The Computational Complexity of Choice Sets

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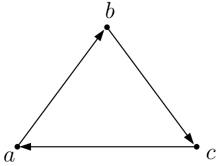
Introduction

- Social choice functions:
  - m alternatives $A = \{a_1, \ldots, a_m\}$
  - n voters with preferences $(\succsim_1, \ldots, \succsim_n)$ over A
  - Social choice function f: $f(\succsim_1, \ldots, \succsim_n) \in A$, for all $(\succsim_1, \ldots, \succsim_n)$
- Majority rule and the dominance relation (notation: $a \succ b$)
- Condorcet winner and Condorcet paradox
- Social choice sets: Smith Set, Schwartz Set, Stable Sets
- Relations between and issues concerning the computational complexity of choice sets

1: $a \succsim b \succsim c$
2: $c \succsim a \succsim b$
3: $b \succsim c \succsim a$



Tournaments, Dominance, and McGarvey's Theorem

Theorem (*McGarvey, 1953*) *Any* dominance relation can be realized by a particular preference profile, even if the individual preferences are linear.

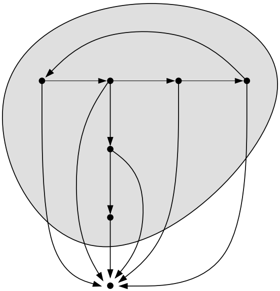
- Assumption: set of preference relations includes linear preferences.
- A *tournament* is a *complete* dominance graph.
- Analyses usually restricted to tournaments

  (*e.g.*, Laffont et.al. (1995), Hudry (2006)).
- However: Ties do occur!
- Our approach: consider all anti-symmetric dominance graphs.

Smith Property and Smith set

Definition

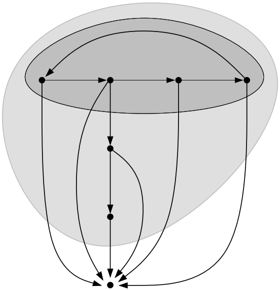
- X has the *Smith property* if: $x \succ y$, for all $x \in X$ and all $y \notin X$.
- The *Smith set* is the smallest non-empty set with the Smith property.



Schwartz Property and Schwartz Set

Definition

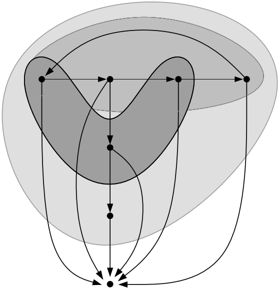
- X has the *Schwartz property* if: $y \not> x$, for all $y \notin X$ and all $x \in X$.
- The *Schwartz set* is the union of the minimal (w.r.t. $\subseteq$) non-empty sets with the Schwartz property.



Von Neumann-Morgenstern Stable Sets

Definition (Stable Sets) A set U is *stable* if both:

- $x \not\succ y$, for all $x, y \in U$ (*internal stability*),
- for all $y \notin U$, there is some $x \in U$ with $x \succ y$ (*external stability*).



Remarks:

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- Relatively unknown as a solution concept in social choice.
- Stable sets need not exist or be unique.

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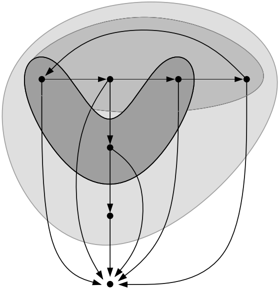

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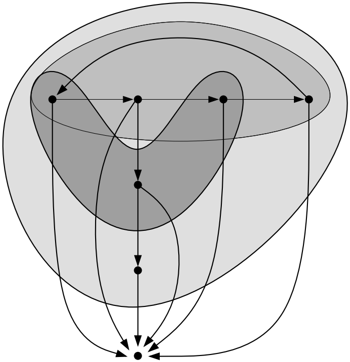
Some Properties of Choice Sets

- All sets contain the Condorcet winner as only element, if there is one.
- The Smith and Schwartz sets coincide in tournaments.
- In tournaments stable sets are equivalent to Condorcet winner.
- In general dominance graphs all sets may differ.
- The Schwartz set and every stable set are contained in the Smith set.
- Stable sets intersects with the Schwartz set.
- Also results for Copeland, Banks, and uncovered set.



Problems and Complexity Classes

Problems:

IS-CONDORCET	is *a* the Condorcet winner?
IN-SCHWARTZ	is *a* in the Schwartz set?
IN-SMITH, IN-STABLE	analogous to IN-SCHWARTZ

Complexity Classes:

$$TC^0 \subseteq L \subseteq NL \subseteq P \subseteq NP$$

Complete problems:

TC^0	majority of 1's in a bitstring
L	undirected graph reachability
NL	directed graph reachability
P	Horn SAT
NP	SAT

Computational Results

Observation IS-CONDORCET is TC^0-complete, even in the two alternative tournament case.
Proof is straightforward. Majority gate required to construct dominance graph.

