- For majority logic, distributivity of $\Box$ fails. One semantics for modal logic that deals with this adequately is provided by NEIGHBOURHOOD STRUCTURES.
- **definition.** A (monotonic) NEIGHBOURHOOD FRAME (N.F.) is a pair (S, ν), S is a nonempty set of states, $\nu : S \to \mathscr{P}(\mathscr{P}(S))$ is the neighbourhood function; for each $s \in S$, $\nu(s)$ is closed under supersets. A NEIGHBOURHOOD MODEL (N.M.), $\mathfrak{M} = (S, \nu, V)$, is a n.f. paired with a valuation $V : W \to \mathscr{P}(\mathsf{Q})$.
- **observation.** If the neighbourhood function is a constant function, then a neighbourhood frame is just a simple game.
- **observation.** The sematics for Neutral Monotonic UD SAFs sits 'inside' neighbourhood semantics.

Comparing Modal Logic and Majority Logic: Structures

Some natural constructions known from GAME THEORY have MODAL ANALOGUES:

Comparing Modal Logic and Majority Logic: Structures

Some natural constructions known from GAME THEORY have MODAL ANALOGUES:

Example. Let $\Omega = (N, W)$ and $\Omega' = (N', W')$. The PRODUCT GAME $\Omega \otimes \Omega$ is given by:

$$(N \cup N', \{X \subseteq \mathscr{P}(N \cup N') \mid X \cap N \in W \text{ and } X \cap N' \in W'\})$$

The BICAMERAL MEET $\Omega \sqcap \Omega'$ is the special case where N and N' are disjoint sets.

Modal analogue when looking at Simple Games: behaves a bit like DISJOINT UNION, in particular φ is *valid* in $\Omega \otimes \Omega$ iff it is valid in *both* underlying games.

Comparing Modal Logic and Majority Logic: Structures

Some natural constructions known from GAME THEORY have MODAL ANALOGUES:

Example. RK-projection of a game Ω.

Modal analogue at the level of Simple Games: BOUNDED MORPHISMS and GENERATED SUBFRAMES. In particular φ is valid in the projection Ω' iff it is valid in the original game Ω.

Axioms and Definability Results (1)

- One occupation of social of social choice to pin down a CLASS OF SOCIAL WELFARE FUNCTIONS by looking at "desired" behavioural properties of ("axiomatic approach").
- Modal logicians pin down a CLASS OF FRAMES by looking at formulae that are VALID throughout the class.
- For $\mathscr{L}_\Box$, such formulae express the behavioural properties of the logic of decision making throughout the class.
- The two perspectives are just the same thing!

Examples for $\mathscr{L}_\Box$.

- A simple game is PROPER if and only if $\Box p \rightarrow \neg\Box\neg p$ is valid.
- A simple game is STRONG if and only if $\neg\Box\neg p \rightarrow \Box p$ is valid.

Axioms and Definability Results (1)

- One occupation of social of social choice to pin down a CLASS OF SOCIAL WELFARE FUNCTIONS by looking at "desired" behavioural properties of ("axiomatic approach").
- Modal logicians pin down a CLASS OF FRAMES by looking at formulae that are VALID throughout the class.
- For $\mathscr{L}_\Box$, such formulae express the behavioural properties of the logic of decision making throughout the class.
- The two perspectives are just the same thing!

Examples for $\mathscr{L}_\Box$.

- A simple game is PROPER if and only if $\Box p \to \neg\Box\neg p$ is valid.
- A simple game is STRONG if and only if $\neg\Box\neg p \to \Box p$ is valid.

Axioms and Definability Results (1)

- One occupation of social of social choice to pin down a CLASS OF SOCIAL WELFARE FUNCTIONS by looking at "desired" behavioural properties of ("axiomatic approach").
- Modal logicians pin down a CLASS OF FRAMES by looking at formulae that are VALID throughout the class.
- For $\mathscr{L}_{\Box}$, such formulae express the behavioural properties of the logic of decision making throughout the class.
- The two perspectives are just the same thing!

Examples for $\mathscr{L}_{\Box}$.

- A simple game is PROPER if and only if $\Box p \rightarrow \neg\Box\neg p$ is valid.
- A simple game is STRONG if and only if $\neg\Box\neg p \rightarrow \Box p$ is valid.

