On Complexity of Lobbying in Multiple Referenda

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Complexity in Social Sciences

Recently there was a surge of interest for complexity in some areas of economics and political science.

- As "bounded rationality" stems from inherent limits to human information-processing capabilities, complexity allows us to have an insight into this misterious concept and, to the extent, quantify it.
- It is recognised that computational limits have direct economic implications. Complexity of the problem is directly related to the costs of solving it.
- Complexity might work in our favour protecting integrity of social choice from manipulation.

What do we aim to achieve?

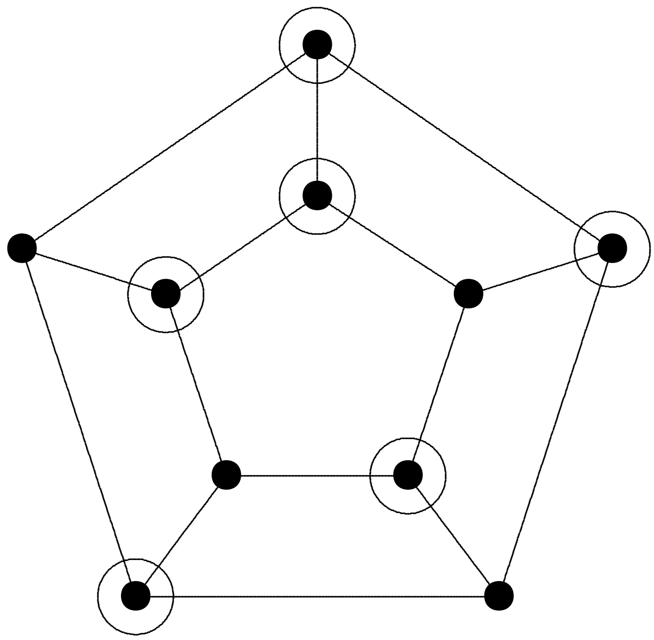
In this talk we will discuss lobbying under direct and representative democracy.

Our goals are broader:

- to emphasise the role of parameterized complexity analysis for naturally parameterized problems whose important practical applications have small parameter values;
- to introduce a problem complete for the class of parameterized complexity problems $W[2]$ and formulated in terms of social sciences.

NP-Completeness Can Be Misleading. Vertex Cover

Definition 1 *A vertex cover for a graph* $G = (V, E)$ *is a set of vertices* $V' \subseteq V$ *such that for every edge* $uv \in E$*,* $u \in V'$*, or* $v \in V'$ *(or both).*



Garey and Johnson singled out six model NP-complete problems. VERTEX COVER is one of them.

Vertex Cover (continued)

Problem: VERTEX COVER
Instance: A graph $G = (V, E)$ and a positive integer k.
Question: Does G have a vertex cover of size at most k?

VERTEX COVER has an algorithm with a running time $O(1.2738^k + kn)$ and can be solved practically for all n, when $k \leq 500$.

Conclusion:

NP-completeness, is not sufficient alone to describe intractability questions for naturally parameterized problems whose important applications have small parameter values. It must be complemented with a parameterized complexity analysis.

Parameterized Problems and Fixed-Parameter Tractability

Parameterized complexity analysis deals with problems which have a distinguished parameter k.

Definition 2 *A parameterized problem is a set* $P \subseteq \Sigma^* \times \mathbb{N}$*, where* Σ *is a finite alphabet.*

If $(x, k) \in \Sigma^* \times \mathbb{N}$ is an instance of a parameterized problem, we refer to x as the input and k as the parameter.

A problem P is said to be Fixed Parameter Tractable (FPT) if there is an algorithm, that given a pair $(x, k) \in \Sigma^* \times \mathbb{N}$ decides whether or not $(x, k) \in P$ in at most

$$f(k)|x|^c$$

steps, where f is an arbitrary computable function and c does not depend on k.

Fixed-Parameter Reducibilities

Before talking about completeness I have to explain reducibilities first.

Definition 3 *Let* $P \subseteq \Sigma^* \times \mathbb{N}$ *and* $P' \subseteq \Sigma'^* \times \mathbb{N}$ *be two parameterized problems. An FPT-reduction from* P *to* P' *is an algorithm that computes for every instance* (x, k) *of* P *an instance* (x', k') *of* P' *in time* $g(k) \cdot |x|^c$ *such that* $k' \leq h(k)$ *and*

$$(x, k) \in P \Longleftrightarrow (x', k') \in P'$$

for some computable functions $g, h\colon \mathbb{N} \to \mathbb{N}$ *and a constant* $c \in \mathbb{N}$.

W-Hierarchy

There is a natural hierarchy of parameterized complexity classes

$$FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots .$$

intuitively based on the complexity of circuits required to check a solution.

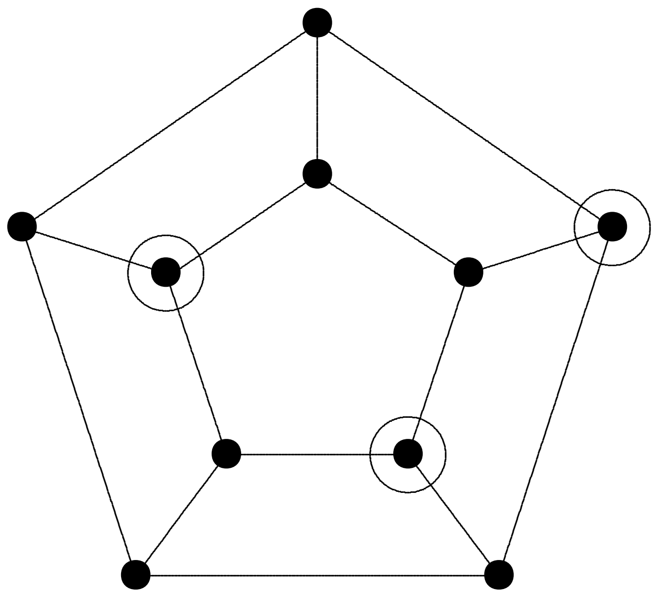
If $P = NP$, the W-hierarchy also collapces.

k-INDEPENDENT SET is an example of a $W[1]$-complete propblem. It is believed to be not FPT.