Theorem IN-SMITH is TC^0-complete.

Theorem IN-SCHWARTZ is NL-complete.

N.B.: For tournaments IN-SCHWARTZ=IN-SMITH and hence TC^0-complete.

Theorem IN-STABLE is NP-complete, even if the existence of a stable set is guaranteed.

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IN-SMITH is TC^0-Complete

Theorem IN-SMITH is TC^0-complete.

Proof of hardness: IN-SMITH equivalent to IS-CONDORCET in the two alternative tournament case.

Proof of membership:

- **Observation**: *if there is set X with Smith property of size k then for all x:*

$$outdeg(x) \geq n - k \quad \text{iff} \quad x \in X.$$

- Check in parallel for $k = 1$, $k = 2$,... whether $\{x \in A \mid outdeg(x) \geq n - k\}$ has Smith property.
- Check whether $a \in \{x \in A \mid outdeg(x) \geq n - k\}$.
- This can be done in TC^0 (*i.e.*, with constant depth threshold circuits).

IN-SCHWARTZ is *NL*-Complete

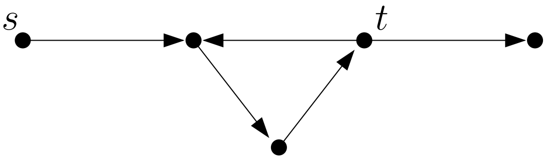
Theorem IN-SCHWARTZ is *NL*-complete.

Proof of membership:

- **Lemma:** *An alternative a is in the Schwartz set iff for all $b \in A$ with a path from b to a, there also is a path from a to b.*
- Check for each $b \in A$ whether b reachable from a.
- If so, check if a is reachable from b.
- This can be done in *NL*.

IN-SCHWARTZ is *NL*-Complete

Proof of hardness: Reduction from directed graph reachability.



Node *t* reachable from *s* iff *s* is in Schwartz set.

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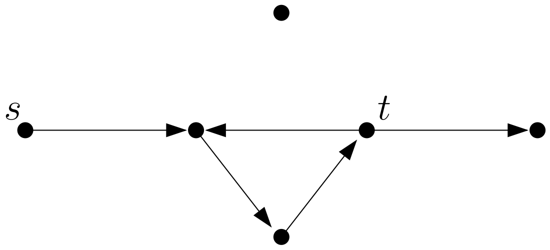

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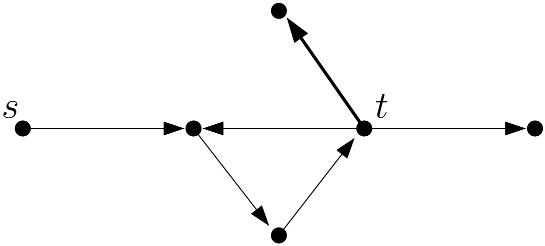

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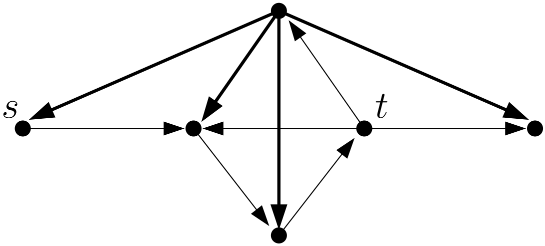

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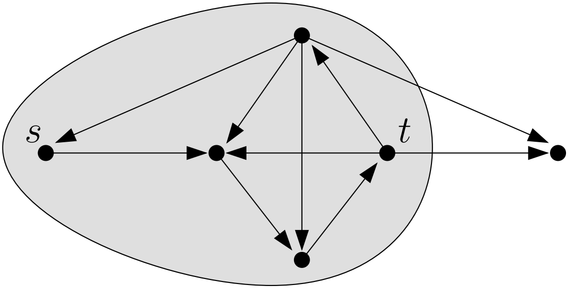

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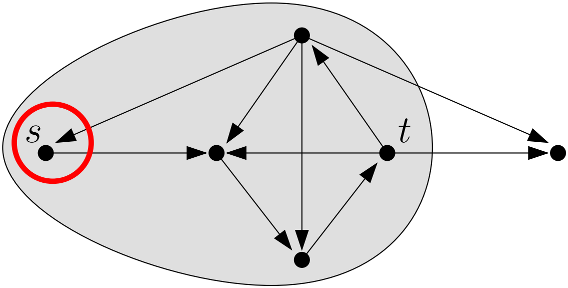

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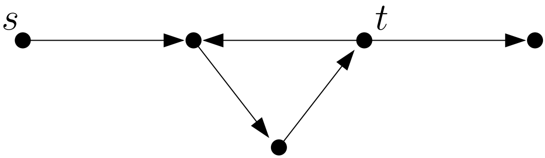