Axioms and Definability Results (1)

- One occupation of social of social choice to pin down a CLASS OF SOCIAL WELFARE FUNCTIONS by looking at "desired" behavioural properties of ("axiomatic approach").
- Modal logicians pin down a CLASS OF FRAMES by looking at formulae that are VALID throughout the class.
- For $\mathscr{L}_\Box$, such formulae express the behavioural properties of the logic of decision making throughout the class.
- The two perspectives are just the same thing!

Examples for $\mathscr{L}_\Box$.

- A simple game is PROPER if and only if $\Box p \rightarrow \neg\Box\neg p$ is valid.
- A simple game is STRONG if and only if $\neg\Box\neg p \rightarrow \Box p$ is valid.

Axioms and Definability Results (1)

- One occupation of social of social choice to pin down a CLASS OF SOCIAL WELFARE FUNCTIONS by looking at "desired" behavioural properties of ("axiomatic approach").
- Modal logicians pin down a CLASS OF FRAMES by looking at formulae that are VALID throughout the class.
- For $\mathscr{L}_\Box$, such formulae express the behavioural properties of the logic of decision making throughout the class.
- The two perspectives are just the same thing!

Examples for $\mathscr{L}_\Box$.

- A simple game is PROPER if and only if $\Box p \to \neg\Box\neg p$ is valid.
- A simple game is STRONG if and only if $\neg\Box\neg p \to \Box p$ is valid.

Axioms and Definability Results (2)

Another example for $\mathscr{L}_\Box$. Suppose we want the majority logic to behave 'consistently', i.e. just like classical propositional logic. What we need is the two formulae above, and $\Box$ to distribute:

$$\Box p \wedge \Box q \leftrightarrow \Box(p \wedge q)$$

- These three formulae pin down the class of ultrafilters, hence the DICTATORIAL M-N-UD-SAFs.
- So, the impossibility results obtained in social choice theory emerge.

Axioms and Definability Results

Because the constructions introduced for simple games preserve validity of formulae, they have consequences for the definability of classes of SAFs in the language $\mathscr{L}_\Box$, just like in modal logic.

Theorem

Let K *a class of M-N-UD-SAFs.* K *is definable by a set of* $\mathscr{L}_\Box$*-formulae only if it is closed under* RK*-projections and bicameral meet.*

Other direction?

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (1)

Example for $\mathscr{L}_{\Box\Box}$.

- Of course, the modal language $\mathscr{L}_{\Box\Box}$ can also be used to express properties of SAFs.
- F is a consensus SAFiff F satisfies M, N, and UD and $F \Vdash \Box p \rightarrow p$.
- The class of consensus SAFs is not definable by $\Box$, since it is not closed under RK-projection.
- What separates the expressivity of $\mathscr{L}_{\Box}$ from $\mathscr{L}_{\Box\Box}$?

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (1)

Example for $\mathscr{L}_{\Box\Box}$.

- Of course, the modal language $\mathscr{L}_{\Box\Box}$ can also be used to express properties of SAFs.
- F is a consensus SAFiff F satisfies M, N, and UD and $F \Vdash \Box p \rightarrow p$.
- The class of consensus SAFs is not definable by $\Box$, since it is not closed under RK-projection.
- What separates the expressivity of $\mathscr{L}_{\Box}$ from $\mathscr{L}_{\Box\Box}$?

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (1)

Example for $\mathscr{L}_{\Box\Box}$.

- Of course, the modal language $\mathscr{L}_{\Box\Box}$ can also be used to express properties of SAFs.
- F is a consensus SAFiff F satisfies M, N, and UD and $F \Vdash \Box p \rightarrow p$.
- The class of consensus SAFs is not definable by $\Box$, since it is not closed under RK-projection.
- What separates the expressivity of $\mathscr{L}_{\Box}$ from $\mathscr{L}_{\Box\Box}$?

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (1)

Example for $\mathscr{L}_{\Box\Box}$.