Definition 4 *An independent set for a graph* $G = (V, E)$ *is a set of vertices* $V' \subseteq V$ *such that for no edge* $uv \in E$ *both* $u \in V'$*, and* $v \in V'$*.*

Independent Set

Problem: Independent Set
Instance: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Does G have an independent set of size k?



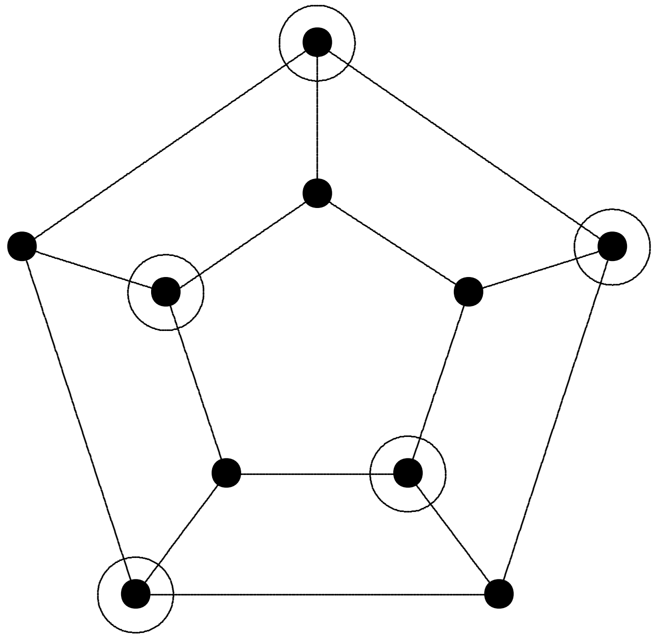
3-independent set

It is also NP-complete, of course.

Dominating Set

The best known complete problem for $W[2]$ is k-dominating set.

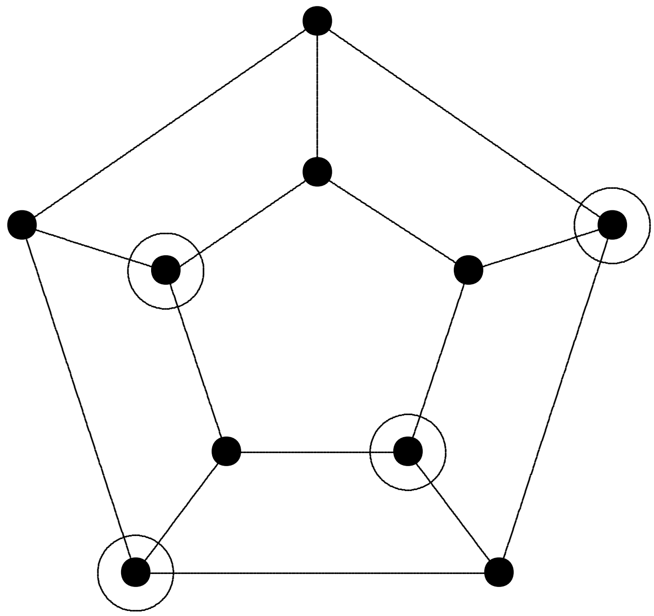
Definition 5 *A dominating set for a graph* $G = (V, E)$ *is a set of vertices* $V' \subseteq V$ *such that for every vertex* $u \in V$ *either* $u \in V'$ *or there exists an edge* $uv \in E$ *with* $v \in V'$.



5-dominating set

Independent Dominating Set

Problem: INDEPENDENT DOMINATING SET
Instance: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Does G have an independent dominating set of size k?



4-independent dominating set

This problem is also $W[2]$-complete.

Optimal Lobbying Assumptions

- n voters are voting in m referenda, in each of them they have to vote “Yes” (1) or “No” (0).
- Information about voters’ inclinitions towards the m issues voted in referenda is known to the Lobby.
- Lobby wants a specific outcome for each of the referenda.
- Lobby has a limited budget and can buy any k voters (and tell them how to vote).

Optimal Lobbying (Example)

	1	**2**	**3**	**4**	**5**
Voter 1	0	1	1	1	0
Voter 2	0	0	1	0	1
Voter 3	0	0	0	1	1
Voter 4	1	1	1	0	0
Voter 5	0	0	0	1	0
Voter 6	1	0	1	0	0
Voter 7	1	0	1	0	0
Result	0	0	1	0	0
Lobby	1	1	1	1	1

Voters 5 and 6 can be bribed to achieve the desired result.

Optimal Lobbying

Problem: OPTIMAL LOBBYING

Instance: An n by m 0/1 matrix $\mathcal{E}$, a positive integer k, and a length m 0/1 target vector $\mathbf{x}$.

Parameter: k

Question: Is there a choice of k rows of the matrix, such that these rows can be edited so that in each column of the resulting matrix, a majority vote in that column yields the outcome shown in the target vector $\mathbf{x}$?

Theorem 1 OPTIMAL LOBBYING *is* $W[2]$*-complete.*

Our Reductions

INDEPENDENT $\boldsymbol{k}$-DOMINATING SET

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OPTYMAL LOBBYING

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$\boldsymbol{k}$-DOMINATING SET