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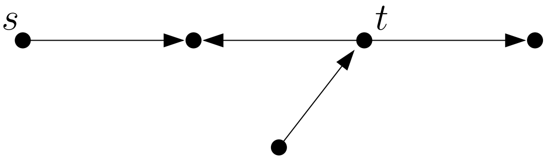

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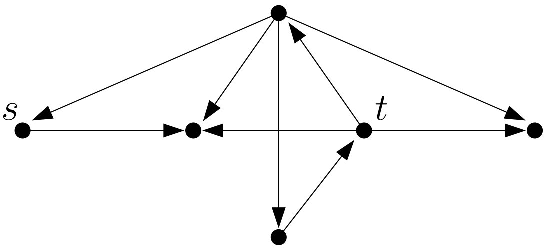

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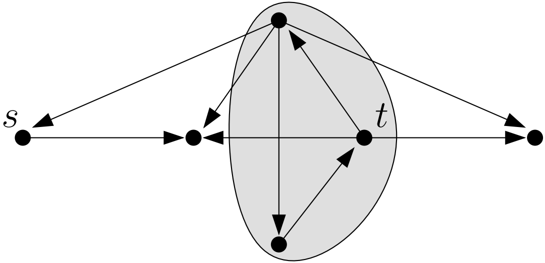

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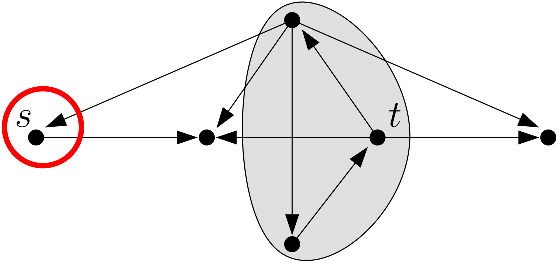

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Proof of membership: Straightforward.

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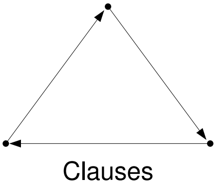

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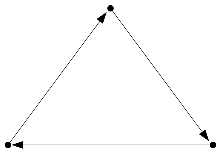
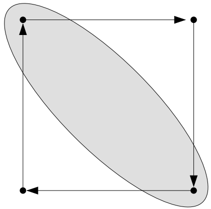


Clauses



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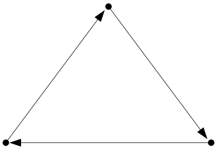

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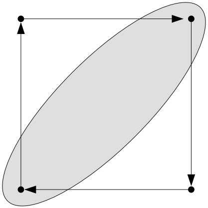

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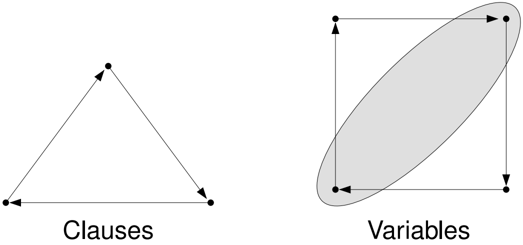
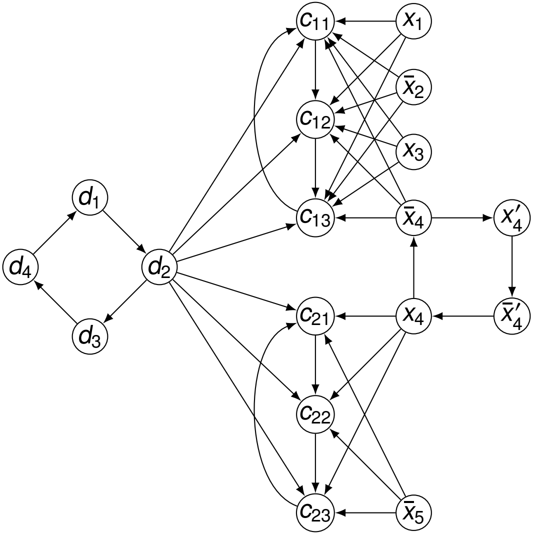


(Based on a similar construction by Chvátal, 1973).

IN-STABLE is *NP*-Complete



Dominance graph for $(x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_4 \vee \bar{x}_5)$

Summary

- Various choice sets taking over the role of maximum in dominance graphs.
- The formal properties of choice sets differ for tournaments and general dominance graphs, also w.r.t. computational complexity.
-

	tournaments	general dominance graphs
IS-CONDORCET IN-SMITH	TC^0-complete	TC^0-complete
IN-SCHWARTZ		NL-complete
IN-STABLE		NP-complete

- Generic hardness results for social choice functions with the social choice in a particular social choice set.