- Of course, the modal language $\mathscr{L}_{\Box\Box}$ can also be used to express properties of SAFs.
- F is a consensus SAFiff F satisfies M, N, and UD and $F \Vdash \Box p \rightarrow p$.
- The class of consensus SAFs is not definable by $\Box$, since it is not closed under RK-projection.
- What separates the expressivity of $\mathscr{L}_{\Box}$ from $\mathscr{L}_{\Box\Box}$?

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (2)

RK-INVARIANCE. Let (F, π) and (F, π') be simple models. A formula $\varphi \in \mathscr{L}_{\Box\Box}$ is RK-invariant iff whenever $(F, \pi), i \Vdash \varphi$ and $(F, \pi) \leq^{\mathrm{M}}_{\mathrm{RK}} (F', \pi')$, then there is a state (or agent) i' in the model (F', π') such that $(F', \pi'), i' \Vdash \varphi$. In words, satisfaction of φ is preserved under RK-projection.

Theorem

Let $\varphi \in \mathscr{L}_{\Box\Box}$. Then φ is equivalent to a formula $\psi \in \mathscr{L}_{\Box}$ on all models that are generated by simple games if and only if φ is RK-invariant.

Expressivity of $\mathscr{L}_{\Box}$ as a fragment of $\mathscr{L}_{\Box\Box}$ (2)

RK-Invariance. Let (F, π) and (F, π') be simple models. A formula $\varphi \in \mathscr{L}_{\Box\Box}$ is RK-invariant iff whenever $(F, \pi), i \Vdash \varphi$ and $(F, \pi) \leq^{\mathrm{M}}_{\mathrm{RK}} (F', \pi')$, then there is a state (or agent) i' in the model (F', π') such that $(F', \pi'), i' \Vdash \varphi$. In words, satisfaction of φ is preserved under RK-projection.

Theorem

Let $\varphi \in \mathscr{L}_{\Box\Box}$. Then φ is equivalent to a formula $\psi \in \mathscr{L}_{\Box}$ on all models that are generated by simple games if and only if φ is RK*-invariant.*

Concluding Remarks

- Applying tools from Modal Logic. In particular frame definability results correspond to the axiomatic approach in an interesting way.
- Trivialising Neighbourhood Semantics?
- . . . Operations on structures behave subtly different. An analogue of disjoint union is missing.
- Further directions: An infinite language might give a full GT theorem for finite or so called "core complete" SAFs.
- Further directions: Relaxing Neutrality? Using modalities for each formula ("[φ]").
- Thank you!

Concluding Remarks

- Applying tools from Modal Logic. In particular frame definability results correspond to the axiomatic approach in an interesting way.
- Trivialising Neighbourhood Semantics?
- ...Operations on structures behave subtly different. An analogue of disjoint union is missing.
- Further directions: An infinite language might give a full GT theorem for finite or so called "core complete" SAFs.
- Further directions: Relaxing Neutrality? Using modalities for each formula ("[φ]").
- THANK YOU!

Concluding Remarks

- Applying tools from Modal Logic. In particular frame definability results correspond to the axiomatic approach in an interesting way.
- Trivialising Neighbourhood Semantics?
- ...Operations on structures behave subtly different. An analogue of disjoint union is missing.
- Further directions: An infinite language might give a full GT theorem for finite or so called "core complete" SAFs.
- Further directions: Relaxing Neutrality? Using modalities for each formula ("$[\varphi]$").
- THANK YOU!

Concluding Remarks

- Applying tools from Modal Logic. In particular frame definability results correspond to the axiomatic approach in an interesting way.
- Trivialising Neighbourhood Semantics?
- ...Operations on structures behave subtly different. An analogue of disjoint union is missing.
- Further directions: An infinite language might give a full GT theorem for finite or so called “core complete” SAFs.
- Further directions: Relaxing Neutrality? Using modalities for each formula ("$[\varphi]$").
- THANK YOU!

Concluding Remarks

- Applying tools from Modal Logic. In particular frame definability results correspond to the axiomatic approach in an interesting way.
- Trivialising Neighbourhood Semantics?
- ...Operations on structures behave subtly different. An analogue of disjoint union is missing.
- Further directions: An infinite language might give a full GT theorem for finite or so called "core complete" SAFs.
- Further directions: Relaxing Neutrality? Using modalities for each formula ("[φ]").
- THANK YOU!